EIGENVALUE CONDITION NUMBERS AND A FORMULA OF BURKE, LEWIS AND OVERTON*

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Abstract. In a paper by Burke, Lewis and Overton, a first order expansion has been given for the minimum singular value of $A - zI$, $z \in \mathbb{C}$, about a nonderogatory eigenvalue λ of $A \in \mathbb{C}^{n \times n}$. This note investigates the relationship of the expansion with the Jordan canonical form of A . Furthermore, formulas for the condition number of eigenvalues are derived from the expansion.

Key words. Eigenvalue condition numbers, Jordan canonical form, Singular values.

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1. Introduction. By $\pi_\Sigma(A)$ we denote the product of the nonzero singular values of the matrix $A \in \mathbb{C}^{n \times m}$, counting multiplicities. For the zero matrix $0 \in \mathbb{C}^{n \times m}$ we set $\pi_\Sigma(0) = 1$. If A is square then $\Lambda(A)$ denotes the spectrum and $\pi_\Lambda(A)$ stands for the product of the nonzero eigenvalues, counting multiplicities. If all eigenvalues of A are zero then we set $\pi_\Lambda(A) = 1$. The subject of this note is the ratio

$$q(A, \lambda) := \frac{\pi_\Sigma(A - \lambda I_n)}{|\pi_\Lambda(A - \lambda I_n)|}, \quad \lambda \in \Lambda(A).$$

In [1] the following first order expansion has been given for the function

$$z \mapsto \sigma_{\min}(A - zI_n), \quad z \in \mathbb{C},$$

where $\sigma_{\min}(\cdot)$ denotes the minimum singular value and I_n is the $n \times n$ identity matrix.

THEOREM 1.1. *Let $\lambda \in \mathbb{C}$ be a nonderogatory eigenvalue of algebraic multiplicity m of the matrix $A \in \mathbb{C}^{n \times n}$. Then*

$$\sigma_{\min}(A - zI_n) = \frac{|z - \lambda|^m}{q(A, \lambda)} + \mathcal{O}(|z - \lambda|^{m+1}), \quad z \in \mathbb{C}.$$

The relevance of this result for the perturbation theory of eigenvalues is as follows. The closed ϵ -pseudospectrum of $A \in \mathbb{C}^{n \times n}$ with respect to the spectral norm, $\|\cdot\|$, is defined by

$$\Lambda_\epsilon(A) = \{ z \in \mathbb{C} \mid z \in \Lambda(A + \Delta), \Delta \in \mathbb{C}^{n \times n}, \|\Delta\| \leq \epsilon \}.$$

In words, $\Lambda_\epsilon(A)$ is the set of all eigenvalues of all matrices of the form $A + \Delta$ where the spectral norm of the perturbation Δ is bounded by $\epsilon > 0$. It is well known [10] that

$$\Lambda_\epsilon(A) = \{ z \in \mathbb{C} \mid \sigma_{\min}(A - zI) \leq \epsilon \}.$$

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Theorem 1.1 yields an estimate for the size of pseudospectra for small ϵ : Roughly speaking if ϵ is small enough then the connected component of $\Lambda_\epsilon(A)$ that contains the eigenvalue λ is approximately a disk of radius $(q(A, \lambda)\epsilon)^{1/m}$ about λ . It follows that $q(A, \lambda)^{1/m}$ is the Hölder condition number of λ . We discuss this in detail in Section 4.

However, the main concern of this note is to establish the relationship of $q(A, \lambda)$ with the Jordan decomposition of A . For a simple eigenvalue the relationship is as follows. Let $x, y \in \mathbb{C}^n \setminus \{0\}$ be a right and a left eigenvector of A to the eigenvalue λ respectively, i.e. $Ax = \lambda x$, $y^*A = \lambda y^*$, where y^* denotes the conjugate transpose of y . Then

$$P = (y^*x)^{-1}xy^* \in \mathbb{C}^{n \times n}$$

is a projection onto the one dimensional eigenspace $\mathbb{C}x$. The kernel of P is the direct sum of all generalized eigenspaces belonging to the eigenvalues different from λ . As is well known [5, p.490],[3, p.202],[9, p.186], the condition number of λ equals the norm of P . Combined with the considerations above this yields that

$$q(A, \lambda) = \|P\|. \quad (1.1)$$

In Section 3 we give an elementary proof of the identity (1.1) without using Theorem 1.1. Furthermore, we show that for a nondegeneratory eigenvalue of algebraic multiplicity $m \geq 2$,

$$q(A, \lambda) = \|N^{m-1}\|, \quad (1.2)$$

where N is the nilpotent operator associated with λ in the Jordan decomposition of A . The formulas (1.1) and (1.2) are the main results of this note. The proofs also show that the assumption that λ is nondegeneratory is necessary.

The next section contains some preliminaries about the computation of the two products $\pi_\Sigma(A)$ and $\pi_\Lambda(A)$ and about the relationship of the Schur form of A with the Jordan decomposition.

Throughout this note, $\|\cdot\|$ stands for the spectral norm.

2. Preliminaries. Below we list some easily verified properties of $\pi_\Lambda(A)$, the product of the nonzero eigenvalues of A , and of $\pi_\Sigma(A)$, the product of the nonzero singular values of A . In the sequel A^T and A^* denote the transpose and the conjugate transpose of A respectively.

- (a) If $A \in \mathbb{C}^{n \times n}$ is nonsingular then $\pi_\Lambda(A) = \det(A)$.
- (b) For any $A \in \mathbb{C}^{n \times n}$: $\pi_\Lambda(A^T) = \pi_\Lambda(A)$ and $\pi_\Lambda(A^*) = \overline{\pi_\Lambda(A)}$.
- (c) Let $S \in \mathbb{C}^{n \times n}$ be nonsingular. Then for any $A \in \mathbb{C}^{n \times n}$, $\pi_\Lambda(SAS^{-1}) = \pi_\Lambda(A)$.
- (d) Let $A_{11} \in \mathbb{C}^{n \times n}$, $A_{22} \in \mathbb{C}^{m \times m}$ and $A_{12} \in \mathbb{C}^{n \times m}$. Then

$$\pi_\Lambda \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right) = \pi_\Lambda(A_{11}) \pi_\Lambda(A_{22}).$$

- (e) For any $A \in \mathbb{C}^{n \times m}$, $\pi_\Sigma(A)^2 = \pi_\Lambda(A^*A) = \pi_\Lambda(AA^*)$.

- (f) If $A \in \mathbb{C}^{n \times n}$ is nonsingular then $\pi_\Sigma(A) = |\det(A)| = |\pi_\Lambda(A)|$.
 (g) Let $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ be unitary. Then for any $A \in \mathbb{C}^{n \times m}$,
 $\pi_\Sigma(UAV) = \pi_\Sigma(A)$.

In the next section we need the lemmas below.

LEMMA 2.1. Let $M \in \mathbb{C}^{n \times n}$ be nonsingular, $X \in \mathbb{C}^{m \times n}$ and $Y = XM^{-1}$. Then

$$\pi_\Sigma \left(\begin{bmatrix} M \\ X \end{bmatrix} \right) = \pi_\Sigma(M) \sqrt{\det(I_n + Y^*Y)}.$$

Proof. We have

$$\begin{aligned} \pi_\Sigma \left(\begin{bmatrix} M \\ X \end{bmatrix} \right)^2 &= \pi_\Lambda \left(\begin{bmatrix} M^* & X^* \end{bmatrix} \begin{bmatrix} M \\ X \end{bmatrix} \right) \\ &= \det(M^*M + X^*X) \\ &= \det(M^*(I_n + Y^*Y)M) \\ &= \det(M^*) \det(M) \det(I_n + Y^*Y) \\ &= \pi_\Sigma(M)^2 \det(I_n + Y^*Y). \quad \square \end{aligned}$$

LEMMA 2.2. Let $Y \in \mathbb{C}^{m \times n}$. Then $\|I_n + Y^*Y\| = \|I_m + YY^*\|$ and $\det(I_n + Y^*Y) = \det(I_m + YY^*)$.

Proof. The case $Y = 0$ is trivial. Let $Y \neq 0$. The matrices Y and Y^* have the same nonzero singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p > 0$ say. The eigenvalues different from 1 of both $I_n + Y^*Y$ and $I_m + YY^*$ are $1 + \sigma_1^2 \geq 1 + \sigma_2^2 \geq \dots \geq 1 + \sigma_p^2$. Thus $\|I_n + Y^*Y\| = \|I_m + YY^*\| = 1 + \sigma_1^2$ and $\det(I_n + Y^*Y) = \det(I_m + YY^*) = \prod_{k=1}^p (1 + \sigma_k^2)$. $\square$

We proceed with remarks on the Jordan decomposition. Let $\lambda_1, \dots, \lambda_\kappa$ be the pairwise different eigenvalues of $A \in \mathbb{C}^{n \times n}$. Let $\mathcal{X}_j = \ker(A - \lambda_j I_n)^n$ be the generalized eigenspaces. By the Jordan decomposition theorem we have

$$A = \sum_{j=1}^{\kappa} (\lambda_j P_j + N_j), \quad (2.1)$$

where $P_1, \dots, P_\kappa \in \mathbb{C}^{n \times n}$ are the projectors of direct decomposition $\mathbb{C}^n = \bigoplus_{j=1}^{\kappa} \mathcal{X}_j$, i.e.

$$P_j^2 = P_j, \quad \text{range}(P_j) = \mathcal{X}_j, \quad \ker(P_j) = \bigoplus_{k=1, k \neq j}^{\kappa} \mathcal{X}_k,$$

and $N_1, \dots, N_\kappa \in \mathbb{C}^{n \times n}$ are the nilpotent matrices $N_j = (A - \lambda_j I_n)P_j$. The eigenvalue λ_j is said to be

- semisimple (nondefective) if $\mathcal{X}_j = \ker(A - \lambda_j I_n)$,
- simple if $\dim \mathcal{X}_j = 1$,
- nonderogatory if $\dim \ker(A - \lambda_j I_n) = 1$.

In the following m denotes the algebraic multiplicity of λ_j . Note that if $m \geq 2$ then λ_j is nonderogatory if and only if $N_j^{m-1} \neq 0$. We now recall how to obtain the operators P_j and N_j from a Schur form of A . We only consider the nontrivial case that A has at least two different eigenvalues. By the Schur decomposition theorem there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$U^*AU = \begin{bmatrix} \lambda_j I_m + T & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{12} \in \mathbb{C}^{m \times (n-m)}$, $A_{22} \in \mathbb{C}^{(n-m) \times (n-m)}$, $\Lambda(A_{22}) = \Lambda(A) \setminus \{\lambda_j\}$ and $T \in \mathbb{C}^{n \times n}$ is strictly upper triangular,

$$T = \begin{bmatrix} 0 & t_{12} & \dots & \dots & t_{1m} \\ & \ddots & t_{23} & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & t_{m-1,m} \\ & & & & 0 \end{bmatrix}.$$

If $m = 1$ (i.e. λ_j is simple) then T is the 1×1 zero matrix. Since the spectra of T and $A_{22} - \lambda_j I_{n-m}$ are disjoint the Sylvester equation

$$R(A_{22} - \lambda_j I_{n-m}) - TR = A_{12}. \quad (2.2)$$

has a unique solution $R \in \mathbb{C}^{m \times (n-m)}$.

PROPOSITION 2.3. *With the notation above the projector onto the generalized eigenspace and the nilpotent operator associated with λ_j are given by*

$$P_j = U \begin{bmatrix} I_m & -R \\ 0 & 0 \end{bmatrix} U^*, \quad \text{and} \quad N_j = U \begin{bmatrix} T & -TR \\ 0 & 0 \end{bmatrix} U^*.$$

For any integer $\ell \geq 1$ we have

$$N_j^\ell = U \begin{bmatrix} T^\ell & -T^\ell R \\ 0 & 0 \end{bmatrix} U^*. \quad (2.3)$$

The spectral norms of P_j and of N_j^ℓ satisfy

$$\|P_j\| = \|I_m + RR^*\|^{1/2} \quad (2.4)$$

$$\|N_j^\ell\| = \|T^\ell(I_m + RR^*)(T^*)^\ell\|^{1/2}. \quad (2.5)$$

Proof. Let $X_1 := U \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in \mathbb{C}^{n \times m}$, $X_2 := U \begin{bmatrix} R \\ I_{n-m} \end{bmatrix} \in \mathbb{C}^{n \times (n-m)}$. Then obviously $\mathbb{C}^n = \text{range}(X_1) \oplus \text{range}(X_2)$ and

$$AX_1 = X_1(\lambda_j I_m + T). \quad (2.6)$$

Furthermore, (2.2) yields that

$$A X_2 = X_2 A_{22}. \quad (2.7)$$

Hence, $\text{range}(X_1)$ and $\text{range}(X_2)$ are complementary invariant subspaces of A . The relations (2.6) and (2.7) imply that for any $\lambda \in \mathbb{C}$ and any integer $\ell \geq 1$,

$$\begin{aligned} (A - \lambda I_n)^\ell X_1 &= X_1((\lambda_j - \lambda)I_m + T)^\ell, \\ (A - \lambda I_n)^\ell X_2 &= X_2(A_{22} - \lambda I_{n-m})^\ell. \end{aligned} \quad (2.8)$$

Using this and the fact that $\lambda_j \notin \Lambda(A_{22})$ it is easily verified that $\text{range}(X_1) = \ker(A - \lambda_j I_n)^n$ and $\text{range}(X_2) = \bigoplus_{k=1, k \neq j}^{\kappa} \ker(A - \lambda_k I_n)^n$. The matrix

$$P_j = U \begin{bmatrix} I_m & -R \\ 0 & 0 \end{bmatrix} U^*, \quad (2.9)$$

satisfies $P_j^2 = P_j$, $P_j X_1 = X_1$ and $P_j X_2 = 0$. Hence, P_j is the Jordan projector onto the generalized eigenspace $\ker(A - \lambda_j I_n)^n$. For the associated nilpotent matrix N_j one obtains

$$N_j = (A - \lambda_j I_n)P_j = U \begin{bmatrix} T & -T R \\ 0 & 0 \end{bmatrix} U^*. \quad (2.10)$$

The formulas (2.3), (2.4) and (2.5) are immediate from (2.9) and (2.10). $\square$

We give an expression for $\|N_j^{m-1}\|$ which is a bit more explicit than formula (2.5). First note that if λ_j has algebraic multiplicity $m \geq 2$ then

$$T^{m-1} = \begin{bmatrix} 0 & \dots & 0 & \tau \\ \vdots & & \vdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}, \quad \text{where } \tau = \prod_{k=1}^{m-1} t_{k,k+1}.$$

Let $e_m^T = [0 \dots 0 \ 1]^T \in \mathbb{C}^m$ and $r = e_m^T R$. Then r is the lower row of R . Since the lower row of TR is zero it follows from the Sylvester equation (2.2) that

$$r = e_m^T A_{12}(A_{22} - \lambda_j I_m)^{-1}. \quad (2.11)$$

From (2.3) or (2.5) we obtain

PROPOSITION 2.4. *Suppose λ_j has algebraic multiplicity $m \in \{2, \dots, n-1\}$. Then*

$$\|N_j^{m-1}\| = |\tau| \sqrt{1 + \|r\|^2}.$$

3. Main result. We are now in a position to state and prove our main result on the ratio

$$q(A, \lambda_j) = \frac{\pi_\Sigma(A - \lambda_j I_n)}{|\pi_\Lambda(A - \lambda_j I_n)|}, \quad \lambda_j \in \Lambda(A). \quad (3.1)$$

THEOREM 3.1. *Let $\lambda_j \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$. Let P_j and N_j be the eigenprojector and the nilpotent operator associated with λ_j . Then the following holds.*

- (a) *If λ_j is a semisimple eigenvalue then $q(A, \lambda_j) = \pi_\Sigma(P_j)$.*
- (b) *If λ_j is a simple eigenvalue then $q(A, \lambda_j) = \|P_j\|$.*
- (c) *If λ_j is a nonderogatory eigenvalue of algebraic multiplicity $m \geq 2$ then*

$$q(A, \lambda_j) = \|N_j^{m-1}\|.$$

Proof. First, we treat the case that A has at least two different eigenvalues. In view of Proposition 2.3 and since the products $\pi_\Sigma(A - \lambda_j I_n)$, $\pi_\Lambda(A - \lambda_j I_n)$ are invariant under unitary similarity transformations we may assume that

$$A = \begin{bmatrix} \lambda_j I_m + T & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad P_j = \begin{bmatrix} I_m & -R \\ 0 & 0 \end{bmatrix},$$

where $\Lambda(A_{22}) = \Lambda(A) \setminus \{\lambda_j\}$, $T \in \mathbb{C}^{n \times n}$ is strictly upper triangular and $R \in \mathbb{C}^{m \times (n-m)}$ is the solution of the Sylvester equation $R(A_{22} - \lambda_j I_{n-m}) - TR = A_{12}$.

(a). Suppose λ_j is semisimple. Then $T = 0$ and $R(A_{22} - \lambda_j I_{n-m}) = A_{12}$. Thus,

$$\begin{aligned} (A - \lambda_j I_n)^*(A - \lambda_j I_n) &= \begin{bmatrix} 0 & 0 \\ 0 & (A_{22} - \lambda_j I_{n-m})^*(A_{22} - \lambda_j I_{n-m}) + A_{12}^* A_{12} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (A_{22} - \lambda_j I_{n-m})^*(I_{n-m} + R^* R)(A_{22} - \lambda_j I_{n-m}) \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \pi_\Sigma(A - \lambda_j I_n)^2 &= \pi_\Lambda((A - \lambda_j I_n)^*(A - \lambda_j I_n)) \\ &= \det((A_{22} - \lambda_j I_{n-m})^*(I_{n-m} + R^* R)(A_{22} - \lambda_j I_{n-m})) \end{aligned} \quad (3.2)$$

$$\begin{aligned} &= |\det(A_{22} - \lambda_j I_{n-m})|^2 \det(I_{n-m} + R^* R) \\ &= |\pi_\Lambda(A - \lambda_j I_n)|^2 \det(I_{n-m} + R^* R). \end{aligned} \quad (3.3)$$

Furthermore we have $P_j P_j^* = \begin{bmatrix} I_m + R R^* & 0 \\ 0 & 0 \end{bmatrix}$ and hence

$$\pi_\Sigma(P_j)^2 = \det(I_m + R R^*) = \det(I_{n-m} + R^* R). \quad (3.4)$$

The latter equation holds by Lemma 2.2. By combining (3.3) and (3.4) we obtain (a).
(b). If $m = 1$ then P_j has rank 1 and hence, $\pi_\Sigma(P_j) = \|P_j\|$. Thus (b) follows from (a).

(c) Suppose $m \geq 2$ and λ_j is nonderogatory. Then $T = \begin{bmatrix} 0 & D \\ \vdots & \\ 0 & \dots & 0 \end{bmatrix}$, where $D \in \mathbb{C}^{(m-1) \times (m-1)}$ is upper triangular and nonsingular. In the following we write $A_{12} = \begin{bmatrix} \tilde{A} \\ a \end{bmatrix}$, where a is the lower row of A_{12} . Let r denote the lower row of R . By Formula (2.11) we have

$$r = a(A_{22} - \lambda_j I)^{-1}. \quad (3.5)$$

Let us determine $\pi_\Sigma(A)$. Since removing of a column of zeros and a permutation of rows does not change the nonzero singular values of a matrix we have

$$\pi_\Sigma(A - \lambda_j I_n) = \pi_\Sigma \left(\begin{bmatrix} 0 & D & \tilde{A} \\ \vdots & & \\ 0 & \dots & 0 & a \\ 0 & & A_{22} - \lambda_j I_{n-m} \end{bmatrix} \right) = \pi_\Sigma \left(\begin{bmatrix} D & \tilde{A} \\ 0 & A_{22} - \lambda_j I_{n-m} \\ 0 & \dots & 0 & a \end{bmatrix} \right).$$

Lemma 2.1 yields

$$\begin{aligned} \pi_\Sigma \left(\begin{bmatrix} D & \tilde{A} \\ 0 & A_{22} - \lambda_j I_{n-m} \\ 0 & \dots & 0 & a \end{bmatrix} \right) &= \pi_\Sigma \left(\begin{bmatrix} D & \tilde{A} \\ 0 & A_{22} - \lambda_j I \end{bmatrix} \right) \sqrt{\det(1 + yy^*)} \\ &= |\det(D)\det(A_{22} - \lambda_j I)| \sqrt{1 + \|y\|^2} \\ &= |\pi_\Lambda(A - \lambda_j I)| |\det(D)| \sqrt{1 + \|y\|^2}, \end{aligned}$$

where

$$y = \begin{bmatrix} 0 & \dots & 0 & a \end{bmatrix} \begin{bmatrix} D & \tilde{A} \\ 0 & A_{22} - \lambda_j I \end{bmatrix}^{-1}.$$

From (3.5) it follows that $y = \begin{bmatrix} 0 & \dots & 0 & r \end{bmatrix}$ and hence, $\|y\| = \|r\|$. In summary,

$$\pi_\Sigma(A - \lambda_j I_n) = |\pi_\Lambda(A - \lambda_j I_n)| |\det(D)| \sqrt{1 + \|r\|^2}.$$

But $|\det(D)| \sqrt{1 + \|r\|^2} = \|N_j^{m-1}\|$ by Proposition 2.4. Hence, (c) holds.

Finally, we treat the case that λ_1 is the only eigenvalue of A . Let $U^*AU = \lambda_1 I_n + T$ be a Schur decomposition. The eigenprojection is $P_1 = I_n$ and the nilpotent operator is $N_1 = A - \lambda_1 I_n = UTU^*$. Since all eigenvalues of $A - \lambda_1 I_n$ are zero we have $\pi_\Lambda(A - \lambda_1 I_n) = 1$ by definition. If λ_1 is semisimple then also $\pi_\Sigma(A - \lambda_1 I_n) = \pi_\Sigma(0) = 1$. Hence, $q(A, \lambda_1) = 1 = \pi_\Sigma(P_1)$. Suppose $n \geq 2$ and λ_1 is nonderogatory. Then

$$q(A, \lambda_1) = \pi_\Sigma(A - \lambda_1 I_n) = \pi_\Sigma(T) = |\det(D)| = \|T^{n-1}\| = \|N_1^{n-1}\|,$$

where $T = \begin{bmatrix} 0 & D \\ \vdots & \\ 0 & \dots & 0 \end{bmatrix}$. $\square$

4. Condition numbers. In this section we show that $q(A, \lambda)^{1/m}$ equals the Hölder condition number of the nonderogatory eigenvalue λ of algebraic multiplicity m . To this end we introduce some additional notation. By $\mathcal{D}_\lambda(r)$ we denote the closed disk of radius $r > 0$ about $\lambda \in \mathbb{C}$. If $\lambda \in \Lambda(A)$, $A \in \mathbb{C}^{n \times n}$, then $\mathcal{C}_\lambda(\epsilon)$ denotes the connected component of the ϵ -pseudospectrum, $\Lambda_\epsilon(A)$, that contains λ . We define

$$R_\lambda^+(\epsilon) := \inf\{r > 0 \mid \mathcal{C}_\lambda(\epsilon) \subseteq \mathcal{D}_\lambda(r)\},$$

$$R_\lambda^-(\epsilon) := \sup\{r > 0 \mid \mathcal{D}_\lambda(r) \subseteq \mathcal{C}_\lambda(\epsilon)\}.$$

Then

$$\mathcal{D}_\lambda(R_\lambda^-(\epsilon)) \subseteq \mathcal{C}_\lambda(\epsilon) \subseteq \mathcal{D}_\lambda(R_\lambda^+(\epsilon)).$$

THEOREM 4.1. *Let $\lambda \in \Lambda(A)$ be a nonderogatory eigenvalue of algebraic multiplicity m . Then*

$$R_\lambda^\pm(\epsilon) = q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}). \quad (4.1)$$

The proof uses Theorem 1.1 and the lemma below.

LEMMA 4.2. *Let $U \subseteq \mathbb{C}^n$ be an open neighborhood of $z_0 \in \mathbb{C}^n$. Let $f, g : U \rightarrow [0, \infty)$ be continuous functions. For $\epsilon \geq 0$ let $S_f(\epsilon)$ and $S_g(\epsilon)$ denote the connected component containing z_0 of the sublevel set $\{z \in U \mid f(z) \leq \epsilon\}$ and $\{z \in U \mid g(z) \leq \epsilon\}$ respectively. Assume that $0 = g(z_0)$ is an isolated zero of g , and*

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = 1. \quad (4.2)$$

Then there exists an $\epsilon_0 > 0$ and functions $h_\pm : [0, \epsilon_0] \rightarrow [0, \infty)$ with $\lim_{\epsilon \rightarrow 0} h_\pm(\epsilon) = 1$ such that for all $\epsilon \in [0, \epsilon_0]$,

$$S_g(h_-(\epsilon)\epsilon) \subseteq S_f(\epsilon) \subseteq S_g(h_+(\epsilon)\epsilon). \quad (4.3)$$

We postpone the proof of the lemma to the end of this section.

Proof of Theorem 4.1: Let in Lemma 4.2, $z_0 = \lambda$ and

$$f(z) = \sigma_{\min}(A - zI_n), \quad g(z) = \frac{|z - \lambda|^m}{q(A, \lambda)}, \quad z \in \mathbb{C}.$$

Then $S_f(\epsilon) = \mathcal{C}_\lambda(\epsilon)$ and $S_g(\epsilon) = \mathcal{D}_\lambda((q(A, \lambda)\epsilon)^{1/m})$. Theorem 1.1 yields $\lim_{z \rightarrow \lambda} \frac{f(z)}{g(z)} = 1$. Hence, by the lemma there are functions $h_\pm$ with $\lim_{\epsilon \rightarrow 0} h_\pm(\epsilon) = 1$ and

$$\mathcal{D}_\lambda((q(A, \lambda)h_-(\epsilon)\epsilon)^{1/m}) \subseteq \mathcal{C}_\lambda(\epsilon) \subseteq \mathcal{D}_\lambda((q(A, \lambda)h_+(\epsilon)\epsilon)^{1/m}).$$

This shows (4.1). $\square$

Now, we give the definition for the Hölder condition number of an eigenvalue of arbitrary multiplicity (see [2]). For $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$ and $\tilde{A} \in \mathbb{C}^{n \times n}$ we set

$$d_m(\tilde{A}, \lambda) := \min\{r \geq 0 \mid \mathcal{D}_\lambda(r) \text{ contains at least } m \text{ eigenvalues of } \tilde{A}\}.$$

If λ is an eigenvalue of $A \in \mathbb{C}^{n \times n}$ of algebraic multiplicity m then the Hölder condition number of λ to the order $\alpha > 0$ is defined by

$$\text{cond}_\alpha(A, \lambda) = \lim_{\epsilon \searrow 0} \sup_{\|\Delta\| \leq \epsilon} \frac{d_m(A + \Delta, \lambda)}{\|\Delta\|^\alpha}.$$

It is easily seen that $0 \neq \text{cond}_\alpha(A, \lambda) \neq \infty$ for at most one order $\alpha > 0$.

THEOREM 4.3. *Let $\lambda \in \Lambda(A)$ be a nonderogatory eigenvalue of multiplicity m . Then*

$$\text{cond}_{1/m}(A, \lambda) = q(A, \lambda)^{1/m} = \begin{cases} \|P\| & \text{if } m = 1, \\ \|N^{m-1}\|^{1/m} & \text{otherwise,} \end{cases} \quad (4.4)$$

where $P \in \mathbb{C}^{n \times n}$ is the eigenprojector onto the generalized eigenspace $\ker(A - \lambda I_n)^m$, and $N = (A - \lambda I_m)P$.

Proof. Let $\Delta \in \mathbb{C}^{n \times n}$ with $\|\Delta\| \leq \epsilon$. Then the continuity of eigenvalues yields, that for any $t \in [0, 1]$ at least m eigenvalues of $A + t\Delta$ are contained in $\mathcal{C}_\lambda(\epsilon)$ counting multiplicities. Hence

$$d_m(A + \Delta, \lambda) \leq R_\lambda^+(\epsilon) = q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}).$$

By letting $\epsilon = \|\Delta\|$ we obtain that for all $\Delta \in \mathbb{C}^{n \times n}$,

$$\frac{d_m(A + \Delta, \lambda)}{\|\Delta\|^{1/m}} \leq q(A, \lambda)^{1/m} + o(\|\Delta\|^{1/m}) \|\Delta\|^{-(1/m)}.$$

This yields

$$\text{cond}_{1/m}(A, \lambda) \leq q(A, \lambda)^{1/m}.$$

Let $r > 0$ be such that $\mathcal{D}_\lambda(r) \cap \Lambda(A) = \{\lambda\}$. Then by the continuity of eigenvalues there is an ϵ_0 such that the following holds for all $\epsilon < \epsilon_0$,

- (a) $\mathcal{D}_\lambda(r) \cap \Lambda_\epsilon(A) = \mathcal{C}_\lambda(\epsilon)$.
- (b) For any $\Delta \in \mathbb{C}^{n \times n}$ with $\|\Delta\| \leq \epsilon$, the set $\mathcal{C}_\lambda(\epsilon)$ contains precisely m eigenvalues of $A + \Delta$ counting multiplicities.

Let $\epsilon < \epsilon_0$ and let $z_\epsilon \in \mathbb{C}$ be a boundary point of $\mathcal{C}_\lambda(\epsilon)$. Then $\sigma_{\min}(A - z_\epsilon I_n) = \epsilon$. Let $\Delta_\epsilon = -\epsilon u v^*$, where $u, v \in \mathbb{C}^n$ is a pair of normalized left and right singular vectors of $A - z_\epsilon I_n$ belonging to the minimum singular value, i.e.

$$(A - z_\epsilon I_n) v = \epsilon u, \quad u^*(A - z_\epsilon I_n) = \epsilon v^*, \quad \|u\| = \|v\| = 1.$$

Then $\|\Delta_\epsilon\| = \epsilon$ and $z_\epsilon \in \Lambda(A + \Delta_\epsilon)$ since $(A + \Delta_\epsilon)v = z_\epsilon v$. Thus, by (a) and (b),

$$\begin{aligned} d_m(A + \Delta_\epsilon, \lambda) &\geq |z_\epsilon - \lambda| \\ &\geq R_\lambda^-(\epsilon) \\ &= q(A, \lambda)^{1/m} \epsilon^{1/m} + o(\epsilon^{1/m}). \end{aligned}$$

and therefore

$$\frac{d_m(A + \Delta_\epsilon, \lambda)}{\|\Delta_\epsilon\|^{1/m}} \geq q(A, \lambda)^{1/m} + o(\epsilon^{1/m})\epsilon^{-(1/m)}.$$

Hence, $\text{cond}_{1/m}(A, \lambda) \geq q(A, \lambda)^{1/m}$. $\square$

REMARK 4.4. In [7] (see also [2, 4]) the following generalization of Theorem 4.3 has been shown. Let λ be an *arbitrary* eigenvalue of A . If λ is semisimple then

$$\text{cond}_1(A, \lambda) = \|P\|.$$

If λ is not semisimple then

$$\text{cond}_{1/m}(A, \lambda) = \|N^{m-1}\|^{1/m},$$

where m denotes the index of nilpotency of N , i.e. $N^m = 0$, $N^{m-1} \neq 0$.

Proof of Lemma 4.2: By B_r we denote the closed ball of radius $r > 0$ about z_0 . The condition that z_0 is an isolated zero of g combined with (4.2) yields that z_0 is also an isolated zero of f . Hence, there is an $r_0 > 0$ such that $f(z) > 0$ for all $z \in B_{r_0} \setminus \{z_0\}$. This implies that $\epsilon_r := \min_{z \in \partial B_r} f(z) > 0$ for any $r \in (0, r_0]$. If $\epsilon < \epsilon_r$ then ∂B_r does not intersect the sublevel sets $\{z \in U \mid f(z) \leq \epsilon\}$. Thus $S_f(\epsilon)$ is contained in the interior of B_r . Note that $S_f(\epsilon)$ being a connected component of a closed set is closed. It follows that $S_f(\epsilon)$ is compact if $\epsilon < \epsilon_{r_0}$. Now, let

$$\phi_\pm(z) := \begin{cases} (1 \pm \|z - z_0\|)^{\frac{g(z)}{f(z)}} & z \in B_{r_0} \setminus \{z_0\}, \\ 1, & z = z_0. \end{cases}$$

Condition (4.2) yields that the functions $\phi_\pm : U \rightarrow \mathbb{R}$ are continuous. For $\epsilon < \epsilon_{r_0}$ let

$$h_-(\epsilon) := \min_{z \in S_f(\epsilon)} \phi_-(z), \quad h_+(\epsilon) := \max_{z \in S_f(\epsilon)} \phi_+(z).$$

Then we have for all $\epsilon < \epsilon_r$,

$$\min_{z \in B_r} \phi_\pm(z) \leq h_\pm(\epsilon) \leq \max_{z \in B_r} \phi_\pm(z).$$

As r tends to 0 the max and the min tend to $\phi_\pm(z_0) = 1$. This yields $\lim_{\epsilon \rightarrow 0} h_\pm(\epsilon) = 1$. If $z \in \partial S_f(\epsilon)$ then $f(z) = \epsilon$ and $g(z) > (1 - \|z - z_0\|)^{\frac{g(z)}{f(z)}} f(z) \geq h_-(\epsilon)\epsilon$. Thus $\partial S_f(\epsilon)$ does not intersect $E := \{z \in U \mid g(z) \leq h_-(\epsilon)\epsilon\}$. Thus $S_g(h_-(\epsilon)\epsilon)$ being a connected component of E is either contained in the interior of $S_f(\epsilon)$ or in the complement of $S_f(\epsilon)$. The latter is impossible since $z_0 \in S_f(\epsilon) \cap S_g(h_-(\epsilon)\epsilon)$. Hence, $S_g(h_-(\epsilon)\epsilon) \subset S_f(\epsilon)$. This proves the first inclusion in (4.3). To prove the second suppose $z_0 \neq z \in \partial S_g(h_+(\epsilon)\epsilon) \cap S_f(\epsilon)$. Then $g(z) = h_+(\epsilon)\epsilon$ and $0 < f(z) \leq \epsilon$. Hence $g(z)/f(z) \geq h_+(\epsilon)$, a contradiction. Thus $S_f(\epsilon)$ is contained in the interior of $S_g(h_+(\epsilon)\epsilon)$. $\square$

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