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LINEARIZATIONS OF POLYNOMIAL MATRICES WITH SYMMETRIES AND THEIR APPLICATIONS*

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Abstract. In an earlier paper by the present authors, a new family of companion forms associated with a regular polynomial matrix was presented, generalizing similar results by M. Fiedler who considered the scalar case. This family of companion forms preserves both the finite and infinite elementary divisor structure of the original polynomial matrix, thus all its members can be seen as linearizations of the corresponding polynomial matrix. In this note, its applications on polynomial matrices with symmetries, which appear in a number of engineering fields, are examined.

Key words. Polynomial matrix, Companion form, Linearization, Self-adjoint polynomial matrix.

AMS subject classifications. 15A21, 15A22, 15A23, 15A57.

1. Preliminaries. We consider polynomial matrices of the form

$$T(s) = T_n s^n + T_{n-1}s^{n-1} + ... + T_0, \tag{1.1}$$

with $T_i \in \mathbb{C}^{p\times p}$. A polynomial matrix $T(s)$ is said to be *regular* iff $\det T(s) \neq 0$ for almost every $s \in \mathbb{C}$. The associated with $T(s)$ matrix pencil

$$P(s) = sP_1 - P_0,$$

where

$$P_1 = \begin{bmatrix} T_n & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_p \end{bmatrix}, P_0 = \begin{bmatrix} -T_{n-1} & -T_{n-2} & \cdots & -T_0 \\ I_p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_p & 0 \end{bmatrix} \tag{1.2}$$

is known as the *first companion form* of $T(s)$. The first companion form is well known to be a linearization of the polynomial matrix $T(s)$ (see [5]), that is there exist unimodular polynomial matrices $U(s)$ and $V(s)$ such that

$$P(s) = U(s)diag\left\{T(s), I_{p(n-1)}\right\} V(s).$$

An immediate consequence of the above relation is that the first companion form has the same finite elementary divisors structure with $T(s)$. However, in [13], [10], this

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important property of the first companion form of $T(s)$ has been shown to hold also for the infinite elementary divisors structures of $P(s)$ and $T(s)$.

Motivated by the preservation of both finite and infinite elementary divisors structure, a notion of strict equivalence between a polynomial matrix and a pencil has been proposed in [13]. According to this definition, a polynomial matrix is said to be strictly equivalent to a matrix pencil iff they possess identical finite and infinite elementary divisors structure, which in the special case where both matrices are of degree one (i.e., pencils) reduces to the standard definition of [3].

Similar results hold for the *second companion form* of $T(s)$ defined by

$$\hat{P}(s) = sP_1 - \hat{P}_0,$$

where P_0 is defined in (1.2) and

$$\hat{P}_1 = \begin{bmatrix} -T_{n-1} & I_p & \cdots & 0 \\ -T_{n-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & I_p \\ -T_0 & 0 & \cdots & 0 \end{bmatrix}.$$

It can be easily seen that $\det T(s) = \det P(s) = \det \hat{P}(s)$, so the matrix pencils $P(s)$, $\hat{P}(s)$ are regular iff $T(s)$ is regular.

The new family of companion forms presented in [1] can be parametrized by products of elementary constant matrices, an idea appeared recently in [2] for the scalar case. Surprisingly, this new family contains apart form the first and second companion forms, many new ones, unnoticed in the subject's bibliography. Companion forms of polynomial matrices (or even scalar polynomials) are of particular interest in many research fields as a theoretical or computational tool. First order representations are in general easier to manipulate and provide better insight on the underlying problem. In view of the variety of forms arising from the proposed family of linearizations, one may choose particular ones that are better suited for specific applications (for instance when dealing with self-adjoint polynomial matrices [4], [5], [9], [6], [7] or the quadratic eigenvalue problem [12]).

The content is organized as follows: in section 2, we review the main results of [1]. In section 3, we present the application of a particular member of this family of linearizations to the special case of systems described by polynomial matrices with certain symmetries. Finally in section 4, we summarize our results and briefly discuss subjects for further research and applications.

2. A new family of companion forms. In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively and $\mathbb{K}^{p\times m}$ where $\mathbb{K}$ is a field, stands for the set of $p \times m$ matrices with elements in $\mathbb{K}$. The transpose (resp. conjugate transpose) of a matrix A will be denoted by $A^\top$ (resp. A^*), $\det A$ is the determinant and $\ker A$ is the right null-space or kernel of the matrix A. A standard assumption throughout the paper is the regularity of the polynomial matrix $T(s)$, i.e., $\det T(s) \neq 0$ for almost every $s \in \mathbb{C}$.

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Along the lines of [2], we define the matrices (notice that the indices are ordered reversely comparing to those in [2] and [1])

$$A_n = diag\{T_n, I_{p(n-1)}\}, \tag{2.1}$$

$$A_k = \begin{bmatrix} I_{p(n-k-1)} & 0 & \cdots \\ 0 & C_k & \ddots \\ \vdots & \ddots & I_{p(k-1)} \end{bmatrix},\ k = 1, 2, \ldots, n-1, \tag{2.2}$$

$$A_0 = diag\{I_{p(n-1)}, -T_0\}, \tag{2.3}$$

where

$$C_k = \begin{bmatrix} -T_k & I_p \\ I_p & 0 \end{bmatrix}. \tag{2.4}$$

The above defined sequence of matrices $A_i,\ i = 0, 1, 2, \ldots, n$ can be easily shown to provide an easy way to derive the first and second companion forms of the polynomial matrix $T(s)$.

Lemma 2.1. [1] *The first and second companion forms of $T(s)$ are given respectively by*

$$P(s) = sA_n - A_{n-1}A_{n-2} \ldots A_0, \tag{2.5}$$

$$\hat{P}(s) = sA_n - A_0 \ldots A_{n-2}A_{n-1}. \tag{2.6}$$

The following theorem will serve as the main tool for the construction of the new family of companion forms of $T(s)$.

Theorem 2.2. [1] *Let $P(s)$ be the first companion form of a regular polynomial matrix $T(s)$. Then for every possible permutation $(i_1, i_2, \ldots, i_n)$ of the n-tuple $(0, 2, \ldots, n-1)$ the matrix pencil $Q(s) = sA_n - A_{i_1}A_{i_2} \ldots A_{i_n}$ is strictly equivalent to $P(s)$, i.e., there exist non-singular constant matrices M and N such that*

$$P(s) = MQ(s)N, \tag{2.7}$$

where $A_i,\ i = 0, 1, 2, \ldots, n$ are defined in (2.1), (2.2) *and* (2.3).

The above theorem states that any matrix pencil of the form $Q(s) = sA_0 - A_{i_1}A_{i_2} \ldots A_{i_n}$ has identical finite and infinite elementary divisor structure with $T(s)$. Thus for any permutation $(i_1, i_2, \ldots, i_n)$ of the n-tuple $(0, 2, \ldots, n-1)$ the resulting companion matrices are by transitivity strictly equivalent amongst each other. Furthermore the companion forms arising from theorem 2.2 can be considered to be strictly equivalent to the polynomial matrix $T(s)$ in the sense of [13]. Notice, that the members of the new family of companion forms cannot in general be produced by permutational similarity transformations of $P(s)$ not even in the scalar case (see [2]).

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In view of the asymmetry in the distribution of A_i's in the constant and first order terms of $Q(s)$, it is natural to expect more freedom in the construction of companion forms. In this sense the following corollary is an improvement of theorem 2.2.

COROLLARY 2.3. [1] *Let $P(s)$ be the first companion form of a regular polynomial matrix $T(s)$. For any four ordered sets of indices $I_k = (i_{k,1}, i_{k,2}, \ldots, i_{k,n_k})$, $k = 1,2,3,4$ such that $I_i \cap I_j = \varnothing$ for $i \neq j$ and $\bigcup_{k=1}^{4} I_k = \{1,2,3,\ldots,n-1\}$ the matrix pencil*

$$R(s) = sA_{I_1}^{-1}A_nA_{I_2}^{-1} - A_{I_3}A_0A_{I_4},$$

is strictly equivalent to $P(s)$, where $A_{I_k} = A_{i_{k,1}}A_{i_{k,2}}\ldots A_{i_{k,n_k}}$ for $I_k \neq \varnothing$ and $A_{I_k} = I$ for $I_k = \varnothing$.

Notice that the inverses of A_k, $k = 1,2,\ldots,n-1$ have a particularly simple form, that is,

$$A_k^{-1} = \begin{bmatrix} I_{p(n-k-1)} & 0 & \cdots \\ 0 & C_k^{-1} & \ddots \\ \vdots & \ddots & I_{p(k-1)} \end{bmatrix},$$

with

$$C_k^{-1} = \begin{bmatrix} 0 & I_p \\ I_p & T_k \end{bmatrix}.$$

In view of this simple inversion formula, corollary 2.3 produces a broader class of companion forms than the one derived from theorem 2.2, which are strictly equivalent (in the sense of [13]) to the polynomial matrix $T(s)$.This is justified by the fact that the "middle" coefficients of $T(s)$ can be chosen to appear either on the constant or first-order term of the companion pencil $R(s)$.

The following example illustrates such a case.

EXAMPLE 2.4. Let $T(s) = T_3s^3 + T_2s^2 + T_1s + T_0$. We can choose to move the coefficients T_1, T_2 on any term of the companion matrix $R(s)$. For instance we can have T_2 on the first order term and T_1 on the constant term of $R(s)$, i.e.,

$$R(s) = sA_3A_2^{-1} - A_1A_0,$$

or

$$R(s) = s\begin{bmatrix} 0 & T_3 & 0 \\ I & T_2 & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & -T_1 & -T_0 \\ 0 & I & 0 \end{bmatrix}.$$

3. Applications to systems described by polynomial matrices with symmetries. We now focus on polynomial matrices of the form (1.1) where the coefficients are real and either symmetric or skew symmetric. We shall further assume that

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the leading coefficient matrix of $T(s)$ is non-singular, i.e., $\det(T_0) \neq 0$.We introduce a linearization of such a polynomial matrix of particular importance.

Let $T(s)$ be a polynomial matrix of degree n, with $\det T_0 \neq 0$. Then the companion form of $T(s)$,

$$R_s(s) = sA_{odd}^{-1} - A_{even}, \tag{3.1}$$

where

$$A_{even} = A_0 A_2 \ldots A_n^{-1}, A_{odd} = A_1 A_3 \ldots A_{n-1}, \text{ for } n \text{ even}$$

and

$$A_{even} = A_0 A_2 \ldots A_{n-1}, A_{odd} = A_1 A_3 \ldots A_n^{-1}, \text{ for } n \text{ odd},$$

is obviously a member of the family of linearizations introduced in corollary 2.3.

It easy to see that the above linearization of $T(s)$ has a particularly simple form as shown in the following example:

EXAMPLE 3.1. We illustrate the form of $R_s(s)$ for $n = 4$ and $n = 5$ respectively. For $n = 4$

$$R_s(s) = s\begin{bmatrix} 0 & I_p & & \\ I_p & T_3 & & \\ & & 0 & I_p \\ & & I_p & T_1 \end{bmatrix} - \begin{bmatrix} T_4^{-1} & & & \\ & -T_2 & I_p & \\ & I_p & 0 & \\ & & & -T_0 \end{bmatrix}$$

For $n = 5$

$$R_s(s) = s\begin{bmatrix} T_5 & & & & \\ & 0 & I_p & & \\ & I_p & T_3 & & \\ & & & 0 & I_p \\ & & & I_p & T_1 \end{bmatrix} - \begin{bmatrix} -T_4 & I_p & & & \\ I_p & 0 & & & \\ & & -T_2 & I_p & \\ & & I_p & 0 & \\ & & & & -T_0 \end{bmatrix}.$$

Obviously, the above linearization has the advantage of preserving the (skew) symmetric structure of the polynomial matrix $T(s)$, i.e., the resulting pencil has (skew) symmetric coefficients as well. This is a desirable feature in many applications where such polynomial matrices appear. The fact that any linearization preserves the (possible) special eigenstructure of the polynomial matrix, in general does not allow the use of special numerical methods exploiting the (skew) symmetric structure of the original coefficients. In the following we present two such cases.

3.1. Systems with symmetric coefficients. Consider the system described by the differential equation

$$\sum_{i=0}^{n} T_i \frac{d^i x}{dt^i} = Bu$$

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with $T_i^\top = T_i$. Typical applications of such models involve second order mechanical, vibrational, vibro-acoustics, fluid mechanics, constrained least-square and signal processing systems [12] of the form

$$M\ddot{x} + C\dot{x} + Kx = Bu,$$

where the M, C and K are symmetric matrices and possibly holding certain definiteness properties. The linearization of the associated polynomial matrix $T(s) = s^2M + sC + K$ in this case is given by (3.1) as follows

$$R_s(s) = s\begin{bmatrix} 0 & I \\ I & C \end{bmatrix} - \begin{bmatrix} M^{-1} & 0 \\ 0 & -K \end{bmatrix}.$$

Obviously, the coefficient matrices of the pencil $R_s(s)$ are symmetric too. In the special case of second order systems, there are other symmetry preserving linearizations known in the literature [12]. However there is not a general linearization method for matrices of degree more than two, having this appealing property. For instance, the numerical solution of vibration problems by the dynamic element method (see example 6 in [8]) requires the solution of cubic eigenvalue problem of the form

$$(\lambda^3F_3 + \lambda^2F_2 + \lambda F_1 + F_0)v = 0,$$

where $F_i = F_i^\intercal,\ i = 0, 1, 2, 3$. Our symmetric linearization in this case is given by (3.1)

$$R_s(s) = \lambda\begin{bmatrix} F_3 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & F_1 \end{bmatrix} - \begin{bmatrix} -F_2 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & -F_0 \end{bmatrix}.$$

3.2. Systems with alternating coefficients. Consider the polynomial $T(s)$ of the form (1.1), where now the coefficients T_i alternate between symmetric and skew symmetric [8], i.e.,

$$T_i^\intercal = (-1)^iT_i,\ \text{for } i = 0, 1, 2, ..., n \tag{3.2}$$

or

$$T_i^\intercal = (-1)^{i+1}T_i,\ \text{for } i = 0, 1, 2, ..., n. \tag{3.3}$$

Again the proposed matrix pencil (3.1) appears to be suitable for the linearization of polynomial matrices with alternating symmetry after a slight sign modification. Define

$$P_i = diag\left\{I_{(i-1)p}, -I_p, I_{(n-i)p}\right\}, 0 < i \leq n$$

$P_i \in \mathbb{R}^{np\times np}$ and

$$M_i = \prod_{j=0}^{\left\lfloor\frac{n-i}{4}\right\rfloor} P_{4j+i}.$$

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1. If (3.2) holds: the pencil $L(s)$ is also a strict equivalent linearization of $T(s)$ with the first order term being skew symmetric and the constant one being symmetric.
   (a) *n even*: $L(s) = M_2 R_s(s) M_3$.
   (b) *n odd*: $L(s) = M_3 R_s(s) M_4$.
2. If (3.3) holds: Similarly the following linearizations of $T(s)$ have their first order terms symmetric and the constant ones skew symmetric.
   (a) *n even*: $L(s) = M_3 R_s(s) M_4$.
   (b) *n odd*: $L(s) = M_2 R_s(s) M_3$.

Higher order systems of differential equations with alternating coefficients are of particular importance, since they can be used in the modelling of several mechanical systems and they are strongly related to the Hamiltonian eigenvalue problem (see examples 1,2 and 3 in [8]).

EXAMPLE 3.2. [8] Consider the mechanical system governed by the differential equation

$$M\ddot{x} + C\dot{x} + Kx = Bu,$$

where x and u are state and control variables. The computation of the optimal control u that minimizes the cost functional

$$\int_{t_0}^{t_1} (x^\intercal Q_0 x + \dot{x}^\intercal Q_1 \dot{x} + u^\intercal R u)\, dt$$

is associated with the eigenvalue problem

$$\lambda^2 \begin{bmatrix} M & 0 \\ -Q_1 & -M^\intercal \end{bmatrix} + \lambda \begin{bmatrix} C & 0 \\ 0 & C^\intercal \end{bmatrix} + \begin{bmatrix} K & -BR^{-1}B^\intercal \\ Q_0 & -K^\intercal \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0. \tag{3.4}$$

The coefficient matrices are from left to right Hamiltonian, skew Hamiltonian and again Hamiltonian. A matrix H is said to be Hamiltonian (skew Hamiltonian) iff $(JH)^\intercal = JH$ (respectively, $(JH)^\intercal = -JH$), where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Obviously $J^{-1} = J^\intercal = -J$. Premultiplying (3.4) by J, we obtain the equivalent eigenvalue problem

$$\lambda^2 \begin{bmatrix} Q_1 & M^\intercal \\ M & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -C^\intercal \\ C & 0 \end{bmatrix} + \begin{bmatrix} -Q_0 & K^\intercal \\ K & -BR^{-1}B^\intercal \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0,$$

where now the coefficient matrices are respectively symmetric, skew symmetric and again symmetric. In order to linearize the above problem using case 1a, we obtain the equivalent first order matrix pencil

$$\lambda \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & C^\intercal \\ 0 & -I & -C & 0 \end{bmatrix} - \begin{bmatrix} 0 & M^{-1} & 0 & 0 \\ M^{-\intercal} & -M^{-\intercal} Q_1 M^{-1} & 0 & 0 \\ 0 & 0 & -Q_0 & K^\intercal \\ 0 & 0 & K & -BR^{-1}B^\intercal \end{bmatrix}$$

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which has a skew symmetric first order coefficient matrix and a symmetric constant term. The preservation of the alternating symmetry of the original higher order problem is very important for computational purposes. The spectrum of the proposed first order pencil has the Hamiltonian structure, while additionally its coefficients have the desirable alternating symmetry. A similar approach using a different linearization and its significance in spectral computations, has been presented in [8].

4. Conclusions. In this paper we present a number of applications of the results appearing in [1], using a particular member of the proposed family of linearizations of a regular polynomial matrix. Throughout the variety of forms arising from this family, a particular one seems to be of special interest, since it preserves the symmetric or alternating symmetry structure of the underlying polynomial matrix. The present note aims to present only preliminary results regarding this new family of companion forms, leaving many theoretical and computational aspects to be the subject of further research.

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