Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

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SUBDIRECT SUMS OF S-STRICTLY DIAGONALLY DOMINANT MATRICES[*]

RAFAEL BRU[†], FRANCISCO PEDROCHE[†], AND DANIEL B. SZYLD[‡]

Abstract. Conditions are given which guarantee that the k-subdirect sum of S-strictly diagonally dominant matrices (S-SDD) is also S-SDD. The same situation is analyzed for SDD matrices. The converse is also studied: given an SDD matrix C with the structure of a k-subdirect sum and positive diagonal entries, it is shown that there are two SDD matrices whose subdirect sum is C.

AMS subject classifications. 15A48, 15A18, 65F15.

Key words. Subdirect sum, Diagonally dominant matrices, Overlapping blocks.

1. Introduction. The concept of k-subdirect sum of square matrices emerges naturally in several contexts. For example, in matrix completion problems, overlapping subdomains in domain decomposition methods, global stiffness matrix in finite elements, etc.; see, e.g., [1], [2], [5], and references therein.

Subdirect sums of matrices are generalizations of the usual sum of matrices (a k-subdirect sum is formally defined below in section 2). They were introduced by Fallat and Johnson in [5], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric M-matrices, is positive definite or symmetric M-matrices, respectively. They also showed that this is not the case for M-matrices: the subdirect sum of two M-matrices may not be an M-matrix, and therefore the subdirect sum of two H-matrices may not be an H-matrix.

In this paper we show that for a subclass of H-matrices the k-subdirect sum of matrices belongs to the same class. We show this for certain strictly diagonally dominant matrices (SDD) and for S-strictly diagonally dominant matrices (S-SDD), introduced in [4]; see also [3], [9], for further properties and analysis. We also show that the converse holds: given an SDD matrix C with the structure of a k-subdirect sum and positive diagonal entries, then there are two SDD matrices whose subdirect sum is C.

2. Subdirect sums. Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.1}$$

[*]Received by the editors 29 March 2006. Accepted for publication 10 July 2006. Handling Editor: Angelika Bunse-Gerstner.

[†]Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València. Camí de Vera s/n. 46022 València. Spain (rbru@imm.upv.es, pedroche@imm.upv.es). Supported by Spanish DGI grant MTM2004-02998.

[‡]Department of Mathematics, Temple University, Philadelphia, PA 19122-6094, U.S.A. (szyld@temple.edu). Supported in part by the U.S. Department of Energy under grant DE-FG02-05ER25672.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

where A_{22} and B_{11} are square matrices of order k. Following [5], we call the square matrix of order $n = n_1 + n_2 - k$ given by

$$C = \begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & A_{22} + B_{11} & B_{12} \\ O & B_{21} & B_{22} \end{bmatrix} \tag{2.2}$$

the k-subdirect sum of A and B and denote it by $C = A \oplus_k B$.

It is easy to express each element of C in terms of those of A and B. To that end, let us define the following set of indices

$$\begin{array}{rcl} S_1 & = & \{1, 2, \ldots, n_1 - k\}, \\ S_2 & = & \{n_1 - k + 1, n_1 - k + 2, \ldots, n_1\}, \\ S_3 & = & \{n_1 + 1, n_1 + 2, \ldots, n\}. \end{array} \tag{2.3}$$

Denoting $C = (c_{ij})$ and $t = n_1 - k$, we can write

$$c_{ij} = \begin{cases} a_{ij} & i \in S_1, \quad j \in S_1 \cup S_2 \\ 0 & i \in S_1, \quad j \in S_3 \\ a_{ij} & i \in S_2, \quad j \in S_1 \\ a_{ij} + b_{i-t,j-t} & i \in S_2, \quad j \in S_2 \\ b_{i-t,j-t} & i \in S_2, \quad j \in S_3 \\ 0 & i \in S_3, \quad j \in S_1 \\ b_{i-t,j-t} & i \in S_3, \quad j \in S_2 \cup S_3. \end{cases} \tag{2.4}$$

Note that $S_1 \cup S_2 \cup S_3 = \{1, 2, \ldots, n\}$ and that $n = t + n_2$; see Figure 2.1.

$$C = \begin{array}{c} \begin{array}{ccc} \overleftrightarrow{\;S_1\;} & \overleftrightarrow{\qquad\qquad S_2 \qquad\qquad} & \overleftrightarrow{\quad S_3 \quad} \end{array} \\ \begin{pmatrix} a_{11} & \cdots & a_{1p} & \cdots & a_{1,n_1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{p,1} & \cdots & a_{p,p} + b_{1,1} & \cdots & a_{p,n_1} + b_{1,n_1-t} & \cdots & b_{1,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,p} + b_{n_1-t,1} & \cdots & a_{n_1,n_1} + b_{n_1-t,n_1-t} & \cdots & b_{n_1-t,n-t} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & b_{n-t,1} & \cdots & b_{n-t,n_1-t} & \cdots & b_{n-t,n-t} \end{pmatrix} \end{array} \begin{array}{l} S_1 \\ \\ S_2 \\ \\ S_3 \end{array}$$

FIG. 2.1. *Sets for the subdirect sum $C = A \oplus_k B$, with $t = n_1 - k$ and $p = t + 1$; cf.* (2.4).

3. Subdirect sums of S-SDD matrices. We begin with some definitions which can be found, e.g., in [4], [9].

DEFINITION 3.1. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, let us define the ith deleted absolute row sum as

$$r_i(A) = \sum_{j \neq i,\, j=1}^{n} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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and the ith deleted absolute row-sum with columns in the set of indices $S = \{i_1, i_2, \ldots\} \subseteq N := \{1, 2, \ldots, n\}$ as

$$r_i^S(A) = \sum_{j\neq i,\, j\in S} |a_{ij}|, \quad \forall i = 1, 2, \ldots, n.$$

Given any nonempty set of indices $S \subseteq N$ we denote its complement in N by $\bar{S} := N\backslash S$. Note that for any $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ we have that $r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A)$.

Definition 3.2. Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$ and given a nonempty subset S of $\{1, 2, \ldots, n\}$, then A is an S-strictly diagonally dominant matrix if the following two conditions hold:

$$\left.\begin{array}{lll} i) & |a_{ii}| > r_i^S(A) & \forall i \in S, \\ ii) & (|a_{ii}| - r_i^S(A))\,(|a_{jj}| - r_j^{\bar{S}}(A)) > r_i^{\bar{S}}(A)\, r_j^S(A) & \forall i \in S, \forall j \in \bar{S}. \end{array}\right\} \tag{3.1}$$

It was shown in [4] that an S-strictly diagonally dominant matrix (S-SDD) is a nonsingular H-matrix. In particular, when $S = \{1, 2, \ldots, n\}$, then $A = (a_{ij}) \in C^{n\times n}$ is a strictly diagonally dominant matrix (SDD). It is easy to show that an SDD matrix is an S-SDD matrix for any proper subset S, but the converse is not always true as we show in the following example.

Example 3.3. Consider the following matrix

$$A = \begin{bmatrix} 2.6 & -0.4 & -0.7 & -0.2 \\ -0.4 & 2.6 & -0.5 & -0.7 \\ -0.6 & -0.7 & 2.2 & -1.0 \\ -0.8 & -0.7 & -0.5 & 2.2 \end{bmatrix},$$

which is a $\{1, 2\}$-SDD matrix but is not an SDD matrix. A natural question is to ask if the subdirect sum of S-SDD matrices is in the class, but in general this is not true. For example, the 2-subdirect sum $C = A \oplus_2 A$ gives

$$C = \begin{bmatrix} 2.6 & -0.4 & -0.7 & -0.2 & 0 & 0 \\ -0.4 & 2.6 & -0.5 & -0.7 & 0 & 0 \\ -0.6 & -0.7 & 4.8 & -1.4 & -0.7 & -0.2 \\ -0.8 & -0.7 & -0.9 & 4.8 & -0.5 & -0.7 \\ 0 & 0 & -0.6 & -0.7 & 2.2 & -1.0 \\ 0 & 0 & -0.8 & -0.7 & -0.5 & 2.2 \end{bmatrix}$$

which is not a $\{1, 2\}$-SDD matrix: condition ii) of (3.1) fails for the matrix C for the cases $i = 1, j = 5$ and $i = 2, j = 5$. It can also be observed that C is not an SDD matrix.

This example motivates the search of conditions such that the subdirect sum of S-SDD matrices is in the class of S-SDD matrices (for a fixed set S).

We now proceed to show our first result. Let A and B be matrices of order n_1 and n_2, respectively, partitioned as in (2.1) and consider the sets S_i defined in (2.3). Then we have the following relations

$$\left.\begin{array}{rcl} r_i^{S_1}(C) & = & r_i^{S_1}(A) \\ r_i^{S_2\cup S_3}(C) & = & r_i^{S_2}(A) \end{array}\right\}, \quad i \in S_1, \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

which are easily derived from (2.4).

Theorem 3.4. *Let A and B be matrices of order n_1 and n_2, respectively. Let $n_1 \geq 2$, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$, which defines the sets S_1, S_2, S_3 as in* (2.3)*. Let A and B be partitioned as in* (2.1)*. Let S be a set of indices of the form $S = \{1, 2, \ldots\}$. Let A be S-strictly diagonally dominant, with $card(S) \leq card(S_1)$, and let B be strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is S-strictly diagonally dominant, and therefore nonsingular.*

Proof. We first prove the case when $S = S_1$. Since A is S_1-strictly diagonally dominant, we have that

$$\left.\begin{array}{lll} i) & |a_{ii}| > r_i^{S_1}(A) & \forall i \in S_1, \\ ii) & (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)) > r_i^{S_2}(A)\, r_j^{S_1}(A) & \forall i \in S_1, \forall j \in S_2. \end{array}\right\} \tag{3.3}$$

Note that A is of order n_1 and then the complement of S_1 in $\{1, 2, \ldots, n_1\}$ is S_2.

We want to show that C is also an S_1-strictly diagonally dominant matrix, i.e., we have to show that

$$\begin{array}{ll} 1) & |c_{ii}| > r_i^{S_1}(C) \quad \forall i \in S_1, \quad \text{and} \\ 2) & (|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2 \cup S_3}(C)) > r_i^{S_2 \cup S_3}(C)\, r_j^{S_1}(C) \;\; \forall i \in S_1,\, \forall j \in S_2 \cup S_3. \end{array} \tag{3.4}$$

Note that since C is of order n, the complement of S_1 in $\{1, 2, \ldots, n\}$ is $S_2 \cup S_3$.

To see that 1) holds we use equations (2.4), (3.2) and part i) of (3.3) (see also Figure 2.1) to obtain

$$|c_{ii}| = |a_{ii}| > r_i^{S_1}(A) = r_i^{S_1}(C), \quad \forall i \in S_1.$$

To see that 2) holds we distinguish two cases: $j \in S_2$ and $j \in S_3$. If $j \in S_2$, from (2.4) we have the following relations (recall that $t = n_1 - k$):

$$r_j^{S_1}(C) = \sum_{j \neq k,\, k \in S_1} |c_{jk}| = \sum_{j \neq k,\, k \in S_1} |a_{jk}| = r_j^{S_1}(A), \tag{3.5}$$

$$\begin{aligned} r_j^{S_2 \cup S_3}(C) &= \sum_{j \neq k,\, k \in S_2 \cup S_3} |c_{jk}| = \sum_{j \neq k,\, k \in S_2} |c_{jk}| + \sum_{j \neq k,\, k \in S_3} |c_{jk}| \\ &= r_j^{S_2}(C) + r_j^{S_3}(C), \end{aligned} \tag{3.6}$$

$$r_j^{S_2}(C) = \sum_{j \neq k,\, k \in S_2} |a_{jk} + b_{j-t,k-t}|, \tag{3.7}$$

$$r_j^{S_3}(C) = \sum_{j \neq k,\, k \in S_3} |b_{j-t,k-t}| = r_{j-t}^{S_3}(B), \tag{3.8}$$

$$c_{jj} = a_{jj} + b_{j-t,j-t}. \tag{3.9}$$

Therefore we can write

$$\begin{aligned} &(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2 \cup S_3}(C)) = \\ &(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj} + b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C)),\ \forall i \in S_1,\ \forall j \in S_2, \end{aligned} \tag{3.10}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

where we have used that $c_{ii} = a_{ii}$, for $i \in S_1$ and equations (3.2), (3.6) and (3.9). Using now that A_{22} and B_{11} have positive diagonal (or both negative diagonal) we have that $|a_{jj} + b_{j-t,j-t}| = |a_{jj}| + |b_{j-t,j-t}|$ and therefore we can rewrite (3.10) as

$$\begin{aligned}(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) = \\ (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C)),\ \forall i \in S_1,\ \forall j \in S_2.\end{aligned} \tag{3.11}$$

Let us now focus on the second term of the right hand side of (3.11). Observe that from (3.7) and the triangle inequality we have that

$$\begin{aligned} r_j^{S_2}(C) = \sum_{j\neq k,\, k\in S_2} |a_{jk} + b_{j-t,k-t}| &\leq \sum_{j\neq k,\, k\in S_2} |a_{jk}| + \sum_{j\neq k,\, k\in S_2} |b_{j-t,k-t}| \\ = r_j^{S_2}(A) + r_{j-t}^{S_2}(B) & \end{aligned} \tag{3.12}$$

and using (3.8), from (3.12) we can write the inequality

$$|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(A) - r_{j-t}^{S_2}(B) - r_{j-t}^{S_3}(B).$$

Since we have $r_{j-t}^{S_2}(B) + r_{j-t}^{S_3}(B) = r_{j-t}^{S_2\cup S_3}(B)$, we obtain

$$|a_{jj}| + |b_{j-t,j-t}| - r_j^{S_2}(C) - r_j^{S_3}(C) \geq |a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B),$$

which allows us to transform (3.11) into the following inequality

$$\begin{aligned}(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) \geq \\ (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B)),\ \forall i \in S_1,\ \forall j \in S_2,\end{aligned} \tag{3.13}$$

where we have used that $(|a_{ii}| - r_i^{S_1}(A))$ is positive since A is S_1-strictly diagonally dominant. Observe now that $|b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B)$ is also positive since B is strictly diagonally dominant, and thus we can write

$$|a_{jj}| - r_j^{S_2}(A) + |b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B) > |a_{jj}| - r_j^{S_2}(A)$$

which jointly with (3.13) leads to the strict inequality

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) > (|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)), \tag{3.14}$$

for all $i \in S_1$ and for all $j \in S_2$, Finally, using (ii) of (3.3) (i.e., the fact that A is S_1-strictly diagonally dominant) and equations (3.2) and (3.5) we can write the inequality

$$(|a_{ii}| - r_i^{S_1}(A))\,(|a_{jj}| - r_j^{S_2}(A)) > r_i^{S_2}(A)\, r_j^{S_1}(A) = r_i^{S_2\cup S_3}(C)\, r_j^{S_1}(C)$$

for all $i \in S_1$ and for all $j \in S_2$, which allows to transform equation (3.14) into the inequality

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) > r_i^{S_2\cup S_3}(C)\, r_j^{S_1}(C),\ \forall i \in S_1,\ \forall j \in S_2.$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

Therefore we have proved condition 2) for the case $j \in S_2$.

In the case $j \in S_3$, we have from (2.4) that

$$r_j^{S_1}(C) = \sum_{j\neq k,\, k\in S_1} |c_{jk}| = 0.$$

Therefore the condition 2) of (3.4) becomes

$$(|c_{ii}| - r_i^{S_1}(C))\,(|c_{jj}| - r_j^{S_2\cup S_3}(C)) > 0, \quad \forall i \in S_1, \forall j \in S_3, \tag{3.15}$$

and it is easy to show that this inequality is fulfilled. The first term is positive since, as before, we have that $|c_{ii}| - r_i^{S_1}(C) = |a_{ii}| - r_i^{S_1}(A) > 0$. The second term of (3.15) is also positive since we have that $c_{jj} = b_{j-t,j-t}$ for all $j \in S_3$ and

$$r_j^{S_2\cup S_3}(C) = \sum_{j\neq k,\, k\in S_2\cup S_3} |c_{jk}| = \sum_{j\neq k,\, k\in S_2\cup S_3} |b_{j-t,k-t}| = r_{j-t}^{S_2\cup S_3}(B),\ \forall j \in S_3,$$

and since B is strictly diagonally dominant we have

$$|b_{j-t,j-t}| - r_{j-t}^{S_2\cup S_3}(B) > 0,\ \forall j \in S_3.$$

Therefore equation (3.15) is fulfilled and the proof for the case $S = S_1$ is completed.

When $card(S) < card(S_1)$ the proof is analogous. We only indicate that the key point in this case is the subcase $j \in S_1\backslash S$ for which it is easy to show that a condition similar to 2) for C in (3.4) still holds. $\square$

When $card(S) > card(S_1)$ the preceding theorem is not valid as we show in the following example.

Example 3.5. In this example we show a matrix A that is an S-SDD matrix with $card(S) > card(S_1)$ and a matrix B that is an SDD matrix but the subdirect sum C is not an S-SDD matrix. Let the following matrices A and B be partitioned as

$$A = \left[\begin{array}{r|rrr} 1.0 & -0.3 & -0.4 & -0.5 \\ \hline -0.9 & 1.6 & -0.4 & -0.7 \\ -0.1 & -0.4 & 1.3 & -0.4 \\ -0.1 & -0.9 & -0.1 & 2.0 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{rrr|r} 2.0 & 0.2 & -0.3 & -0.1 \\ 0.8 & 2.9 & -0.2 & -0.5 \\ -0.5 & -0.1 & 2.4 & -0.9 \\ \hline -0.6 & -0.8 & -0.8 & 2.3 \end{array}\right].$$

We have from (2.3) that $S_1 = \{1\}$, $S_2 = \{2,3,4\}$ and $S_3 = \{5\}$. It is easy to show that A is $\{1,2\}$-SDD, A is not SDD, and B is SDD. The 3-subdirect sum $C = A \oplus_3 B$

$$C = \left[\begin{array}{r|rrr|r} 1.0 & -0.3 & -0.4 & -0.5 & 0 \\ \hline -0.9 & 3.6 & -0.2 & -1.0 & -0.1 \\ -0.1 & 0.4 & 4.2 & -0.6 & -0.5 \\ -0.1 & -1.4 & -0.2 & 4.4 & -0.9 \\ \hline 0 & -0.6 & -0.8 & -0.8 & 2.3 \end{array}\right]$$

is not a $\{1,2\}$-SDD: the corresponding condition ii) for C in equation (3.1) fails for $i = 1, j = 5$.

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A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

Remark 3.6. An analogous result to Theorem 3.4 can be obtained when the matrix B is S-strictly diagonally dominant with $S = \{n_1 + 1, n_1 + 2, \ldots\}$, $card(S) \leq card(S_3)$, and the matrix A is strictly diagonally dominant. The proof is completely analogous, and thus we omit the details.

It is easy to show that if A is a strictly diagonally dominant matrix, then A is also an S_1-strictly diagonally dominant matrix. Therefore we have the following corollary.

Corollary 3.7. *Let A and B be matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in* (2.1)*. If A and B are strictly diagonally dominant and all diagonal entries of A_{22} and B_{11} are positive, then the k-subdirect sum $C = A \oplus_k B$ is strictly diagonally dominant, and therefore nonsingular.*

Remark 3.8. In the general case of successive k-subdirect sums of the form

$$(A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 \oplus \cdots$$

when A_1 is S-SDD with $card(S) \leq n_1 - k_1$ and A_2, A_3, ..., are SDD matrices, we have that all the subdirect sums are S-SDD matrices, provided that in each particular subdirect sum the quantity $card(S)$ is no larger than the corresponding overlap, in accordance with Theorem 3.4.

4. Overlapping SDD matrices. In this section we consider the case of square matrices A and B of order n_1 and n_2, respectively, which are principal submatrices of a given SDD matrix, and such that they have a common block with positive diagonals. This situation, as well as a more general case outlined in Theorem 4.1 later in this section, appears in many variants of additive Schwarz preconditioning; see, e.g., [2], [6], [7], [8]. Specifically, let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

be an SDD matrix of order n, with $n = n_1 + n_2 - k$, and with M_{22} a square matrix of order k, such that its diagonal is positive. Let us consider two principal submatrices of M, namely

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}.$$

Therefore the k-subdirect sum of A and B is given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}. \tag{4.1}$$

Since A and B are SDD matrices, according to Corollary 3.7 the subdirect sum given by equation (4.1) is also an SDD matrix. This result can clearly be extended to the sum of p overlapping submatrices of a given SDD matrix with positive diagonal entries. We summarize this result formally as follows; cf. a similar result for

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

M-matrices in [1]. Here, we consider consecutive principal submatrices defined by consecutive indices of the form $\{i, i+1, i+2, \ldots\}$.

THEOREM 4.1. *Let M be an SDD matrix with positive diagonal entries. Let A_i, $i = 1, \ldots, p$, be consecutive principal submatrices of M of order n_i, and consider the $p-1$ k_i-subdirect sums given by*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1$$

in which $C_0 = A_1$, and $k_i < \min(n_i, n_{i+1})$. Then each of the k_i-subdirect sums C_i is an SDD matrix, and in particular

$$C_{p-1} = A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_p} A_p \tag{4.2}$$

is an SDD matrix.

5. SDD matrices with the structure of a subdirect sum. We address the following question. Let C be square of order n, an SDD matrix with positive diagonal entries, and having the structure of a k-subdirect sum. Can we find matrices A and B with the same properties such that $C = A \oplus_k B$? We answer this in the affirmative in the following result.

PROPOSITION 5.1. *Let*

$$C = \begin{bmatrix} C_{11} & C_{12} & O \\ C_{21} & C_{22} & C_{23} \\ O & C_{32} & C_{33} \end{bmatrix},$$

with the matrices C_{ii} of order $n_1 - k$, k, $n_2 - k$, for $i = 1, 2, 3$, respectively, and C an SDD matrix with positive diagonal entries. Then, we can find square matrices A and B of order n_1 and n_2 such that they are SDD matrices with positive diagonal entries, and such that $C = A \oplus_k B$. In other words, we have

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}$$

such that $C_{22} = A_{22} + B_{22}$.

The proof of this proposition resembles that of [5, Proposition 4.1], where a similar question was studied for M-matrices, and we do not repeat it here. We mention that it is immediate to generalize Proposition 5.1 to a matrix C with the structure of a subdirect sum of several matrices such as that of (4.2) of Theorem 4.1.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 201-209, July 2006

ELA

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