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EIGENVALUES AND EIGENVECTORS OF TRIDIAGONAL MATRICES*

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Abstract. This paper is continuation of previous work by the present author, where explicit formulas for the eigenvalues associated with several tridiagonal matrices were given. In this paper the associated eigenvectors are calculated explicitly. As a consequence, a result obtained by Wen-Chyuan Yueh and independently by S. Kouachi, concerning the eigenvalues and in particular the corresponding eigenvectors of tridiagonal matrices, is generalized. Expressions for the eigenvectors are obtained that differ completely from those obtained by Yueh. The techniques used herein are based on theory of recurrent sequences. The entries situated on each of the secondary diagonals are not necessary equal as was the case considered by Yueh.

Key words. Eigenvectors, Tridiagonal matrices.

AMS subject classifications. 15A18.

1. Introduction. The subject of this paper is diagonalization of tridiagonal matrices. We generalize a result obtained in [5] concerning the eigenvalues and the corresponding eigenvectors of several tridiagonal matrices. We consider tridiagonal matrices of the form

$$A_n = \begin{pmatrix} -\alpha+b & c_1 & 0 & 0 & \ldots & 0 \\ a_1 & b & c_2 & 0 & \ldots & 0 \\ 0 & a_2 & b & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \ldots & \ldots & 0 & a_{n-1} & -\beta+b \end{pmatrix}, \tag{1}$$

where $\{a_j\}_{j=1}^{n-1}$ and $\{c_j\}_{j=1}^{n-1}$ are two finite subsequences of the sequences $\{a_j\}_{j=1}^{\infty}$ and $\{c_j\}_{j=1}^{\infty}$ of the field of complex numbers $\mathbb{C}$, respectively, and α, β and b are complex numbers. We suppose that

$$a_j c_j = \begin{cases} d_1^2, & \text{if } j \text{ is odd} \\ d_2^2, & \text{if } j \text{ is even} \end{cases} \quad j = 1, 2, ..., \tag{2}$$

where d_1 and d_2 are complex numbers. We mention that matrices of the form (1) are of circulant type in the special case when $\alpha = \beta = a_1 = a_2 = ... = 0$ and all the entries on the subdiagonal are equal. They are of Toeplitz type in the special case when $\alpha = \beta = 0$ and all the entries on the subdiagonal are equal and those on the superdiagonal are also equal (see U. Grenander and G. Szego [4]). When the

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entries on the principal diagonal are not equal, the calculi of the eigenvalues and the corresponding eigenvectors becomes very delicate (see S. Kouachi [6]).
When $a_1 = a_2 = ... = c_1 = c_2 = ... = 1$, $b = -2$ and $\alpha = \beta = 0$, the eigenvalues of A_n has been constructed by J. F. Elliott [2] and R. T. Gregory and D. Carney [3] to be

$$\lambda_k = -2 + 2\cos\frac{k\pi}{n+1},\ k = 1, 2, ..., n.$$

When $a_1 = a_2 = ... = c_1 = c_2 = ... = 1$, $b = -2$ and $\alpha = 1$ and $\beta = 0$ or $\beta = 1$, the eigenvalues has been reported to be

$$\lambda_k = -2 + 2\cos\frac{k\pi}{n},\ k = 1, 2, ..., n,$$

and

$$\lambda_k = -2 + 2\cos\frac{2k\pi}{2n+1},\ k = 1, 2, ..., n,$$

respectively without proof.
W. Yueh[1] has generalized the results of J. F. Elliott [2] and R. T. Gregory and D. Carney [3] to the case when $a_1 = a_2 = ... = a,\ c_1 = c_2 = ... = c$ and $\alpha = 0,\ \beta = \sqrt{ac}$ or $\alpha = 0,\ \beta = -\sqrt{ac}$ or $\alpha = -\beta = \sqrt{ac}$
$\alpha = \beta = \sqrt{ac}$ or $\alpha = \beta = -\sqrt{ac}$. He has calculated, in this case, the eigenvalues and their corresponding eigenvectors

$$\lambda_k = b + 2\sqrt{ac}\cos\theta_k,\ k = 1, .., n,$$

where $\theta_k = \frac{2k\pi}{2n+1}, \frac{(2k-1)\pi}{2n+1}, \frac{(2k-1)\pi}{2n}, \frac{k\pi}{n}$ and $\frac{(k-1)\pi}{n},\ k = 1, .., n$ respectively.
In S. Kouachi[5], we have generalized the results of W. Yueh [1] to more general matrices of the form (1) for any complex constants satisfying condition

$$a_j c_j = d^2,\ j = 1, 2, ...,$$

where d is a complex number. We have proved that the eigenvalues remain the same as in the case when the a_i's and the c_i's are equal but the components of the eigenvector $u^{(k)}(\sigma)$ associated to the eigenvalue λ_k, which we denote by $u_j^{(k)}(\sigma),\ j = 1, .., n$, are of the form

$$u_j^{(k)}(\sigma) = (-d)^{1-j}\, a_{\sigma_1}...a_{\sigma_{j-1}} u_1^{(k)} \frac{d\sin(n-j+1)\theta_k - \beta\sin(n-j)\theta_k}{d\sin n\theta_k - \beta\sin(n-1)\theta_k},\ j = 1, ..., n,$$

where θ_k is given by formula

$$d^2\sin(n+1)\theta_k - d(\alpha+\beta)\sin n\theta_k + \alpha\beta\sin(n-1)\theta_k = 0,\ k = 1, ..., n.$$

Recently in S. Kouachi [6], we generalized the above results concerning the eigenvalues of tridiagonal matrices (1) satisfying condition (2), but we were unable to calculate the corresponding eigenvectors, in view of the complexity of their expressions. The

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matrices studied by J. F. Elliott [2] and R. T. Gregory and D. Carney [3] are special cases of those considered by W. Yueh[1] which are, at their tour, special cases with regard to those that we have studied in S. Kouachi [5]. All the conditions imposed in the above papers are very restrictive and the techniques used are complicated and are not (in general) applicable to tridiagonal matrices considered in this paper even tough for small n. For example, our techniques are applicable for all the 7×7 matrices

$$A_7 = \begin{pmatrix} 5-4\sqrt{2} & 9 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 8 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & -18 & 5 & 5+i\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 & 5-i\sqrt{7} & 5 & -27i & 0 \\ 0 & 0 & 0 & 0 & 2i & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & -32 & 5-3\sqrt{6} \end{pmatrix}$$

and

$$A_7' = \begin{pmatrix} 5-4\sqrt{2} & 54i & 0 & 0 & 0 & 0 & 0 \\ -i & 5 & -16 & 0 & 0 & 0 & 0 \\ 0 & -2 & 5 & 6i & 0 & 0 & 0 \\ 0 & 0 & -9i & 5 & -8i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 2i\sqrt{2} & 5 & -18i & 0 \\ 0 & 0 & 0 & 0 & 3i & 5 & 2+2i \\ 0 & 0 & 0 & 0 & 0 & 8-8i & 5-3\sqrt{6} \end{pmatrix},$$

and guarantee that they possess the same eigenvalues and in addition they give their exact expressions (formulas (15) lower) since condition (2) is satisfied:

$$\begin{aligned} \lambda_1,\ \lambda_4 &= 5 \pm \sqrt{\left(3\sqrt{6}\right)^2 + \left(4\sqrt{2}\right)^2 + 2\left(3\sqrt{6}\right)\left(4\sqrt{2}\right)\cos\left(\frac{2\pi}{7}\right)}, \\ \lambda_2,\ \lambda_5 &= 5 \pm \sqrt{\left(3\sqrt{6}\right)^2 + \left(4\sqrt{2}\right)^2 + 2\left(3\sqrt{6}\right)\left(4\sqrt{2}\right)\cos\left(\frac{4\pi}{7}\right)}, \\ \lambda_3,\ \lambda_6 &= 5 \pm \sqrt{\left(3\sqrt{6}\right)^2 + \left(4\sqrt{2}\right)^2 + 2\left(3\sqrt{6}\right)\left(4\sqrt{2}\right)\cos\left(\frac{6\pi}{7}\right)}, \\ \lambda_7 &= 5 - \left(3\sqrt{6} + 4\sqrt{2}\right), \end{aligned}$$

whereas the recent techniques are restricted to the limited case when the entries on the subdiagonal are equal and those on the superdiagonal are also equal and the direct calculus give the following characteristic polynomial

$$\begin{aligned} P(\lambda) = \lambda^7 &+ \left(4\sqrt{2} + 3\sqrt{6} - 35\right)\lambda^6 + \left(24\sqrt{3} - 120\sqrt{2} - 90\sqrt{6} + 267\right)\lambda^5 + \\ &\left(684\sqrt{2} - 600\sqrt{3} + 447\sqrt{6} + 2075\right)\lambda^4 + \left(6320\sqrt{2} + 1872\sqrt{3} + 6060\sqrt{6} - 23\,893\right)\lambda^3 + \\ &\left(31\,920\sqrt{3} - 47\,124\sqrt{2} - 33\,891\sqrt{6} - 24\,105\right)\lambda^2 \end{aligned}$$

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$$+\left(369\,185-98\,568\sqrt{3}-114\,090\sqrt{6}-44\,760\sqrt{2}\right)\lambda$$
$$+\left(365\,828\sqrt{2}-239\,160\sqrt{3}+142\,833\sqrt{6}+80\,825\right)$$

for which the roots are very difficult to calculate (degree of $P \geq 5$).

If σ is a mapping (not necessary a permutation) from the set of the integers from 1 to $(n-1)$ into the set of the integers different of zero $\mathbb{N}^*$, we denote by $A_n(\sigma)$ the $n \times n$ matrix

$$A_n(\sigma) = \begin{pmatrix} -\alpha+b & c_{\sigma_1} & 0 & 0 & \ldots & 0 \\ a_{\sigma_1} & b & c_{\sigma_2} & 0 & \ldots & 0 \\ 0 & a_{\sigma_2} & b & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & c_{\sigma_{n-1}} \\ 0 & \ldots & \ldots & 0 & a_{\sigma_{n-1}} & -\beta+b \end{pmatrix} \tag{1.1}$$

and by $\Delta_n(\sigma) = |\Delta_n(\sigma) - \lambda I_n|$ its characteristic polynomial. If $\sigma = i$, where i is the identity, then $A_n(i)$ and its characteristic polynomial $\Delta_n(i)$ will be denoted by A_n and Δ_n respectively. Our aim is to establish the eigenvalues and the corresponding eigenvectors of the matrices $A_n(\sigma)$.

2. The Eigenvalue Problem. Throughout this section we suppose $d_1 d_2 \neq 0$.In the case when $\alpha = \beta = 0$, the matrix $A_n(\sigma)$ and its characteristic polynomial will be denoted respectively by $A_n^0(\sigma)$ and $\Delta_n^0(\sigma)$ and in the general case they will be denoted by A_n and Δ_n. We put

$$Y^2 = d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta, \tag{3}$$

where

$$Y = b - \lambda. \tag{3.1}$$

In S. Kouachi [6], we have proved the following result

THEOREM 2.1. *When $d_1 d_2 \neq 0$, the eigenvalues of the class of matrices $A_n(\sigma)$ on the form (1.1) are independent of the entries ($a_i,\ c_i,\ i = 1,..,n-1$) and of the mapping σ provided that condition (2) is satisfied and their characteristic determinants are given by*

$$\Delta_n = (d_1 d_2)^{m-1} \frac{d_1 d_2 (Y-\alpha-\beta)\sin(m+1)\theta + \left(\alpha\beta Y - \alpha d_1^2 - \beta d_2^2\right)\sin m\theta}{\sin\theta}, \tag{4.a}$$

when $n = 2m+1$ is odd and

$$\Delta_n = (d_1 d_2)^{m-1} \frac{d_1 d_2 \sin(m+1)\theta + \left[\alpha\beta + d_2^2 - (\alpha+\beta)Y\right]\sin m\theta + \alpha\beta \dfrac{d_1}{d_2}\sin(m-1)\theta}{\sin\theta}, \tag{4.b}$$

when $n = 2m$ is even.

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Proof. When $\alpha = \beta = 0$, formulas (4.a) and (4.b) become, respectively

$$\Delta_n^0 = (d_1 d_2)^m \, Y \frac{\sin{(m+1)\,\theta}}{\sin\theta}, \tag{4.a.0}$$

when $n = 2m+1$ is odd and

$$\Delta_n^0 = (d_1 d_2)^m \, \frac{\sin(m+1)\theta + \dfrac{d_2}{d_1}\sin m\theta}{\sin\theta}, \tag{4.b.0}$$

when $n = 2m$ is even. Since the right hand sides of formulas (4.a.0) and (4.b.0) are independent of σ, then to prove that the characteristic polynomial of $A_n^0\,(\sigma)$, which we denote by Δ_n^0, is also, it suffices to prove them for $\sigma = i$. Expanding Δ_n^0 in terms of it's last column and using (2) and (3), we get

$$\Delta_n^0 = Y\Delta_{n-1}^0 - d_2^2\Delta_{n-2}^0, n = 3, ..., \tag{4.a.1}$$

when $n = 2m+1$ is odd and

$$\Delta_n^0 = Y\Delta_{n-1}^0 - d_1^2\Delta_{n-2}^0, n = 3, ..., \tag{4.b.1}$$

when $n = 2m$ is even. Then by writing the expressions of Δ_n^0 for $n = 2m+1,\ 2m$ and $2m-1$ respectively, multiplying Δ_{2m}^0 and Δ_{2m-1}^0 by Y and d_1^2 respectively and adding the three resulting equations term to term, we get

$$\Delta_{2m+1}^0 = \left(Y^2 - d_1^2 - d_2^2\right)\Delta_{2m-1}^0 - d_1^2 d_2^2 \Delta_{2m-3}^0, \tag{4.a.2}$$

We will prove by induction in m that formula (4.a.0) is true.
If $n = 2m+1$ is odd, for $m = 0$ and $m = 1$ formula (4.a.0) is satisfied. Suppose that it is satisfied for all integers $< m$, then from (4.a.2) and using (3), we get

$$\Delta_{2m+1}^0 = Y\,(d_1 d_2)^m \,\frac{2\sin m\theta\cos\theta - \sin{(m-1)\,\theta}}{\sin\theta}.$$

Using the well known trigonometric formula

$$2\sin\eta\cos\zeta = \sin{(\eta+\zeta)} + \sin{(\eta-\zeta)}\,, \tag{*}$$

for $\eta = m\theta$ and $\zeta = \theta$, we deduce formula (4.a.0).
When $n = 2m$ is even, applying formula (4.a.1) for $n = 2m+1$, we get

$$\Delta_{2m}^0 = \frac{\Delta_{2m+1}^0 + d_2^2\Delta_{2m-1}^0}{Y}.$$

By direct application of (4.a.0) two times, for $n = 2m+1$ and $n = 2m-1$, to the right hand side of the last expression, we deduce (4.b.0).

If we suppose that $\alpha \neq 0$ or $\beta \neq 0$, then expanding Δ_n in terms of the first and last columns and using the linear property of the determinants with regard to its columns, we get

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$$\Delta_n = \Delta_n^0 - \alpha \left|E_{n-1}^2\right| - \beta \left|E_{n-1}^1\right| + \alpha\beta \begin{vmatrix} Y & c_2 & 0 & ... & 0 \\ a_2 & Y & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-2} \\ 0 & ... & 0 & a_{n-2} & Y \end{vmatrix},$$

where E_{n-1}^1 and E_{n-1}^2 are the $(n-1)$ square matrices of the form (1)

$$E_{n-1}^i = \begin{pmatrix} Y & c_i & 0 & ... & 0 \\ a_i & Y & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n+i-3} \\ ... & ... & 0 & a_{n+i-3} & Y \end{pmatrix}, \ i = 1, \ 2.$$

Since all the entries a_i's on the subdiagonal and c_i's on the superdiagonal satisfy condition (2), then using formulas (4.a.0) and (4.b.0) and taking in the account the order of the entries a_i's and c_i's, we deduce the general formulas (4.a) and (4.b). □

Before proceeding further, let us deduce from formula (4.b) a proposition for the matrix $B_n(\sigma)$ which is obtained from $A_n(\sigma)$ by interchanging the numbers α and β.

PROPOSITION 2.2. *When n is even, the eigenvalues of $B_n(\sigma)$ are the same as $A_n(\sigma)$.*

Let us see what formula (4) says and what it does not say. It says that if $a_i^{'}, \ c_i^{'}, \ i = 1,..,n-1$ are other constants satisfying condition (2) and

$$A_n^{'} = \begin{pmatrix} -\alpha + b & c_1^{'} & 0 & 0 & ... & 0 \\ a_1^{'} & b & c_2^{'} & 0 & ... & 0 \\ 0 & a_2^{'} & b & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & c_{n-1}^{'} \\ 0 & ... & ... & 0 & a_{n-1}^{'} & -\beta + b \end{pmatrix}$$

then the matrices A_n, $A_n^{'}$ and $A_n(\sigma)$ possess the same characteristic polynomial and hence the same eigenvalues. Therefore we have this immediate consequence of formula (4)

COROLLARY 2.3. *The class of matrices $A_n(\sigma)$, where σ is a mapping from the set of the integers from 1 to $(n-1)$ into $\mathbb{N}^*$ are similar provided that all the entries on the subdiagonal and on the superdiagonal satisfy condition (2).*

The components of the eigenvector $u^{(k)}(\sigma)$, $k = 1,...,n$ associated to the eigenvalue λ_k, $k = 1,...,n$, which we denote by $u_j^{(k)}$, $j = 1,..,n$, are solutions of the

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linear system of equations

$$\left\{ \begin{array}{c} (-\alpha+\xi_k)u_1^{(k)}+c_{\sigma_1}u_2^{(k)}=0, \\ a_{\sigma_1}u_1^{(k)}+\xi_k u_2^{(k)}+c_{\sigma_2}u_3^{(k)}=0, \\ \ldots \\ a_{\sigma_{n-1}}u_{n-1}^{(k)}+(-\beta+\xi_k)u_n^{(k)}=0, \end{array} \right. \tag{5}$$

where $\xi_k = Y$ is given by formula (3) and $\theta_k,\ k = 1, ..., n$ are solutions of

$$d_1 d_2 \left(\xi_k - \alpha - \beta\right) \sin\left(m+1\right)\theta_k + \left(\alpha\beta\xi_k - \alpha d_1^2 - \beta d_2^2\right) \sin m\theta_k = 0, \tag{6.a}$$

when $n = 2m + 1$ is odd and

$$d_1 d_2 \sin(m+1)\theta_k + \left[\alpha\beta + d_2^2 - (\alpha+\beta)\,\xi_k\right] \sin m\theta_k + \alpha\beta\frac{d_1}{d_2}\sin(m-1)\theta_k = 0, \tag{6.b}$$

when $n = 2m$ is even.

Since these n equations are linearly dependent, then by eliminating the first equation we obtain the following system of $(n-1)$ equations and $(n-1)$ unknowns, written in a matrix form as

$$\begin{pmatrix} \xi_k & c_{\sigma_2} & 0 & \ldots & 0 \\ a_{\sigma_2} & \xi_k & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{\sigma_{n-1}} \\ 0 & \ldots & 0 & a_{\sigma_{n-1}} & (-\beta+\xi_k) \end{pmatrix} \begin{pmatrix} u_2^{(k)} \\ u_3^{(k)} \\ \vdots \\ \vdots \\ u_n^{(k)} \end{pmatrix} = \begin{pmatrix} -a_{\sigma_1}u_1^{(k)} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

The determinant of this system is given by formulas (4) for $\alpha = 0$ and n replaced by $n-1$ and equal to

$$\Delta_{n-1}^{(k)} = (d_1 d_2)^{m-1}\,\frac{d_1 d_2 \sin(m+1)\theta_k + \left[d_1^2 - \beta\xi_k\right]\sin m\theta_k}{\sin\theta_k}, \tag{8.a}$$

when $n = 2m + 1$ is odd and

$$\Delta_{n-1}^{(k)} = (d_1 d_2)^{m-1}\,\frac{(\xi_k - \beta)\sin m\theta_k - \beta\dfrac{d_1}{d_2}\sin(m-1)\theta_k}{\sin\theta_k}, \tag{8.b}$$

when $n = 2m$ is even, for all $k = 1, ..., n$.

$$u_j^{(k)}\left(\sigma\right) = \frac{\Gamma_j^{(k)}\left(\sigma\right)}{\Delta_{n-1}^{(k)}},\ j,\ k = 1, ..., n, \tag{9}$$

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where

$$\Gamma_j^{(k)}(\sigma) = \begin{vmatrix} \xi_k & c_{\sigma_2} & 0 & ... & -a_{\sigma_1} u_1^{(k)} & 0 & ... & 0 \\ a_{\sigma_2} & \xi_k & \ddots & \ddots & 0 & 0 & ... & \vdots \\ 0 & \ddots & \ddots & c_{\sigma_{j-2}} & \vdots & \vdots & ... & \vdots \\ 0 & 0 & a_{\sigma_{j-2}} & \xi_k & 0 & 0 & ... & \vdots \\ \vdots & \vdots & \ddots & a_{\sigma_{j-1}} & 0 & c_{\sigma_j} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \xi_k & \ddots & 0 \\ \vdots & \vdots & ... & \ddots & \vdots & a_{\sigma_j+1} \ddots & \ddots & c_{\sigma_{n-1}} \\ 0 & ... & ... & ... & 0 & 0 & a_{\sigma_{n-1}} & (-\beta+\xi_k) \end{vmatrix},$$

$j = 2, ..., n,\quad k = 1, ..., n$. By permuting the $j-2$ first columns with the $(j-1)$-th one and using the properties of the determinants, we get

$$u_j^{(k)}(\sigma) = (-1)^{j-2} \frac{\Lambda_j^{(k)}(\sigma)}{\Delta_{n-1}},\ j = 2, ..., n, \tag{10}$$

where $\Lambda_j^{(k)}(\sigma)$ is the determinant of the matrix

$$C_j^{(k)}(\sigma) = \begin{pmatrix} T_{j-1}^{(k)}(\sigma) & \mathbf{0} \\ \mathbf{0} & S_{n-j}^{(k)}(\sigma) \end{pmatrix},$$

where

$$T_{j-1}^{(k)}(\sigma) = \begin{pmatrix} -a_{\sigma_1} u_1^{(k)} & \xi_k & c_{\sigma_2} & 0 & \cdots & 0 \\ 0 & a_{\sigma_2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & c_{\sigma_{j-2}} \\ \vdots & 0 & 0 & \ddots & \ddots & \xi_k \\ 0 & \cdots & \cdots & \cdots & 0 & a_{\sigma_{j-1}} \end{pmatrix}$$

is the supertriangular matrix of order $j-1$ with diagonal $(-a_{\sigma_1} u_1^{(k)}, a_{\sigma_2}, ..., a_{\sigma_{j-1}})$ and

$$S_{n-j}^{(k)}(\sigma) = \begin{pmatrix} \xi_k & c_{\sigma_{j+1}} & 0 & \cdots & 0 \\ a_{\sigma_j+1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{\sigma_{n-1}} \\ 0 & \cdots & 0 & a_{\sigma_{n-1}} & (-\beta+\xi_k) \end{pmatrix},$$

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is a tridiagonal matrix of order $n-j$ belonging to the class of the form (1.1) and satisfying condition (2). Thus

$$\begin{aligned}\left|C_j^{(k)}(\sigma)\right| &= \left|T_{j-1}^{(k)}(\sigma)\right|\left|S_{n-j}^{(k)}(\sigma)\right| \\ &= -a_{\sigma_1}...a_{\sigma_{j-1}}u_1^{(k)}\Delta_{n-j}^{(k)},\ j=2,...,n \text{ and } k=1,...,n,\end{aligned} \tag{11}$$

where $\Delta_{n-j}^{(k)}(\sigma)$ is given by formulas (4) for $\alpha=0$ and $n-1$ replaced by $n-j$

$$\Delta_{n-j}^{(k)} = \begin{cases} (d_1d_2)^{\frac{n-j}{2}-1}\dfrac{d_1d_2\sin(\frac{n-j}{2}+1)\theta_k+\left(d_1^2-\beta\xi_k\right)\sin\frac{n-j}{2}\theta_k}{\sin\theta_k}, \text{when } j \text{ is odd,} \\ (d_1d_2)^{\frac{n-j-1}{2}}\dfrac{(\xi_k-\beta)\sin(\frac{n-j+1}{2})\theta_k-\beta\frac{d_2}{d_1}\sin(\frac{n-j-1}{2})\theta_k}{\sin\theta_k}, \text{ when } j \text{ is even,}\end{cases} \tag{12.a}$$

when n is odd and

$$\Delta_{n-j}^{(k)} = \begin{cases} (d_1d_2)^{\frac{n-j-1}{2}}\dfrac{(\xi_k-\beta)\sin(\frac{n-j-1}{2})\theta_k-\beta\dfrac{d_1}{d_2}\sin(\frac{n-j-1}{2})\theta_k}{\sin\theta_k}, \text{when } j \text{ is odd,} \\ (d_1d_2)^{\frac{n-j}{2}-1}\dfrac{d_1d_2\sin(\frac{n-j}{2}+1)\theta_k+\left(d_2^2-\beta\xi_k\right)\sin\frac{n-j}{2}\theta_k}{\sin\theta_k}, \text{ when } j \text{ is even,}\end{cases} \tag{12.b}$$

when n is even, for all $j=2,...,n$ and $k=1,...,n$. Using formulas (9)-(12), we get

$$u_j^{(k)}(\sigma) = (-1)^{j-1}a_{\sigma_1}...a_{\sigma_{j-1}}u_1^{(k)}\frac{\Delta_{n-j}^{(k)}}{\Delta_{n-1}^{(k)}},\ j=2,...,n \text{ and } k=1,...,n. \tag{13}$$

Finally

$$u_j^{(k)}(\sigma) = \mu_j(\sigma)\,u_1^{(k)}\begin{cases} \dfrac{d_1d_2\sin(\frac{n-j}{2}+1)\theta_k+\left(d_1^2-\beta\xi_k\right)\sin\frac{n-j}{2}\theta_k}{d_1d_2\sin(\frac{n+1}{2})\theta_k+\left(d_1^2-\beta\xi_k\right)\sin\left(\frac{n-1}{2}\right)\theta_k}, \text{when } j \text{ is odd,} \\ \sqrt{d_1d_2}\dfrac{(\xi_k-\beta)\sin(\frac{n-j+1}{2})\theta_k-\beta\frac{d_2}{d_1}\sin(\frac{n-j-1}{2})\theta_k}{d_1d_2\sin(\frac{n+1}{2})\theta_k+\left(d_1^2-\beta\xi_k\right)\sin\left(\frac{n-1}{2}\right)\theta_k}, \text{ when } j \text{ is even,}\end{cases} \tag{13.a}$$

when n is odd and

$$u_j^{(k)}(\sigma) = \mu_j(\sigma)\,u_1^{(k)}\begin{cases} \dfrac{(\xi_k-\beta)\sin(\frac{n-j-1}{2})\theta_k-\beta\dfrac{d_1}{d_2}\sin(\frac{n-j-1}{2})\theta_k}{(\xi_k-\beta)\sin\frac{n}{2}\theta_k-\beta\dfrac{d_1}{d_2}\sin(\frac{n}{2}-1)\theta_k}, \text{when } j \text{ is odd,} \\ \frac{1}{\sqrt{d_1d_2}}\dfrac{d_1d_2\sin(\frac{n-j}{2}+1)\theta_k+\left(d_2^2-\beta\xi_k\right)\sin\frac{n-j}{2}\theta_k}{(\xi_k-\beta)\sin\frac{n}{2}\theta_k-\beta\dfrac{d_1}{d_2}\sin(\frac{n}{2}-1)\theta_k}, \text{ when } j \text{ is even,}\end{cases} \tag{13.b}$$

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for all $j = 2, ..., n$ and $k = 1, ..., n$, when n is even, where

$$\mu_j(\sigma) = \left(-\sqrt{d_1 d_2}\right)^{1-j} a_{\sigma_1} ... a_{\sigma_{j-1}}, \; j = 2, ..., n, \tag{†}$$

$\xi_k = Y$ and θ_k are given respectively by (3) and formulas (6).

3. Special Cases. From now on, we put

$$\rho_j(\sigma) = \left(-\sqrt{d_1 d_2}\right)^{n-1} \mu_j(\sigma), \; j = 1, ..., n, \tag{‡}$$

where $\mu_j(\sigma)$ is given by (†).

3.1. Case when n is odd. If $\alpha = \beta = 0$, we have

Theorem 3.1. *If $\alpha = \beta = 0$, the eigenvalues $\lambda_k(\sigma)$, $k = 1, ..., n$ of the class of matrices $A_n(\sigma)$ on the form (1.1) are independent of the entries $(a_i, c_i, i = 1, .., n-1)$ and of σ provided that condition (2) is satisfied and they are given by*

$$\lambda_k = \begin{cases} b + \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \; k = 1, ..., m, \\ b - \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \; k = m+1, ..., 2m, \\ b, \; k = n. \end{cases} \tag{14}$$

The corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma), ..., u_n^{(k)}(\sigma)\right)^t$, $k = 1, .., n-1$ are given by

$$u_j^{(k)}(\sigma) = \rho_j(\sigma) \begin{cases} d_1 d_2 \sin(\frac{n-j}{2} + 1)\theta_k + d_1^2 \sin\frac{n-j}{2}\theta_k, \text{ when is } j \text{ odd}, \\ \sqrt{d_1 d_2}(b - \lambda_k) \sin(\frac{n-j+1}{2})\theta_k, \text{ when is } j \text{ even}, \end{cases} \tag{14.a}$$

and

$$u_j^{(k)}(\sigma) = \begin{cases} a_{\sigma_1} ... a_{\sigma_{j-1}} \left(-d_2^2\right)^{\frac{n-j}{2}}, \text{ when } j \text{ is odd}, \\ 0, \text{ when } j \text{ is even}, \end{cases} \tag{14.b}$$

$j = 1, ...n$, where $\rho_j(\sigma)$ is given by (‡) and

$$\theta_k = \begin{cases} \frac{2k\pi}{n+1}, \; k = 1, ..., m, \\ \frac{2(k-m)\pi}{n+1}, \; k = m+1, ..., 2m. \end{cases}.$$

Proof, We take $a_{\sigma_0} = a_0 = 1$. The eigenvalues λ_k, $k = 1, ..., 2m$ are trivial consequence of (4) by putting $(m+1)\theta = k\pi$, $k = 1, ..., m$ and using (3). The

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eigenvalue λ_n is a consequence of (4) and (3.1) by putting $Y = 0$. Formula (14.a) is a trivial consequence of (13.a) by taking $\beta = 0$ and choosing

$$u_1^{(k)} = \left(-\sqrt{d_1 d_2}\right)^{n-1}\left[d_1 d_2 \sin(\frac{n+1}{2})\theta_k + d_1^2 \sin\left(\frac{n-1}{2}\right)\theta_k\right],\ k = 1, ..., n-1.$$

Concerning the $n-$th eigenvector, we solve directly system (7) and choose

$$u_1^{(n)} = \left(-d_2^2\right)^{\frac{n-1}{2}}. \square$$

If $\alpha = d_2$ and $\beta = d_1$ or $\beta = -d_1$ and $\alpha = -d_2$, then using the trigonometric formula (*) for $\eta = (\frac{2m+1}{2})\theta_k$ and $\zeta = \frac{\theta_k}{2}$, formula (6.a) becomes

$$(\xi_k \pm (d_1 + d_2)) \sin \frac{(2m+1)}{2}\theta_k \cos \frac{\theta_k}{2} = 0, \tag{6.a.1}$$

then we get

Theorem 3.2. *If $\alpha = d_2$ and $\beta = d_1$ or $\beta = -d_1$ and $\alpha = -d_2$,the eigenvalues $\lambda_k(\sigma)$, $k = 1, ..., n$ of the class of matrices $A_n(\sigma)$ on the form (1.1)are independent of the entries (a_i, c_i, $i = 1, .., n-1$) and of σ provided that condition (2) is satisfied and they are given by*

$$\lambda_k = \begin{cases} b + \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k},\ k = 1, ..., m, \\ b - \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k},\ k = m+1, ..., 2m, \\ b - (\alpha + \beta),\ k = n. \end{cases} \tag{15}$$

The corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma), ..., u_n^{(k)}(\sigma)\right)^t$, $k = 1, .., n$ are given by

$$u_j^{(k)}(\sigma) = \rho_j(\sigma) \begin{cases} d_2 \sin(\frac{n-j}{2} + 1)\theta_k + [d_1 - b + \lambda_k] \sin \frac{n-j}{2}\theta_k, j \text{ is odd}, \\ -\sqrt{\frac{d_2}{d_1}}\left[(d_1 - b + \lambda_k) \sin(\frac{n-j+1}{2})\theta_k + d_2 \sin(\frac{n-j-1}{2})\theta_k\right] j \text{ is even}, \end{cases} \tag{15.a}$$

$k = 1, ..n-1$ and

$$u_j^{(n)}(\sigma) = \rho_j(\sigma) \begin{cases} 1, \text{when } j \text{ is odd}, \\ \sqrt{\frac{d_2}{d_1}},\ \text{when } j \text{ is even}, \end{cases}$$

$j = 1, .., n$, when $\alpha = d_2$ and $\beta = d_1$ and

$$u_j^{(k)}(\sigma) = \rho_j(\sigma) \begin{cases} d_2 \sin(\frac{n-j}{2} + 1)\theta_k + [d_1 + b - \lambda_k] \sin \frac{n-j}{2}\theta_k, j \text{ is odd}, \\ \sqrt{\frac{d_2}{d_1}}\left[(d_1 + b - \lambda_k) \sin(\frac{n-j+1}{2})\theta_k + d_2 \sin(\frac{n-j-1}{2})\theta_k\right] j \text{ is even}, \end{cases} \tag{15.b}$$

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$k = 1, ..n-1$ and

$$u_j^{(n)}(\sigma) = \rho_j(\sigma) \begin{cases} 1, \text{ when } j \text{ is odd,} \\ \\ -\sqrt{\frac{d_2}{d_1}}, \text{ when } j \text{ is even,} \end{cases}$$

$j = 1, .., n$, when $\beta = -d_1$ and $\alpha = -d_2$, $j = 1, .., n$, where $\rho_j(\sigma)$ is given by (‡) and

$$\theta_k = \begin{cases} \frac{2k\pi}{n}, \ k = 1, ..., m, \\ \\ \frac{2(k-m)\pi}{n}, \ k = m+1, ..., 2m. \end{cases}.$$

Proof. Formula (15) is a simple consequence of (6.a.1). Using (13.a), the expressions (15.a) and (15.b) are trivial by choosing

$$u_1^{(k)} = \left(-\sqrt{d_1 d_2}\right)^{n-1} \left[d_2 \sin(\frac{n+1}{2})\theta_k + [d_1 - b + \lambda_k] \sin\left(\frac{n-1}{2}\right)\theta_k\right],$$

if $\alpha = d_2$ and $\beta = d_1$ and

$$u_1^{(k)} = \left(-\sqrt{d_1 d_2}\right)^{n-1} \left[d_2 \sin(\frac{n+1}{2})\theta_k + [d_1 + b - \lambda_k] \sin\left(\frac{n-1}{2}\right)\theta_k\right],$$

when $\beta = -d_1$ and $\alpha = -d_2$.The last eigenvector is obtained by choosing

$$u_1^{(n)}(\sigma) = \left(-\sqrt{d_1 d_2}\right)^{n-1} a_{\sigma_1} ... a_{\sigma_{j-1}}. \ \square$$

When $\alpha = d_2$ and $\beta = -d_1$ or $\alpha = -d_2$ and $\beta = d_1$, then using the trigonometric formula

$$2\cos\eta \sin\zeta = \sin(\eta + \zeta) - \sin(\eta - \zeta), \tag{**}$$

for $\eta = (\frac{2m+1}{2})\theta_k$ and $\zeta = \frac{\theta_k}{2}$, formula (4.a) becomes

$$(\xi_k \pm (d_2 - d_1)) \cos \frac{(2m+1)}{2}\theta_k \sin \frac{\theta_k}{2} = 0, \tag{6.a.2}$$

then we get

Theorem 3.3. *If $\alpha = -d_2$ and $\beta = d_1$ or $\beta = -d_1$ and $\alpha = d_2$,the eigenvalues $\lambda_k(\sigma)$, $k = 1, ..., n$ of the class of matrices $A_n(\sigma)$ on the form (1.1)are independent of the entries (a_i, c_i, $i = 1, .., n-1$) and of σ provided that condition (2) is satisfied and they are given by*

$$\lambda_k = \begin{cases} b + \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \ k = 1, ..., m, \\ \\ b - \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \ k = m+1, ..., 2m, \\ \\ b - (\alpha + \beta), \ k = n. \end{cases} \tag{16}$$

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The corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma), ..., u_n^{(k)}(\sigma)\right)^t$, $k = 1, .., n$ *are given by (15.a) and*

$$u_j^{(n)}(\sigma) = \rho_j(\sigma) \begin{cases} (-1)^{\frac{j-1}{2}}, \text{when } j \text{ is odd}, \\ \sqrt{\frac{d_2}{d_1}}(-1)^{\frac{j+2}{2}}, \text{ when } j \text{ is even}, \end{cases} \quad j = 1, .., n,$$

when $\alpha = -d_2$ *and* $\beta = d_1$ *and (15.b) and*

$$u_j^{(n)}(\sigma) = \rho_j(\sigma) \begin{cases} (-1)^{\frac{j-1}{2}}, \text{when } j \text{ is odd}, \\ \sqrt{\frac{d_2}{d_1}}(-1)^{\frac{j}{2}}, \text{ when } j \text{ is even}, \end{cases} \quad j = 1, .., n,$$

when $\beta = -d_1$ *and* $\alpha = d_2$*,where* $\rho_j(\sigma)$ *is given by (‡) and*

$$\theta_k = \begin{cases} \frac{(2k-1)\pi}{n}, \ k = 1, ..., m, \\ \frac{(2(k-m)-1)\pi}{n}, \ k = m+1, ..., 2m. \end{cases} .$$

Proof. Formula (16) is trivial by solving (6.a.2). Concerning the eigenvectors, following the same reasoning as in the case when $\alpha = d_2$ and $\beta = d_1$ or $\beta = -d_1$ and $\alpha = -d_2$ and since we use formula (13.a) to find the components of the eigenvectors which depend only of β, we deduce the same results. The last eigenvector is obtained by passage to the limit in formula (13.a) when θ_k tends to π and choosing the first component as in the previous case. □

3.2. Case when n is even. If $\alpha\beta = d_2^2$, then using (*) for $\eta = m\theta_k$ and $\zeta = \theta_k$, formula (6.b) becomes

$$\left[2d_1 d_2 \cos\theta_k + \alpha\beta + d_2^2 - (\alpha+\beta)\xi_k\right] \sin m\theta_k = 0.$$

Using (3), we get

$$\left[\xi_k^2 - (\alpha+\beta)\xi_k + d_2^2 - d_1^2\right] \sin m\theta_k = 0, \tag{6.b.1}$$

which gives

$$\sin m\theta_k = 0$$

and

$$\xi_k^2 - (\alpha+\beta)\xi_k + d_2^2 - d_1^2 = 0, \tag{3.2}$$

then we get

Theorem 3.4. *If* $\alpha b = d_2^2$ *,the eigenvalues* $\lambda_k(\sigma)$, $k = 1, ..., n$ *of the class of matrices* $A_n(\sigma)$ *on the form (1.1)are independent of the entries* $(a_i,\ c_i,\ i = 1, .., n-1)$

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and of σ provided that condition (2) is satisfied and they are given by

$$\lambda_k = \begin{cases} b + \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \ k = 1, ..., m-1, \\ b - \sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \cos\theta_k}, \ k = m, ..., 2m-2, \\ b + \dfrac{(\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + 4d_1^2}}{2}, \ k = n-1, \\ b + \dfrac{(\alpha+\beta) - \sqrt{(\alpha-\beta)^2 + 4d_1^2}}{2}, \ k = n. \end{cases} \tag{17}$$

The corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma), ..., u_n^{(k)}(\sigma)\right)^t, \ k = 1, .., n-2$ *are given by*

$$u_j^{(k)}(\sigma) = \rho_j(\sigma) \begin{cases} (b - \lambda_k - \beta)\sin(\frac{n-j-1}{2})\theta_k - \beta\dfrac{d_1}{d_2}\sin(\frac{n-j-1}{2})\theta_k, j \text{ is odd,} \\ \frac{1}{\sqrt{d_1 d_2}}\left[d_1 d_2 \sin(\frac{n-j}{2}+1)\theta_k + \left(d_2^2 - \beta(b-\lambda_k)\right)\sin\frac{n-j}{2}\theta_k\right], j \, even \end{cases} \tag{17.a}$$

where $\rho_j(\sigma)$, $j = 1, ..., n$ *is given by (‡) and*

$$\theta_k = \begin{cases} \frac{2k\pi}{n}, \ k = 1, ..., m-1, \\ \frac{2(k-m+1)\pi}{n}, \ k = m, ..., 2m-2. \end{cases}.$$

The eigenvectors $u^{(n-1)}(\sigma)$ *and* $u^{(n)}(\sigma)$ *associated respectively with the eigenvalues* λ_{n-1} *and* λ_n *are given by formula (13.b), where* θ_k *is given by (3), (3.1) and (3.2).*

Proof. Formula (17) is a consequence of (6.b.1). The eigenvectors are a consequence of formula (13.b) by choosing

$$u_1^{(k)} = \left(-\sqrt{d_1 d_2}\right)^{n-1} \begin{cases} (b - \lambda_k - \beta)\sin\frac{n}{2}\theta_k - \beta\dfrac{d_1}{d_2}\sin(\frac{n}{2}-1)\theta_k, \text{ when } j \text{ is odd,} \\ (b - \lambda_k - \beta)\sin\frac{n}{2}\theta_k - \beta\dfrac{d_1}{d_2}\sin(\frac{n}{2}-1)\theta_k, \text{ when } j \text{ is even.} \end{cases} \quad \square$$

When $\alpha = -\beta = \pm d_2$, then, using (**), formula (6.b) gives

$$2d_1 d_2 \cos m\theta = 0, \tag{6.b.2}$$

then we have

THEOREM 3.5. *If* $\alpha = -\beta = \pm d_2$,*the eigenvalues* $\lambda_k(\sigma)$, $k = 1, ..., n$ *of the class of matrices* $A_n(\sigma)$ *on the form (1.1)are independent of the entries* $(a_i, \ c_i, \ i =$

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$1,..,n-1$*) and of* σ *provided that condition (2) is satisfied and they are given by*

$$\lambda_k = \begin{cases} b+\sqrt{d_1^2+d_2^2+2d_1d_2\cos\theta_k}, & k=1,...,m, \\ b-\sqrt{d_1^2+d_2^2+2d_1d_2\cos\theta_k}, & k=m+1,...,n, \end{cases} \tag{18}$$

The corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma),...,u_n^{(k)}(\sigma)\right)^t$, $k=1,..,n$ *are given by*

$$u_j^{(k)}(\sigma) = \rho_j(\sigma)\begin{cases} (b-\lambda_k-d_2)\sin(\frac{n-j-1}{2})\theta_k - d_1\sin(\frac{n-j-1}{2})\theta_k, j \text{ is odd,} \\ d_1d_2\sin(\frac{n-j}{2}+1)\theta_k + \left[d_2^2-d_2(b-\lambda_k)\right]\sin\frac{n-j}{2}\theta_k, j \text{ is even,} \end{cases} \tag{18.a}$$

when $\alpha=-\beta=-d_2$ *and*

$$u_j^{(k)}(\sigma) = \rho_j(\sigma)\begin{cases} (b-\lambda_k+d_2)\sin(\frac{n-j-1}{2})\theta_k + d_1\sin(\frac{n-j-1}{2})\theta_k, j \text{ is odd,} \\ d_1d_2\sin(\frac{n-j}{2}+1)\theta_k + \left[d_2^2+d_2(b-\lambda_k)\right]\sin\frac{n-j}{2}\theta_k, j \text{ is even,} \end{cases} \tag{18.b}$$

when $\alpha=-\beta=d_2$ *where* $\rho_j(\sigma)$, $j=1,...,n$ *is given by (‡) and*

$$\theta_k = \begin{cases} \frac{(2k-1)\pi}{n}, & k=1,...,m, \\ \frac{(2k-2m-1)\pi}{n}, & k=m+1,...,n. \end{cases}$$

Proof. Formula (18) is a consequence of (6.b.2).The eigenvectors are a consequence of formula (13.b) by choosing

$$u_1^{(k)} = \left(-\sqrt{d_1d_2}\right)^{n-1}\begin{cases} (b-\lambda_k-d_2)\sin\frac{n}{2}\theta_k - d_1\sin(\frac{n}{2}-1)\theta_k, \text{when } j=2l+1 \\ \sqrt{d_1d_2}\left[(b-\lambda_k-d_2)\sin\frac{n}{2}\theta_k - d_1\sin(\frac{n}{2}-1)\theta_k\right], \text{when } j=2l \end{cases}$$

when $\alpha=-\beta=-d_2$

$$u_1^{(k)} = \left(-\sqrt{d_1d_2}\right)^{n-1}\begin{cases} (b-\lambda_k+d_2)\sin\frac{n}{2}\theta_k + d_1\sin(\frac{n}{2}-1)\theta_k, \text{when } j=2l+1 \\ \sqrt{d_1d_2}\left[(b-\lambda_k+d_2)\sin\frac{n}{2}\theta_k + d_1\sin(\frac{n}{2}-1)\theta_k\right], \text{when } j=2l \end{cases}$$

when $\alpha=-\beta=d_2$. □

4. Case when $d_1d_2=0$**.** In this case, we have proved in S. Kouachi [6], the following.

PROPOSITION 4.1. *When* $d_1d_2=0$*,the eigenvalues* $\lambda_k(\sigma)$, $k=1,...,n$ *of the class of matrices* $A_n(\sigma)$ *on the form (1.1) are independent of the entries* $(a_i,\ c_i,\ i=$

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$1,..,n-1$*) and of* σ *provided that condition (2) is satisfied and their characteristic polynomials are given by*

$$\Delta_n = \begin{cases} \left(\xi^2 - d_2^2\right)^{\frac{n-1}{2}-1} (\xi - \alpha) \left(\xi^2 - \beta\xi - d_2^2\right), \text{when } n \text{ is odd;} \\ \\ \left(\xi^2 - d_2^2\right)^{\frac{n}{2}-1} \left(\xi^2 - (\alpha + \beta)\,\xi + \alpha\beta\right), \text{when } n \text{ is even,} \end{cases} \tag{19.a}$$

when $d_1 = 0$ *and*

$$\Delta_n = \begin{cases} \left(\xi^2 - d_1^2\right)^{\frac{n-1}{2}-1} (\xi - \beta) \left(\xi^2 - \alpha\xi - d_1^2\right), \text{when } n \text{ is odd;} \\ \\ \left(\xi^2 - d_1^2\right)^{\frac{n}{2}-2} \left(\xi^2 - \alpha\xi - d_1^2\right) \left(\xi^2 - \beta\xi - d_1^2\right), \text{when } n \text{ is even,} \end{cases} \tag{19.b}$$

when $d_2 = 0$*, where* $\xi = Y$ *is given by (3)*

An immediate consequence of this proposition is

PROPOSITION 4.2. *If* $d_1 d_2 = 0$*,the eigenvalues* $\lambda_k(\sigma)$ *,* $k = 1, ..., n$ *of the class of matrices* $A_n(\sigma)$ *on the form (1.1)are independent of the entries* $(a_i,\ c_i,\ i = 1,..,n-1)$ *and of* σ *provided that condition (2) is satisfied:*
1) When $\alpha = \beta = 0$*, they are reduced to three eigenvalues*

$$\{\ b \pm d_2,\ b\}$$

when $d_1 = 0$ *or when* n *is odd and* $d_2 = 0$

$$\{\ b \pm d_1,\ b\}$$

and only two eigenvalues

$$\{\ b \pm d_1\}$$

when n *is even and* $d_2 = 0$*.*
2) When $\alpha \neq 0$ *or* $\beta \neq 0$*, they are reduced to five eigenvalues*

$$:\left\{\ b \pm d_2,\ b - \alpha,\ b - \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 + 4d_2^2}\right\}$$

when n *is odd and* $d_1 = 0$*, five also*

$$\left\{b \pm d_1, b - \beta,\ b - \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 + 4d_1^2}\right\}$$

when n *is odd and* $d_2 = 0$*, four*

$$\{b \pm d_2, b - \alpha,\ b - \beta\}$$

when n *is even and* $d_1 = 0$ *and six eigenvalues*

$$\left\{b \pm d_1, b - \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 + 4d_1^2},\ b - \frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 + 4d_1^2}\right\}$$

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when n is even and $d_2 = 0$.
3) All the corresponding eigenvectors $u^{(k)}(\sigma) = \left(u_1^{(k)}(\sigma), ..., u_n^{(k)}(\sigma)\right)^t$, $k = 1, .., n$ are are simple and given by
3.1) When λ_k is simple

$$u_j^{(k)}(\sigma) = \nu_j(\sigma) \begin{cases} \begin{cases} \left[(b-\lambda_k)^2 - d_2^2\right]^{\frac{n-j}{2}}, \text{ when } j \text{ is odd,} \\ \left[(b-\lambda_k)^2 - d_2^2\right]^{\frac{n-j-1}{2}} (b-\lambda_k), \text{ when } j \text{ is even,} \end{cases} , n \text{ is odd,} \\ \begin{cases} \left[(b-\lambda_k)^2 - d_2^2\right]^{\frac{1+n-j}{2}}, \text{ when } j \text{ is odd,} \\ \left[(b-\lambda_k)^2 - d_2^2\right]^{\frac{n-j}{2}} (b-\lambda_k), \text{ when } j \text{ is even,} \end{cases} , n \text{ is even,} \end{cases} , \tag{20.a}$$

when $d_1 = 0$ and

$$u_j^{(k)}(\sigma) = \nu_j(\sigma) \begin{cases} \begin{cases} (b-\lambda_k)\left[(b-\lambda_k)^2 - d_1^2\right]^{\frac{n-j}{2}}, \text{ when } j \text{ is odd,} \\ \left[(b-\lambda_k)^2 - d_1^2\right]^{\frac{n-j-1}{2}+1}, \text{ when } j \text{ is even,} \end{cases} , n \text{ is odd,} \\ \begin{cases} (b-\lambda_k)\left[(b-\lambda_k)^2 - d_1^2\right]^{\frac{n-1-j}{2}}, \text{ when } j \text{ is odd,} \\ \left[(b-\lambda_k)^2 - d_1^2\right]^{\frac{n-j}{2}}, \text{ when } j \text{ is even,} \end{cases} , n \text{ is even,} \end{cases} \tag{20.b}$$

when $d_2 = 0$, $j = 2, ..., n$ and $k = 1, ..., n$, where

$$\nu_j(\sigma) = (-1)^{n-j} a_{\sigma_1} ... a_{\sigma_{j-1}}, j = 2, ..., n.$$

3.2) When λ_k is multiple, then all the components are zero except the last four ones at most and which we calculate directly.

Proof. The expressions of the eigenvalues are trivial by annulling the corresponding characteristic determinants. Following the same reasoning as the case $d_1 d_2 \neq 0$, by solving system (7), we get the expressions of the eigenvectors by formulas (13)

$$u_j^{(k)}(\sigma) = (-1)^{n-1} \nu_j(\sigma) u_1^{(k)} \left[\left[\frac{\Delta_{n-j}^{(k)}}{\Delta_{n-1}^{(k)}}\right]\right], \; j = 2, ..., n \text{ and } k = 1, ..., n,$$

where [[.]] denote the reduced fraction.

$$\Delta_{n-1}^{(k)} = \begin{cases} \left(\xi_k^2 - d_2^2\right)^{m-2} \left(\xi_k^2 - d_2^2\right) \left(\xi_k^2 - \beta\xi_k - d_2^2\right), \text{when } n = 2m+1 \text{ is odd,} \\ \left(\xi_k^2 - d_2^2\right)^{m-2} (\xi_k - \beta) \left(\xi_k^2 - d_2^2\right), \text{when } n = 2m \text{ is even,} \end{cases} ,$$

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when $d_1 = 0$ and

$$\Delta_{n-1}^{(k)} = \begin{cases} \left(\xi_k^2 - d_1^2\right)^{m-1} \left(\xi_k^2 - \beta\xi\right)_k, \text{when } n = 2m+1 \text{ is odd,} \\ \left(\xi_k^2 - d_1^2\right)^{m-2} \xi_k \left(\xi_k^2 - \beta\xi_k - d_1^2\right), \text{when } n = 2m \text{ is even,} \end{cases}$$

when $d_2 = 0$.

$$\Delta_{n-j}^{(k)} = \begin{cases} \begin{cases} \left(\xi_k^2 - d_2^2\right)^{\frac{n-j}{2}-2} \left(\xi_k^2 - d_2^2\right) \left(\xi_k^2 - \beta\xi_k - d_2^2\right), \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_2^2\right)^{\frac{n-j-1}{2}-1} \xi_k \left(\xi_k^2 - \beta\xi_k - d_2^2\right), \text{ when } j \text{ is even} \end{cases} n \text{ is odd,} \\ \begin{cases} \left(\xi_k^2 - d_2^2\right)^{\frac{n-j-1}{2}-1} \left(\xi_k - \beta\right) \left(\xi_k^2 - d_2^2\right), \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_2^2\right)^{\frac{n-j}{2}-1} \left(\xi_k^2 - \beta\xi_k\right), \text{ when } j \text{ is even,} \end{cases} n \text{ is even,} \end{cases}$$

when $d_1 = 0$ and

$$\Delta_{n-j}^{(k)} = \begin{cases} \begin{cases} \left(\xi_k^2 - d_1^2\right)^{\frac{n-j}{2}-1} \left(\xi_k^2 - \beta\xi_k\right), \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_1^2\right)^{\frac{n-j-1}{2}-1} \left(\xi_k - \beta\right) \left(\xi_k^2 - d_1^2\right), \text{ when } j \text{ is even} \end{cases} n \text{ is odd;} \\ \begin{cases} \left(\xi_k^2 - d_1^2\right)^{\frac{n-j-1}{2}-1} \xi_k \left(\xi_k^2 - \beta\xi_k - d_1^2\right), \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_1^2\right)^{\frac{n-j}{2}-2} \left(\xi_k^2 - d_1^2\right) \left(\xi_k^2 - \beta\xi_k - d_1^2\right), \text{ when } j \text{ is even} \end{cases} n \text{ is even,} \end{cases}$$

when $d_2 = 0$. Then, when $d_1 = 0$, we have

$$u_j^{(k)}(\sigma) = (-1)^{n-1} \nu_j(\sigma) \begin{cases} \begin{cases} \left(\xi_k^2 - d_2^2\right)^{\frac{1-j}{2}}, \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_2^2\right)^{\frac{-j}{2}} \xi_k, \text{ when } j \text{ is even} \end{cases} n \text{ is odd,} \\ \begin{cases} \left(\xi_k^2 - d_2^2\right)^{\frac{1-j}{2}}, \text{ when } j \text{ is odd,} \\ \left(\xi_k^2 - d_2^2\right)^{\frac{-j}{2}} \xi_k, \text{ when } j \text{ is even,} \end{cases} n \text{ is even.} \end{cases},$$

$j = 2, ..., n$ and $k = 1, ..., n$. By putting $j = n$, calculating $u_1^{(k)}$ according to $u_n^{(k)}(\sigma)$ and choosing

$$u_n^{(k)}(\sigma) = a_{\sigma_1} ... a_{\sigma_{n-1}},$$

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we get (20.a).
Following the same reasoning as in the case when $d_1 = 0$, we deduce (20.b). □

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