Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

ELA

BOUNDED AND STABLY BOUNDED PALINDROMIC DIFFERENCE EQUATIONS OF FIRST ORDER*

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Abstract. Criteria for palindromic difference equations $A^\star x_i + Ax_{i+1} = 0$, where $\star$ stands for either transpose or conjugate transpose, to have bounded or stably bounded solutions are given in terms of the congruent equivalent classes of the matrix A. It is proved that the set of bounded palindromic difference equations is connected in the complex case, and has two connected components corresponding to the sign of the determinant of A in the real case. The connected components of the set of stably bounded palindromic difference equations are characterized.

Key words. Palindromic, Difference equations of first order, Bounded solutions, Stably bounded solutions, Matrix congruence, Canonical forms.

AMS subject classifications. 15A63, 39A99.

1. Introduction. We consider systems of first order difference equations with constant coefficients:

$$A_0x_i + A_1x_{i+1} = 0, \qquad i = 0, 1, \dots, \tag{1.1}$$

where A_0 and A_1 are fixed (real or complex) $n \times n$ matrices, and $\{x_i\}_{i=0}^{\infty}$ is a sequence of vectors to be found.

It will be assumed that A_1 is invertible. Then the solutions of (1.1) are easy to find:

$$x_i = (-A_1^{-1}A_0)^i x_0, \qquad i = 0, 1, \dots. \tag{1.2}$$

Of special interest for us will be situations where every solution sequence $\{x_i\}_{i=0}^{\infty}$ is bounded; in such case we say that the system (1.1) is *bounded*. Using the formula (1.2) and the Jordan form of the matrix $-A_1^{-1}A_0$, it is not difficult to obtain the following well-known criterion:

PROPOSITION 1.1. *The system* (1.1) *is bounded if and only if the eigenvalues of* $-A_1^{-1}A_0$ *have absolute value less than or equal to* 1, *and the partial multiplicities corresponding to the unimodular eigenvalues (if any) are all equal to* 1.

A symmetry assumption will be imposed on the difference equation (1.1). We consider three cases; here and in the sequel, X^T and X^* denote the transpose and the conjugate transpose, respectively, of a (real or complex) matrix X:

(I) $A_0 = A_1^*$, and the matrices are complex;
(II) $A_0 = A_1^T$, and the matrices are real;
(III) $A_0 = A_1^T$, and the matrices are complex.

*Received by the editors 4 September 2005. Accepted for publication 18 January 2006. Handling Editor: Harm Bart.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

The associated matrix polynomial $A_0 + \lambda A_1$ is then *palindromic*, of first degree. This terminology was introduced in [21], by analogy with linguistic palindromes. Bounded and stably bounded palindromic systems of difference equations of even orders for the case (I) have been studied in [8], see also [6], [7], using the matrix polynomial techniques of [9]. Recently, palindromic polynomials of arbitrary degree and their applications have been intensively studied, in particular in terms of linearizations, see [11], [13], [21], [22].

We suppose now that one of the symmetry conditions (I) - (III) holds. Thus, (1.1) can be rewritten in the form

$$A^{\star}x_i + Ax_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{F}^{n\times 1}, \tag{1.3}$$

where $\star$ stands for the conjugate transpose if (I) is assumed, and for the transpose if (II) or (III) is assumed, and where $\mathbb{F} = \mathbb{C}$ (the complex field) if (I) or (III) is assumed, and $\mathbb{F} = \mathbb{R}$ (the real field) if (II) is assumed. Denote $U := A^{-1}A^{\star}$.

We say that the difference equation (1.3) with invertible A is *stably bounded* if there exists $\varepsilon > 0$ such that every solution of any difference equation

$$\widetilde{A}^{\star}x_i + \widetilde{A}x_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{F}^{n\times 1}, \tag{1.4}$$

is bounded provided $\|\widetilde{A} - A\| < \varepsilon$ and $\widetilde{A} \in \mathbb{F}^{n\times n}$. In particular, every solution of the equation (1.3) is bounded.

In this paper we give criteria for boundedness and stable boundedness of (1.3) in terms of the congruence equivalence class of the matrix A, and describe the connected components of the sets of bounded and stably bounded systems. It turns out that the set of bounded systems is connected in the cases of symmetries (I) and (III), and it has two connected components, distinguished by the sign of the determinant of A, under the symmetry (II). In contrast, the set of stably bounded systems, which turns out to be an open set in $\mathbb{F}^{n\times n}$, has a rather involved description of components under the symmetries (I) and (II); there are no nontrivial stably bounded systems under the symmetry (II).

The problems of describing connected components of stably bounded systems, mainly in the context of differential Hamiltonian equations, have a long history, starting with the 1950's [5], [16], [23]; later works include [3], [14], [24]. Some basic parts of this material are exposed in [6], [7]. It is interesting to note that in the case of differential equations as well, the set of stably bounded systems typically has a complicated characterization of components, whereas the set of bounded systems is typically connected.

Besides the introduction, the paper consists of 6 sections. The symmetry (I) is studied in Sections 2 and 3. Preliminary material, largely known, on canonical forms of real matrices under congruence and on Jordan structures of matrices of the form $A^{-1}A^T$, is presented in Section 4. It is used in Sections 5 and 6 to study bounded and stably bounded systems and their connected components under symmetry (II). Finally, in Section 7 palindromic systems with symmetry (III) are considered.

We conclude the introduction with several observations that serve as a starting point of our investigation. First, a remark concerning eigenvalues of matrices of the form $A^{-1}A^{\star}$:

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REMARK 1.2. Let $A \in \mathbb{F}^{n\times n}$ be invertible. If λ is an eigenvalue of $U := A^{-1}A^{\star}$, then so is λ^{-1} (if (II) or (III) holds) or $\overline{\lambda^{-1}}$ (if (I) holds).

This is a general fact valid also for palindromic polynomials of arbitrary degree [21].

PROPOSITION 1.3. *Assume that A is invertible. Then* (1.3) *is bounded if and only if the matrix $U := A^{-1}A^{\star}$ is diagonalizable (over the complex field) with all eigenvalues on the unit circle.*

For the proof use Remark 1.2 together with Proposition 1.1.

PROPOSITION 1.4. *The following statements are equivalent for the system* (1.3) *with an invertible matrix A:*

(1) (1.3) *has a geometrically growing solution $\{x_i\}_{i=0}^{\infty}$, i.e.,*

$$\|x_i\| \geq Kr^i, \qquad i = 0, 1, \ldots,$$

where the constants $K > 0$ and $r > 1$ are independent of i;

(2) *the matrix $U := A^{-1}A^{\star}$ has a nonunimodular eigenvalue;*

(3) *the matrix U has an eigenvalue of absolute value larger than* 1*;*

(4) *the matrix U has an eigenvalue of absolute value smaller than* 1.

The equivalence of (1) and (3) is well known and follows easily from formula (1.2) and the Jordan structure of U; the equivalence of (2), (3), and (4) follows from Remark 1.2.

REMARK 1.5. The boundedness and stable boundedness properties of equation (1.3), as well as the property of having a geometrically growing solution, are invariant under $\star$-congruence transformations of the matrix A. Indeed, if $B = S^{\star}AS$ for some invertible S, then the difference equation

$$B^{\star}y_i + By_{i+1} = 0, \quad i = 0, 1, \ldots,$$

is reduced to (1.3) upon substitution $y_i = S^{-1}x_i$ $(i = 0, 1, 2, \ldots)$.

The following notation will be used throughout: $(\cdot,\cdot)$ stands for the standard inner product in the column vector space $\mathbb{F}^{n\times 1}$, where $\mathbb{F} = \mathbb{C}$, the complex field, or $\mathbb{F} = \mathbb{R}$, the real field. The set of eigenvalues of a matrix X, including complex eigenvalues for real matrices, is denoted by $\sigma(X)$. I_m or I (with the size understood) stands for the $m \times m$ identity matrix. The $m \times m$ upper triangular Jordan block with the (real or complex) eigenvalue a is denoted $J_m(a)$. We use the notation $J_{2m}(a \pm ib)$ or $J_{2m}(a+ib, a-ib)$ to denote the $2m \times 2m$ almost upper triangular real Jordan block with a pair of nonreal complex conjugate eigenvalues $a \pm ib$; here $a, b \in \mathbb{R}$, $b \neq 0$:

$$J_{2m}(a \pm ib) = J_{2m}(a+ib, a-ib) = \begin{bmatrix} T(a,b) & I_2 & 0 & \cdots & 0 & 0 \\ 0 & T(a,b) & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T(a,b) & I_2 \\ 0 & 0 & 0 & \cdots & 0 & T(a,b) \end{bmatrix},$$

where

$$T(a,b) := \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
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2. Symmetry (I). *In this section and in the next one we assume symmetry* (I), i.e., the matrices and vectors are complex and * stands for the conjugate transpose. Recall that two matrices $X, Y \in \mathbb{C}^{n\times n}$ are said to be *congruent* if $X = S^*YS$ for some invertible $S \in \mathbb{C}^{n\times n}$.

In this section we describe bounded and stably bounded systems (1.3) in various ways.

We begin with preliminary facts and remarks needed for the proof of the main theorems in this section, as well as for subsequent results. In view of formula (1.2), Jordan forms of matrices of the form $A^{-1}A^*$ will play a role. These forms are known; see [1], [4], and also [27], where the corresponding problems are addressed in a more general context of fields with involutions.

By Remark 1.5, we may replace the given matrix A by a canonical form obtained from A using a congruence transformation. Such a canonical form is presented next.

PROPOSITION 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be invertible. Then there exist an invertible $S \in \mathbb{C}^{n\times n}$ such that S^*AS has the following form:*

$$
\begin{aligned}
&\delta_1\,(F_{k_1} + iG_{k_1}) \oplus \cdots \oplus \delta_r\,(F_{k_r} + iG_{k_r})\\
&\oplus \eta_1\,((i+\alpha_1)\,F_{\ell_1} + G_{\ell_1}) \oplus \cdots \oplus \eta_q\,\big((i+\alpha_q)\,F_{\ell_q} + G_{\ell_q}\big)\\
&\oplus \begin{bmatrix} 0 & (i+\beta_1)F_{m_1} + G_{m_1} \\ (i+\overline{\beta}_1)F_{m_1} + G_{m_1} & 0 \end{bmatrix}\\
&\oplus \cdots \oplus \begin{bmatrix} 0 & (i+\beta_s)F_{m_s} + G_{m_s} \\ (i+\overline{\beta}_s)F_{m_s} + G_{m_s} & 0 \end{bmatrix}.
\end{aligned} \tag{2.1}
$$

Here, $k_1 \le \cdots \le k_r$, $\ell_1, \cdots, \ell_q$, and $m_1, \cdots, m_q$ are positive integers, α_j are real numbers, β_j are complex nonreal numbers different from $\pm i$, $\delta_1, \ldots, \delta_r, \eta_1, \ldots, \eta_q$ are signs, each equal to $+1$ or -1, and F_m and G_m are $m \times m$ matrices given by

$$
F_m = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \text{and} \quad G_m = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2}
$$

Moreover, the form (2.1) *is unique for a given A, up to a permutation of blocks.*

For the proof see [17, Corollary 6.4].

Other canonical forms for congruence are available, and the literature on this topic in the more general context of sesquilinear forms over commutative fields is quite extensive; we mention here only [12], [25].

Next, a straightforward calculation shows the following fact:

REMARK 2.2. Assume A is in the form (2.1). Then the matrix $U = A^{-1}A^*$ is diagonalizable with all eigenvalues on the unit circle if and only if all k_i's and all ℓ_j's are equal to 1 and the blocks

$$
\begin{bmatrix} 0 & (i+\beta_p)F_{m_p} + G_{m_p} \\ (i+\overline{\beta}_p)F_{m_p} + G_{m_p} & 0 \end{bmatrix}
$$

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A publication of the International Linear Algebra Society
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are absent.

It will be convenient to introduce the following notation for an invertible matrix A:

$$\Sigma(A) := \{z \in \mathbb{C} \, : \, \overline{z}A + zA^* \quad \text{is not invertible}\}.$$

Clearly,

$$\mathbb{C} \setminus \Sigma(A) = \{z \in \mathbb{C} \setminus \{0\} \, : \, -\frac{\overline{z}}{z} \notin \sigma(A^{-1}A^*)\}.$$

In particular, $\mathbb{C} \setminus \Sigma(A)$ is nonempty.

In what follows we use the notation $\nu_+(X)$, resp., $\nu_-(X)$, to denote the number of positive, resp., negative, eigenvalues counted with their multiplicities, of an Hermitian matrix X. For future use we note that for A given by (2.4) we have

$$U := A^{-1}A^* = I_r \oplus \operatorname{diag}\left((i+\alpha_1)^{-1}(-i+\alpha_1), \ldots, (i+\alpha_q)^{-1}(-i+\alpha_q)\right), \tag{2.3}$$

and the number of negative and positive squares $\nu_+(Q_\lambda(z))$ and $\nu_-(Q_\lambda(z))$ of the quadratic form

$$Q_\lambda(z) := ((\overline{z}A + zA^*)x, x), \qquad x \in \operatorname{Ker}(U - \lambda I), \quad \lambda \in \sigma(U),$$

where $z \in \mathbb{C} \setminus \Sigma(A)$, is described in the following remark.

Remark 2.3. (a) Assume

$$\lambda = (i + \alpha_{j_0})^{-1}(-i + \alpha_{j_0}) \neq 1.$$

Then, if the real part of $z(-i + \alpha_{j_0})$ is positive, then $\nu_+(Q_\lambda(z))$ (resp., $\nu_-(Q_\lambda(z))$) is equal to the number of indices j, $(1 \leq j \leq q)$, such that

$$\lambda = (i + \alpha_j)^{-1}(-i + \alpha_j) \quad \text{and} \quad \eta_j = 1 \quad (\text{resp., } \eta_j = -1).$$

If the real part of $z(-i + \alpha_{j_0})$ is negative, then $\nu_+(Q_\lambda(z))$ (resp., $\nu_-(Q_\lambda(z))$) is equal to the number of indices j, $(1 \leq j \leq q)$, such that

$$\lambda = (i + \alpha_j)^{-1}(-i + \alpha_j) \quad \text{and} \quad \eta_j = -1 \quad (\text{resp., } \eta_j = 1).$$

(b) Assume $\lambda = 1$. If the real part of z is positive, then $\nu_+(Q_1(z))$ (resp., $\nu_-(Q_1(z))$) is equal to the number of indices j, $(1 \leq j \leq r)$, such that $\delta_j = 1$ (resp., $\delta_j = -1$), and if the real part of z is negative, then $\nu_+(Q_1(z))$ (resp., $\nu_-(Q_1(z))$) is equal to the number of indices j, $(1 \leq j \leq r)$, such that $\delta_j = -1$ (resp., $\delta_j = 1$).

We need one more piece of preparation. A matrix is said to be *u-diagonalizable* if it is similar to a diagonal matrix with unimodular entries on the diagonal. Let $H \in \mathbb{C}^{n \times n}$ be an invertible Hermitian matrix. An H-unitary matrix U is called *stably u-diagonalizable* if there exists $\varepsilon > 0$ such that every G-unitary matrix V is u-diagonalizable as soon as $G \in \mathbb{C}^{n \times n}$ is Hermitian, V is G-unitary, and the inequality

$$\|G - H\| + \|V - U\| < \varepsilon$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

holds. (This terminology is borrowed from [6], [7], [8].)

THEOREM 2.4. *Let U be an H-unitary matrix. Then U is stably u-diagonalizable if and only if the quadratic form (Hx, x) is either positive definite or negative definite on* $\mathrm{Ker}\,(\lambda_0 I - U)$*, for every $\lambda_0 \in \sigma(U)$.*

See [7, Chapter 9] for the proof of Theorem 2.4. Note that the definiteness of the quadratic form (Hx, x) on $\mathrm{Ker}\,(\lambda_0 I - U)$, where $\lambda_0 \in \sigma(U)$ and U is H-unitary, implies that $|\lambda_0| = 1$ and the partial multiplicities of U corresponding to λ_0 are all equal to 1.

We now are ready to state and prove the main results of this section.

THEOREM 2.5. *The following statements are equivalent for an invertible matrix $A \in \mathbb{C}^{n\times n}$, where $U = A^{-1}A^*$:*

(a) *The equation* (1.3) *is bounded.*

(b) *For every $\lambda \in \sigma(U)$ and every $z \in \mathbb{C} \setminus \Sigma(A)$, the quadratic form $((\overline{z}A + zA^*)x, x)$ is nondegenerate on* $\mathrm{Ker}\,(U - \lambda I)$*, i.e., if $y \in \mathrm{Ker}\,(U - \lambda I)$ is such that $((\overline{z}A + zA^*)x, y) = 0$ for all $x \in \mathrm{Ker}\,(U - \lambda I)$, then necessarily $y = 0$.*

(c) *For every $\lambda \in \sigma(U)$ and some $z \in \mathbb{C} \setminus \Sigma(A)$, the quadratic form $((\overline{z}A + zA^*)x, x)$ is nondegenerate on* $\mathrm{Ker}\,(U - \lambda I)$.

(d) *A is congruent to a diagonal matrix of the form*

$$\text{(2.4)}\quad \mathrm{diag}\,(\delta_1, \ldots, \delta_r) \oplus \mathrm{diag}\,(\eta_1(i + \alpha_1), \ldots, \eta_q(i + \alpha_q)), \quad r + q = n,$$

where r, q are nonnegative integers, α_j are real numbers, and

$$\delta_1, \ldots, \delta_r, \quad \eta_1, \ldots, \eta_q$$

are signs, each equal to $+1$ or -1.

Proof. Using the form (2.1) we show the equivalence of (a) and (d). To this end, combine Remark 2.2 and Proposition 1.3.

Next, note the equality

$$\text{(2.5)}\qquad U^*(\overline{z}A + zA^*)U = \overline{z}A + zA^*, \qquad z \in \mathbb{C}, \quad U = A^{-1}A^*,$$

which can be easily verified. (The equality (2.5) has been observed and used in the literature, see [4], for example.) Thus, for $z \in \mathbb{C} \setminus \Sigma(A)$, the matrix U is unitary in the indefinite inner product

$$[x, y] := ((\overline{z}A + zA^*)x, y), \qquad x, y \in \mathbb{C}^{n\times 1},$$

defined by the Hermitian invertible matrix $\overline{z}A+zA^*$, or in other words, U is $\overline{z}A+zA^*$-unitary. Now the equivalence of (b), (c), and the u-diagonalizability of U follows from the general theory of H-unitary matrices, see [6], [7, Chapter 5]. It remains to use Proposition 1.3 to complete the proof of Theorem 2.5. ◻

THEOREM 2.6. *The following statements are equivalent for an invertible matrix $A \in \mathbb{C}^{n\times n}$, where $U = A^{-1}A^*$:*

(a) *The system* (1.3) *is stably bounded.*

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A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

(b) *There exists $\varepsilon > 0$ such that every system*

$$\widetilde{A}^* x_i + \widetilde{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, x_i \in \mathbb{C}^{n\times 1},$$

with $\|\widetilde{A} - A\| < \varepsilon$ is stably bounded.

(c) *For every $\lambda \in \sigma(U)$ and every $z \in \mathbb{C} \setminus \Sigma(A)$, the condition*

$$x \in \operatorname{Ker}(U - \lambda I) \setminus \{0\} \quad \Longrightarrow \quad ((\overline{z}A + zA^*)x, x) \neq 0 \tag{2.6}$$

holds.

(d) *For every $\lambda \in \sigma(U)$ and some $z \in \mathbb{C} \setminus \Sigma(A)$, the condition* (2.6) *holds.*

(e) *A is congruent to a diagonal matrix of the form*

$$(\pm I_r) \oplus \operatorname{diag}(\eta_1(i + \alpha_1), \ldots, \eta_q(i + \alpha_q)), \quad r + q = n, \tag{2.7}$$

where r, q are nonnegative integers, α_j are real numbers, and $\eta_1, \ldots, \eta_q$ are signs, each equal to $+1$ or -1, subject to the condition that $\alpha_{j_1} = \alpha_{j_2}$ implies $\eta_{j_1} = \eta_{j_2}$.

Proof. (c) $\Longrightarrow$ (d) is trivial. (d) $\Longrightarrow$ (a) follows from Theorem 2.4, using the observation that U is $\overline{z}A+zA^*$-unitary, $z \in \mathbb{C}\setminus\Sigma(A)$. (a) $\Longrightarrow$ (b) follows easily arguing by contradiction. Indeed, if (a) holds, but (b) does not, then in every neighborhood of A there is a matrix $\widehat{A}$ such that the system

$$\widehat{A}^* x_i + \widehat{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{C}^{n\times 1},$$

is not stably bounded, which in turn means that for every such $\widehat{A}$ there exists a $\widehat{B}$ as close as we wish to $\widehat{A}$ for which the corresponding system is not bounded. This leads to a contradiction with (a).

Next, we prove (b) $\Longrightarrow$ (e). We assume arguing by contradiction that (e) does not hold, and we will show that (1.3) is not stably bounded. In other words, we will construct matrices $\widetilde{A}$ by arbitrarily small perturbations of A such that not all solution sequences of

$$\widetilde{A}^* x_i + \widetilde{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, \tag{2.8}$$

are bounded. By Remark 2.2, we may assume that

$$A = \operatorname{diag}(\delta_1, \ldots, \delta_r) \oplus \operatorname{diag}(\eta_1(i + \alpha_1), \ldots, \eta_q(i + \alpha_q)),$$

where the parameters η_j and δ_j are signs ± 1, and the numbers α_j are real. Since (e) does not hold, at least one of the following two situations happens: (1) not all signs $\delta_1, \ldots, \delta_r$ are the same (if 1 is an eigenvalue of U), or (2) there is $\alpha \in \mathbb{R}$ such that $\lambda = (i + \alpha)^{-1}(-i + \alpha)$ is an eigenvalue of U and not all signs η_j such that $\alpha_j = \alpha$ are the same. Therefore, the construction of $\widetilde{A}$ boils down to consideration of the following cases:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} i+\alpha & 0 \\ 0 & -i-\alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}. \tag{2.9}$$

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If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we let

$$\widetilde{A} = \begin{bmatrix} 1 & \varepsilon \\ -\varepsilon & -1 \end{bmatrix},$$

where $\varepsilon > 0$ is small. Then

$$\widetilde{U} := \widetilde{A}^{-1}\widetilde{A}^* = \frac{1}{-1+\varepsilon^2} \begin{bmatrix} -1-\varepsilon^2 & 2\varepsilon \\ 2\varepsilon & -1-\varepsilon^2 \end{bmatrix}$$

has eigenvalues $(1 \pm \varepsilon)^2/(1-\varepsilon^2)$ which are not unimodular. If $A = \begin{bmatrix} i+\alpha & 0 \\ 0 & -i-\alpha \end{bmatrix}$ with $\alpha \neq 0$, we similarly let

$$\widetilde{A} = \begin{bmatrix} i+\alpha & \varepsilon \\ -\varepsilon & -i-\alpha \end{bmatrix}, \quad \varepsilon > 0 \quad \text{small.}$$

Then

$$\widetilde{U} := \widetilde{A}^{-1}\widetilde{A}^* = \frac{1}{-(\alpha+i)^2+\varepsilon^2} \begin{bmatrix} -1-\alpha^2-\varepsilon^2 & 2\alpha\varepsilon \\ 2\alpha\varepsilon & -1-\alpha^2-\varepsilon^2 \end{bmatrix},$$

which has non-unimodular eigenvalues $(1+(\alpha\pm\varepsilon)^2)/((\alpha+i)^2-\varepsilon^2)$. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, we let

$$\widetilde{A} = \begin{bmatrix} i & \varepsilon \\ \varepsilon & -i \end{bmatrix}, \quad \varepsilon > 0 \quad \text{small,}$$

resulting in the matrix

$$\widetilde{U} = \frac{1}{1-\varepsilon^2} \begin{bmatrix} -1-\varepsilon^2 & -2i\varepsilon \\ 2i\varepsilon & -1-\varepsilon^2 \end{bmatrix}$$

with real non-unimodular eigenvalues $-(1 \pm \varepsilon)^2/(1-\varepsilon^2)$.

For the proof of (e) $\Longrightarrow$ (c), observe that if (e) holds then by Remark 2.3 the quadratic form $Q_\lambda(z)$ is definite for every $\lambda \in \sigma(U)$ and every $z \in \mathbb{C} \setminus \Sigma(A)$, and (c) follows. ◻

We note also the following fact:

THEOREM 2.7. *If* (1.3) *is bounded but not stably bounded, then for every* $\varepsilon > 0$ *there exists a system*

$$\widetilde{A}^* x_i + \widetilde{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, x_i \in \mathbb{C}^{n \times 1}, \tag{2.10}$$

with $\|\widetilde{A} - A\| < \varepsilon$ *and such that* (2.10) *has a geometrically growing solution.*

Proof. Just repeat the arguments of the proof of (b) $\Longrightarrow$ (e) in Theorem 2.6. ◻

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3. Connected Components of Stably Bounded Difference Equations. In this section we continue to study the system of first order difference equations

$$A^*x_i + Ax_{i+1} = 0, \qquad i = 0, 1, \ldots, \tag{3.1}$$

where $A \in \mathbb{C}^{n\times n}$ is an invertible matrix.

We now consider the problem of describing the connected components of the set of bounded systems (3.1) and of the set of stably bounded systems (3.1), in the natural topology of matrices. Thus, the open neighborhoods of a system (3.1) have the form

$$\left\{ \{B^*x_i + Bx_{i+1} = 0\}_{i=0}^{\infty} \quad : \quad B \in \mathbb{C}^{n\times n} \text{ and } \quad \|B - A\| < \varepsilon \right\},$$

where $\varepsilon > 0$ is arbitrary but fixed for every neighborhood. The connected components of bounded, resp., stably bounded, systems (3.1) correspond exactly to the connected components of the set of all matrices A that satisfy the conditions of Theorem 2.5, resp., Theorem 2.6.

THEOREM 3.1. *The set of systems* (3.1) *with bounded solutions is arcwise connected.*

Proof. We need to show that the set Ω_n of matrices $A \in \mathbb{C}^{n\times n}$ which are congruent to a matrix of the form (2.4) is connected. Since the group of invertible complex $n \times n$ matrices is connected, it suffices to show that any two matrices of the form (2.4) are connected in Ω_n. Replacing each δ_j in (2.4) by $\delta_j(xi + 1)$, $0 \le x \le 1$, we connect a matrix in the form (2.4) to a matrix in the same form but with $r = 0$ (i.e., the terms $\delta_1, \ldots, \delta_r$ are absent). Next, consider a matrix

$$B := \mathrm{diag}\,(\eta_1(i + \alpha_1), \ldots, \eta_n(i + \alpha_n)), \quad \eta_j \in \{1, -1\}, \quad \alpha_j \in \mathbb{R}.$$

Letting $\alpha_j \longrightarrow +\infty$ if $\eta_j = 1$, $\alpha_j \longrightarrow -\infty$ if $\eta_j = -1$, and scaling

$$B \quad \longrightarrow \quad \mathrm{diag}\,(\sqrt{|\alpha_1|^{-1}}, \ldots, \sqrt{|\alpha_n|^{-1}})B\mathrm{diag}\,(\sqrt{|\alpha_1|^{-1}}, \ldots, \sqrt{|\alpha_n|^{-1}}),$$

we see that $B \longrightarrow I$, and throughout this process all matrices belong to Ω_n. ☐

The situation with connected components of stably bounded systems is more involved. We need some notation to describe the result here. Let SBD_n be the set of all $n \times n$ matrices A that are congruent to a matrix of the form (2.7). Consider the set Ξ_n of all ordered tuples of the form

$$(k; m_1, m_2, \cdots, m_{2\beta}), \tag{3.2}$$

where $1 \le k \le n$ is an integer, and $m_1, \ldots, m_{2\beta}$ are positive integers that sum up to $n - k$. Here, β is a nonnegative integer; if $\beta = 0$, the terms $m_1, \ldots, m_{2\beta}$ do not appear in (3.2). For example, if $n = 5$, then Ξ_5 consists of the following 8 elements:

$$(5); (3; 1, 1); (2; 2, 1); (2; 1, 2); (1; 3, 1); (1; 2, 2); (1; 1, 3); (1; 1, 1, 1, 1).$$

With every ordered tuple

$$\tau := (k; m_1, m_2, \cdots, m_{2\beta}) \in \Xi_n, \tag{3.3}$$

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A publication of the International Linear Algebra Society
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we associate a subset

$$SBD_n(\tau) := SBD_n(k; m_1, m_2, \cdots, m_{2\beta}) \subseteq SBD_n$$

constructed as follows: A matrix $A \in \mathbb{C}^{n\times n}$ belongs to $SBD_n(\tau)$ if and only if either A or $-A$ is congruent to a matrix of the form

$$\begin{aligned} I_k \oplus \left(\oplus_{j=1}^{m_1}(i+\alpha_{1,j})\right) \oplus \left(-\oplus_{j=1}^{m_2}(i+\alpha_{2,j})\right) \oplus \cdots \\ \oplus \left(\oplus_{j=1}^{m_{2\beta-1}}(i+\alpha_{2\beta-1,j})\right) \oplus \left(-\oplus_{j=1}^{m_{2\beta}}(i+\alpha_{2\beta,j})\right), \end{aligned} \tag{3.4}$$

where

$$\alpha_{1,1} \le \cdots \le \alpha_{1,m_1} < \alpha_{2,1} \le \cdots \le \alpha_{2,m_2} < \cdots < \alpha_{2\beta,1} \le \cdots \le \alpha_{2\beta,m_{2\beta}}, \tag{3.5}$$

or to a matrix of the form (3.4) in which I_k is replaced by any one matrix

$$I_{k_1} \oplus \left(-\oplus_{j=1}^{k_3}(i+\kappa_j)\right) \oplus \left(\oplus_{j=1}^{k_2}(i+\gamma_j)\right), \tag{3.6}$$

where k_1, k_2, k_3 are nonnegative integers that sum up to k and

$$\kappa_1 \le \cdots \le \kappa_{k_3} < \alpha_{1,1}, \quad \alpha_{2\beta,m_{2\beta}} < \gamma_1 \le \cdots \le \gamma_{k_2}. \tag{3.7}$$

In these formulas, k, β, and $m_1, \ldots, m_{2\beta}$ are fixed by (3.3), whereas $\alpha_{\ell,j}$, k_1, k_2, k_3, γ_j, and κ_j are variable subject only to the specified restrictions.

In connection with this definition we remark that it will be shown that matrices of the form I_n and $I_{k_1} \oplus (\eta \oplus_{j=1}^{k_3}(i+\kappa_j))$, where $\eta = \pm 1$ and $k_1 + k_3 = n$, are in the same connected component of the set of stably bounded systems (3.1) if and only if $\eta = -1$.

We are now in a position to describe the connected components of the set of stably bounded systems:

THEOREM 3.2. *The classes $SBD_n(\tau)$, $\tau \in \Xi_n$, are the connected components of the set of stably bounded systems of difference equations* (3.1).

Note that since the set of stably bounded systems is open in $\mathbb{C}^{n\times n}$, its connected components coincide with its arcwise connected components.

Thus, the set of 5×5 stably bounded systems (3.1) consists of exactly 8 connected components.

The rest of this section is devoted to the proof of Theorem 3.2. We start with some preliminaries. First, we note the following fact:

PROPOSITION 3.3. *If $A \in SBD_n$, then $\mu A \in SBD_n$ for every $\mu \in \mathbb{C}\setminus\{0\}$.*

The proof follows easily from (e) $\Longleftrightarrow$ (c) in Theorem 2.6.

In particular, A and $-A$ belong to the same connected component of SBD_n.

Next, we prove connectedness of each class $SBD_n(\tau)$:

PROPOSITION 3.4. *For a fixed $\tau \in \Xi_n$, the set $SBD_n(\tau)$ is connected.*

Proof. Let τ be given by (3.3). Since the set of all invertible complex $n\times n$ matrices is connected, and in view of Proposition 3.3, all what we need to show is that the set of matrices of the form (3.4) is connected, and the set of matrices obtained from the

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A publication of the International Linear Algebra Society
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form (3.4) by replacing I_k with any one of the matrices (3.6) subject to $k_1+k_2+k_3=k$ and inequalities (3.7), is connected within $SBD_n(\tau)$ to a matrix of the form (3.4).

The connectivity of the set of matrices of the form (3.4) subject to (3.5) with fixed parameters $m_1, m_2, \ldots, m_{2\beta}$ is easy: If B is a matrix of the form (3.4), and B' is another matrix of the same form but perhaps with different parameters $\alpha'_{\ell,j}$ then let $B(t)$ be the matrix of the form (3.4) with the parameters $t\alpha_{\ell,j}+(1-t)\alpha'_{\ell,j}$, for $0 \le t \le 1$. Clearly, $B(t)$ connects B and B' within the set of matrices of the form (3.4).

Furthermore, consider the matrix

$$\begin{aligned} C := I_{k_1} &\oplus \left(\oplus_{j=1}^{k_2}(i+\gamma_j)\right) \oplus \left(-\oplus_{j=1}^{k_3}(i+\kappa_j)\right) \\ &\oplus \left(\oplus_{j=1}^{m_1}(i+\alpha_{1,j})\right) \oplus \left(-\oplus_{j=1}^{m_2}(i+\alpha_{2,j})\right) \oplus \cdots \\ &\oplus \left(\oplus_{j=1}^{m_{2\beta-1}}(i+\alpha_{2\beta-1,j})\right) \oplus \left(-\oplus_{j=1}^{m_{2\beta}}(i+\alpha_{2\beta,j})\right), \end{aligned}$$

subject to $k_1+k_2+k_3=k$, $k_1>0$, $k_2 \ge 0$, $k_3 \ge 0$, (3.7), and (3.5). We are going to prove that C is connected within $SBD_n(\tau)$ to a matrix of the form (3.4). Without loss of generality we may assume that

$$\kappa_j < 0 \quad \text{for } j=1,2,\ldots,k_3 \quad \text{and} \quad \gamma_\ell > 0 \quad \text{for } \ell=1,2,\ldots,k_1. \tag{3.8}$$

Indeed, we may replace κ_j with $\kappa_j - t$, and γ_ℓ with $\gamma_\ell + t$, for $0 \le t \le t_0$, where t_0 is sufficiently large. This transformation connects C within $SBD_n(\tau)$ with a matrix of the same form as C has, but for which (3.8) is satisfied. Assuming (3.8) holds true, we let

$$\begin{aligned} C(t) := I_{k_1} &\oplus \left(\oplus_{j=1}^{k_2}\left(\frac{i}{t}+\gamma_j\right)\right) \oplus \left(-\oplus_{j=1}^{k_3}\left(\frac{i}{t}+\kappa_j\right)\right) \\ &\oplus \left(\oplus_{j=1}^{m_1}(i+\alpha_{1,j})\right) \oplus \left(-\oplus_{j=1}^{m_2}(i+\alpha_{2,j})\right) \oplus \\ &\cdots \oplus \left(\oplus_{j=1}^{m_{2\beta-1}}(i+\alpha_{2\beta-1,j})\right) \oplus \left(-\oplus_{j=1}^{m_{2\beta}}(i+\alpha_{2\beta,j})\right), \quad 1 \le t \le \infty, \end{aligned}$$

Clearly, $C(t) \in SBD_n(\tau)$, $1 \le t \le \infty$, and $C(\infty)$ is congruent to a matrix of the form (3.4). ∎

PROPOSITION 3.5. (1) *The set SBD_n coincides with the union of the sets $SBD_n(\tau)$ over all $\tau \in \Xi_n$.*

(2) *If $\tau \ne \tau'$, then $SBD_n(\tau) \cap SBD_n(\tau') = \emptyset$.*

(3) *Each set $SBD_n(\tau)$, $\tau \in \Xi_n$, is open in $\mathbb{C}^{n\times n}$.*

Proof. Proof of (1): In view of the the equivalence (a) $\Longleftrightarrow$ (e) in Theorem 2.6, we only need to show that if A is congruent to a diagonal matrix of the form

$$I_r \oplus \operatorname{diag}(\eta_1(i+\alpha_1), \ldots, \eta_n(i+\alpha_n)),$$

where

$$\eta_1 = \cdots = \eta_{\ell_1} \ne \eta_{\ell_1+1}, \quad \eta_n = \eta_{n-1} = \cdots = \eta_{n-\ell_2+1} \ne \eta_{n-\ell_2}, \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n,$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

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subject to $\alpha_{j_1} = \alpha_{j_2} \implies \eta_{j_1} = \eta_{j_2}$, then

$$A \in \bigcup_{\tau \in \Xi_n} SBD_n(\tau).$$

Indeed, assume first that the case $r = 0,\ \eta_1 = 1,\ \eta_n = -1$ does not happen. Then by the definition of the set $SBD_n(\tau)$, A belongs to $SBD_n((r_0; m_1, \dots, m_{2\beta}))$ for some positive integers $m_1, \dots, m_{2\beta}$, where r_0 is given as follows:

$$r_0 = \begin{cases} r & \text{if } \eta_1 = 1,\ \eta_n = -1, \\ r + \ell_1 & \text{if } \eta_1 = -1,\ \eta_n = -1, \\ r + \ell_2 & \text{if } \eta_1 = 1,\ \eta_n = 1, \\ r + \ell_1 + \ell_2 & \text{if } \eta_1 = -1,\ \eta_n = 1. \end{cases}$$

If $r = 0,\ \eta_1 = 1,\ \eta_n = -1$, then $-A$ is congruent to

$$\operatorname{diag}(-\eta_1(i + \alpha_1), \dots, -\eta_n(i + \alpha_n)),$$

therefore $A \in SBD_n(\ell_1 + \ell_2; m_1, \dots, m_{2\beta})$ for some positive integers $m_1, \dots, m_{2\beta}$.

For the proof of (2) use the uniqueness of the form (2.1) up to a permutation of blocks.

We now prove (3). Assume that A or $-A$ is $\mathbb{R}$-congruent to a matrix of the form

$$\begin{aligned} & I_{k_1} \oplus \left(- \oplus_{j=1}^{k_3} (i + \kappa_j)\right) \oplus \left(\oplus_{j=1}^{m_1} (i + \alpha_{1,j})\right) \oplus \left(- \oplus_{j=1}^{m_2} (i + \alpha_{2,j})\right) \oplus \cdots \\ & \oplus \left(\oplus_{j=1}^{m_{2\beta-1}} (i + \alpha_{2\beta-1,j})\right) \oplus \left(- \oplus_{j=1}^{m_{2\beta}} (i + \alpha_{2\beta,j})\right) \oplus \left(\oplus_{j=1}^{k_2} (i + \gamma_j)\right), \end{aligned} \tag{3.9}$$

where $k_1 + k_2 + k_3 = k$, subject to (3.5) and (3.7). Note that $U := A^{-1}A^*$ has eigenvalues

$$\frac{-i + \kappa_j}{i + \kappa_j}, \quad \frac{-i + \alpha_{u,j}}{i + \alpha_{u,j}}, \quad \frac{-i + \gamma_j}{i + \gamma_j},$$

all of them on the unit circle. Note also that

$$\frac{-i + x}{i + x} \longrightarrow 1, \qquad x \in \mathbb{R}, \quad x \longrightarrow \pm\infty.$$

Now combining Remark 2.3 with a perturbation result [7, Theorem 9.8.1] for H-unitary matrices, applied to the $\overline{z}A^* + zA$-unitary matrix U (for a suitable z), yields statement (3). ☐

Proposition 3.6. *Let $A \in SBD_n(\tau)$ and $B \in SBD_n(\tau')$. If $\tau \neq \tau'$, then A and B cannot be connected within the set SBD_n.*

Proof. Arguing by contradiction, assume that A and B can be connected by a continuous curve within SBD_n:

$$A = C(0), \quad B = C(1), \quad C : [0, 1] \to SBD_n,$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

where C is a continuous function. Let t_0 be the supremum of all numbers $t \in [0,1]$ such that

$$C(t') \in SBD_n(\tau) \quad \forall t' \leq t.$$

Then clearly

$$C(t') \in SBD_n(\tau) \quad \forall t' < t_0,$$

and for every $\varepsilon > 0$ there exists $t'(\varepsilon) \in [0,1]$ such that $t_0 \leq t'(\varepsilon) < t_0 + \varepsilon$ and

$$C(t'(\varepsilon)) \notin SBD_n(\tau).$$

We obtain a contradiction with Proposition 3.5(3) applied to $C(t_0)$. $\square$

Proof of Theorem 3.2. The proof is now immediate in view of Propositions 3.4 and 3.6. $\square$

4. Symmetry (II): Preliminaries. *In this and the next two sections we assume that symmetry* (II) *holds*, i.e., the matrices and vectors are real and * stands for the transpose. Real matrices A and B are said to be $\mathbb{R}$-*congruent* if $A = S^T BS$ for some real invertible matrix S.

We start with preliminary results. The main results and their proofs will be given in the next two sections.

Introduce the following standard matrices:

$$\Xi_k := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \iddots & & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ (-1)^{k-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} = (-1)^{k-1}\Xi_k^T \in \mathbb{R}^{k\times k}.$$

Thus, Ξ_k is symmetric if k is odd, and skew-symmetric if k is even.

$$\Upsilon_{2k+1}^{(1)} := G_{2k+1} + \begin{bmatrix} 0 & 0 & F_k \\ 0 & 0_1 & 0 \\ -F_k & 0 & 0 \end{bmatrix}.$$

$$\Upsilon_k^{(2)} := F_k + \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k-1}{2}} \\ 0 & -F_{\frac{k-1}{2}} & 0 \end{bmatrix}, \qquad k \text{ odd}.$$

$$\Upsilon_k^{(3)} := F_k + \begin{bmatrix} 0_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{\frac{k-2}{2}} \\ 0 & 0 & 0_1 & 0 \\ 0 & -F_{\frac{k-2}{2}} & 0 & 0 \end{bmatrix}, \qquad k \text{ even and } k/2 \text{ even}.$$

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$$\Upsilon_\ell^{(4)} := G_\ell + \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \qquad \ell \ \text{ even.}$$

$$\Upsilon_\ell^{(5)} := \begin{bmatrix} 0 & G_{\ell/2} + F_{\ell/2} \\ G_{\ell/2} - F_{\ell/2} & 0 \end{bmatrix}, \qquad \ell \ \text{ even and } \ \ell/2 \ \text{ odd.}$$

$$\Upsilon_\ell^{(6)}(\alpha) := \begin{bmatrix} 0 & (\alpha+1)F_{\ell/2} + G_{\ell/2} \\ (\alpha-1)F_{\ell/2} + G_{\ell/2} & 0 \end{bmatrix}, \qquad \ell \ \text{ even}, \ \alpha > 0.$$

$$\Upsilon_{2m}^{(7)}(\nu) := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \mho(\nu,m) \\ 0 & 0 & \cdots & 0 & -\mho(\nu,m) & -I_2 \\ 0 & 0 & \cdots & \mho(\nu,m) & -I_2 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ (-1)^{m-1}\mho(\nu,m) & -I_2 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where we have denoted by $\mho(\nu,m)$ the 2×2 matrix $\nu\Xi_2^{m+1} + \Xi_2^m$, and where $\nu > 0$.

$$\Upsilon_{4m}^{(8)}(a,b) := \begin{bmatrix} 0 & J_{2m}(a \pm ib)^T + I_{2m} \\ J_{2m}(a \pm ib) - I_{2m} & 0 \end{bmatrix},$$

where $a, b > 0$. In all these matrices, the subscript indicates the size.

THEOREM 4.1. *Let $A \in \mathbb{R}^{n\times n}$. Then A is $\mathbb{R}$-congruent to a matrix of the form*

$$0_{s\times s} \oplus \bigoplus_{c=1}^{x} \Upsilon_{2k'_c+1}^{(1)} \oplus \bigoplus_{j=1}^{r} \delta_j \Upsilon_{k_j}^{(2)} \oplus \bigoplus_{d=1}^{y} \Upsilon_{k''_d}^{(3)} \oplus \bigoplus_{t=1}^{p} \eta_t \Upsilon_{\ell_t}^{(4)} \oplus \bigoplus_{e=1}^{z} \Upsilon_{\ell'_e}^{(5)} \tag{4.1}$$

$$\oplus \bigoplus_{f=1}^{w} \Upsilon_{\ell''_f}^{(6)}(\alpha_f) \oplus \bigoplus_{u=1}^{q} \zeta_u \Upsilon_{2m_u}^{(7)}(\nu_u) \oplus \bigoplus_{g=1}^{v} \Upsilon_{4m'_g}^{(8)}(a_g, b_g). \tag{4.2}$$

Here, $\delta_j, \eta_t, \zeta_u$ are signs ± 1. Some types of blocks may be absent in (4.1), (4.2).

Moreover, the form (4.1), (4.2) *is unique, for a given A, up to a permutation of blocks.*

A result equivalent to Theorem 4.1 was proved in [19].

The proof follows from a well known canonical form for real matrix pencils $B+\lambda C$, where $B = B^T$, $C = -C^T$, upon applying this form to $B := \frac{A+A^T}{2}$, $C := \frac{A-A^T}{2}$. The canonical form can be found in many sources, for example [18] and [26], where historical remarks and further references are given. We have used here the canonical form as presented in [18].

THEOREM 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be invertible. Then the matrix $U := A^{-1}A^T$ is similar to a matrix of the following form:*

$$\bigoplus_{j=1}^{r} J_{k_j}(1) \oplus \bigoplus_{d=1}^{y} (J_{k''_d/2}(1) \oplus J_{k''_d/2}(1)) \oplus \bigoplus_{t=1}^{p} J_{\ell_t}(-1) \oplus \tag{4.3}$$

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A publication of the International Linear Algebra Society
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$$\bigoplus_{e=1}^{z}(J_{\ell'_e/2}(-1)\oplus J_{\ell'_e/2}(-1))\oplus\bigoplus_{f=1}^{w}(J_{\ell''_f/2}(\beta_f)\oplus J_{\ell''_f/2}(\beta_f^{-1}))\oplus\bigoplus_{u=1}^{q}J_{2m_u}(z_u,\overline{z_u}) \tag{4.4}$$

$$\oplus\bigoplus_{g=1}^{v}\left(J_{2m'_g}(z'_g,\overline{z'_g})\oplus J_{2m'_g}((z'_g)^{-1},(\overline{z'_g})^{-1})\right). \tag{4.5}$$

Here k_j's are odd integers, $k''_d/2$'s are even integers, ℓ_t's are even integers, $\ell'_e/2$'s are odd integers, $\ell''_f/2$'s are integers (even or odd), β_f's are nonzero real numbers with absolute values less than 1, z_u's are unimodular numbers with positive imaginary parts, and z'_g's are complex numbers with positive imaginary parts and absolute values less than 1.

Conversely, if $U\in\mathbb{R}^{n\times n}$ is similar to a matrix in the form (4.3) – (4.5)*, then there exists a real invertible matrix A such that $U=A^{-1}A^T$.*

Proof. For the direct statement, we need only to verify the statement for each of the constituent blocks in (4.1) - (4.2). Note that the parameters in (4.3) - (4.5) are obtained from the parameters in (4.1) - (4.2) as follows:

$$\beta_f=\frac{\alpha_f+1}{\alpha_f-1};\quad \alpha_f>0,\ \ \alpha_f\neq 1;\quad z_u=-\frac{1-i\nu_u}{1+i\nu_u};$$

and z'_g is one of the four numbers

$$(a_g-1\pm ib_g)^{-1}(a_g+1\pm ib_g),\quad (a_g+1\pm ib_g)^{-1}(a_g-1\pm ib_g).$$

The verification is straightforward: First, note that the blocks $\Upsilon^{(1)}_{2k+1}$ cannot appear because A is assumed to be invertible. For blocks of other types we have:

$$U^{(2)}_k:=(\Upsilon^{(2)}_k)^{-1}(\Upsilon^{(2)}_k)^T=K^{(2)}_k, \tag{4.6}$$

where $K^{(2)}_k$ is an upper triangular matrix with 1's on the main diagonal and $(2,2,\dots,2,-2,-2,\dots,-2)$ on the next superdiagonal (2 and -2 appear $(k-1)/2$ times each);

$$U^{(3)}_k:=(\Upsilon^{(3)}_k)^{-1}(\Upsilon^{(3)}_k)^T=K^{(3a)}_{k/2}\oplus K^{(3b)}_{k/2} \tag{4.7}$$

where $K^{(3a)}_{k/2}$ and $K^{(3b)}_{k/2}$ are upper triangular matrices with 1's on the main diagonal and 2's (for $K^{(3a)}_{k/2}$) or -2's (for $K^{(3b)}_{k/2}$) on the next superdiagonal;

$$U^{(4)}_\ell:=(\Upsilon^{(4)}_\ell)^{-1}(\Upsilon^{(4)}_\ell)^T=K^{(4)}_\ell, \tag{4.8}$$

where $K^{(4)}_\ell$ is a lower triangular matrix with -1's on the main diagonal and $(-2,-2,\dots,-2,2,2,\dots,2)$ on the next subdiagonal (-2 appears $(\ell/2)-1$ times and 2 appears $\ell/2$ times);

$$U^{(5)}_\ell:=(\Upsilon^{(5)}_\ell)^{-1}(\Upsilon^{(5)}_\ell)^T=K^{(5a)}_{\ell/2}\oplus K^{(5b)}_{\ell/2} \tag{4.9}$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

where $K^{(5a)}_{\ell/2}$ and $K^{(5b)}_{\ell/2}$ are lower triangular matrices with -1's on the main diagonal and -2's (for $K^{(5a)}_{\ell/2}$) or 2's (for $K^{(5b)}_{\ell/2}$) on the next subdiagonal;

$$U^{(6)}_{\ell}(\alpha) := (\Upsilon^{(6)}_{\ell}(\alpha))^{-1}(\Upsilon^{(6)}_{\ell}(\alpha))^T = K^{(6a)}_{\ell/2}(\alpha) \oplus K^{(6b)}_{\ell/2}(\alpha); \tag{4.10}$$

here $\alpha > 0$ (because of the general hypothesis concerning the block $\Upsilon^{(6)}_{\ell}(\alpha)$), $\alpha \neq 1$ (because A is assumed to be invertible), $K^{(6a)}_{\ell/2}(\alpha)$ is a lower triangular matrix with $(\alpha+1)/(\alpha-1)$ on the main diagonal and $-2(\alpha-1)^{-2}$ on the next subdiagonal, whereas $K^{(6b)}_{\ell/2}(\alpha)$ is a lower triangular matrix with $(\alpha-1)/(\alpha+1)$ on the main diagonal and $2(\alpha+1)^{-2}$ on the next subdiagonal;

$$U^{(7)}_{2m}(\nu) := (\Upsilon^{(7)}_{2m}(\nu))^{-1}(\Upsilon^{(7)}_{2m}(\nu))^T = K^{(7)}_{2m}(\nu), \tag{4.11}$$

where ν is a positive parameter, and

$$K^{(7)}_{2m}(\nu) := \begin{bmatrix} J_2(y,\overline{y}) & J_2(x,\overline{x}) & 0 & \cdots & 0 & 0 \\ 0 & J_2(y,\overline{y}) & -J_2(x,\overline{x}) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & J_2(y,\overline{y}) & \pm J_2(x,\overline{x}) \\ 0 & 0 & 0 & \cdots & 0 & J_2(y,\overline{y}) \end{bmatrix},$$

with

$$y := (-1)^{m-1}z^{-1}\overline{z}, \quad x := z^{-2}\overline{z} + (-1)^m z^{-1}, \quad z := i^m(1+i\nu); \tag{4.12}$$

$$U^{(8)}_{4m}(a,b) := (\Upsilon^{(8)}_{4m}(a,b))^{-1}(\Upsilon^{(8)}_{4m}(a,b))^T = K^{(8)}_{4m}(a,b), \tag{4.13}$$

where a and b are positive parameters and

$$\begin{aligned} K^{(8)}_{4m}(a,b) = {} & (J_{2m}(a \pm ib) - I)^{-1}(J_{2m}(a \pm ib) + I) \\ & \oplus (J_{2m}(a \pm ib)^T + I)^{-1}(J_{2m}(a \pm ib)^T - I). \end{aligned}$$

In each instance, it is easy to see that the real Jordan forms of the left hand sides of (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), and (4.13) conform to formulas (4.3) - (4.5). Note that $x \neq 0$ in (4.12). Also note that the eigenvalues of $U^{(8)}_{4m}(a,b)$ are

$$\frac{a+1+ib}{a-1+ib}, \quad \frac{a+1-ib}{a-1-ib}, \quad \frac{a-1+ib}{a+1+ib}, \quad \frac{a-1-ib}{a+1-ib}.$$

For the proof of the converse statement, we replace (without loss of generality) the matrix U by a similar matrix which is a block diagonal with diagonal blocks given by matrices $K^{(2)}_k$, $K^{(3a)}_{k/2} \oplus K^{(3b)}_{k/2}$, $K^{(4)}_{\ell}$, $K^{(5a)}_{\ell/2} \oplus K^{(5b)}_{\ell/2}$, $K^{(6a)}_{\ell/2}(\alpha) \oplus K^{(6b)}_{\ell/2}(\alpha)$, $K^{(7)}_{2m}(\nu)$, $K^{(8)}_{4m}(a,b)$ (there may be several, or no, matrices on the block diagonal of U of any given type, and different parameters k, ℓ, m, ν, a, b may occur for diagonal blocks of

Electronic Journal of Linear Algebra ISSN 1081-3810
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the same type). Now use formulas (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), and (4.13) to write U in the form $U = A^{-1}A^T$. ◻

In another formulation, Jordan forms of real matrices of the form $A^{-1}A^T$ have been described in [1], and see [10] for a corresponding result in the context of bilinear forms.

Note that the proof of Theorem 4.2 establishes a correspondence between the blocks in the canonical form of A under $\mathbb{R}$-congruence in Theorem 4.1 and the blocks of the canonical form of $U = A^{-1}A^T$ under similarity in Theorem 4.2.

THEOREM 4.3. *Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then $U = A^{-1}A^T$ is diagonalizable (over $\mathbb{C}$) with only unimodular eigenvalues if and only if the matrix A is $\mathbb{R}$-congruent to a matrix of the following form:*

$$\operatorname{diag}(\delta_1, \ldots, \delta_r) \oplus \bigoplus_{i=1}^{z} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \bigoplus_{u=1}^{q} \zeta_u \begin{bmatrix} -\nu_u & 1 \\ -1 & -\nu_u \end{bmatrix}. \tag{4.14}$$

Here the parameters δ_j and ζ_u are signs ± 1, and the numbers ν_u are positive.

The proof follows by inspection from Theorem 4.2 and its proof.

5. Boundedness and stable boundedness. Consider the system of difference equations

$$A^T x_i + A x_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{R}^{n\times 1}, \tag{5.1}$$

where A is invertible real $n \times n$ matrix.

Combining Theorem 4.3 and Proposition 1.3 we obtain:

THEOREM 5.1. *The system* (5.1) *is bounded if and only if A is $\mathbb{R}$-congruent to a matrix of the form* (4.14).

Stably bounded systems (5.1) are described in the next theorem. It will be convenient to introduce the following notation: If $X \in \mathbb{R}^{m\times m}$ and if λ is a (possibly nonreal) eigenvalue of X, we let

$$\operatorname{Ker}_{\mathbb{R}}(X; \lambda) = \operatorname{Ker}(X - \lambda I) \subseteq \mathbb{R}^{m\times 1}$$

if λ is real, and

$$\operatorname{Ker}_{\mathbb{R}}(X; \lambda) = \operatorname{Ker}(X^2 - (\lambda + \overline{\lambda})X + |\lambda|^2 I) \subseteq \mathbb{R}^{m\times 1}$$

if $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

THEOREM 5.2. *The following statements are equivalent for an invertible $A \in \mathbb{R}^{n\times n}$:*

(1) *The system* (5.1) *is stably bounded;*

(2) *There exists $\varepsilon > 0$ such that every system*

$$\widetilde{A}^T x_i + \widetilde{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{R}^{n\times 1},$$

with $\widetilde{A} \in \mathbb{R}^{n\times n}$ and $\|\widetilde{A} - A\| < \varepsilon$ is stably bounded;

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(3) *A is $\mathbb{R}$-congruent to a matrix of the form*

$$\pm I_r \oplus \bigoplus_{u=1}^{q} \zeta_u \begin{bmatrix} -\nu_u & 1 \\ -1 & -\nu_u \end{bmatrix}, \tag{5.2}$$

where the ν_u's are positive numbers, and the ζ_u's are signs ± 1 subject to the condition

$$\nu_{u_1} = \nu_{u_2} \quad \Longrightarrow \quad \zeta_{u_1} = \zeta_{u_2}. \tag{5.3}$$

(4) *The matrix $A + A^T$ is invertible, and for every $\lambda \in \sigma(A^{-1}A^T)$ the condition*

$$x \in \mathrm{Ker}_{\mathbb{R}}\,(A^{-1}A^T; \lambda) \setminus \{0\} \quad \Longrightarrow \quad ((A + A^T)x, x) \neq 0 \tag{5.4}$$

holds.

Proof. Proof of (1) $\Longrightarrow$ (3). By Theorem 4.3 we may and do assume that

$$A = \mathrm{diag}\,(\delta_1, \dots, \delta_r) \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \bigoplus_{u=1}^{q} \zeta_u \begin{bmatrix} -\nu_u & 1 \\ -1 & -\nu_u \end{bmatrix}, \tag{5.5}$$

as in (4.14). We assume that (5.1) is stably bounded. Let $B := \begin{bmatrix} x & 1 \\ -1 & -x \end{bmatrix}$, where $x > 0$ is close to zero. Note that

$$B^{-1}B^T = \frac{1}{1 - x^2} \begin{bmatrix} -x^2 - 1 & 2x \\ 2x & -x^2 - 1 \end{bmatrix},$$

and the matrix $B^{-1}B^T$ has nonunimodular eigenvalues. Since B is close to the block $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, it follows from the stable boundedness of (5.1) (arguing by contradiction) that the blocks $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are absent in (5.5). Also, if $C := \begin{bmatrix} 1 & x \\ -x & -1 \end{bmatrix}$, where $x > 0$ is close to zero, then the matrix

$$C^{-1}C^T = \frac{1}{-1 + x^2} \begin{bmatrix} -x^2 - 1 & 2x \\ 2x & -x^2 - 1 \end{bmatrix}$$

has nonunimodular eigenvalues. Using the stable boundedness of (5.1) again, and arguing by contradiction, we conclude that all signs δ_j are the same. Finally, assume that (5.3) is not satisfied. Say, $\nu := \nu_1 = \nu_2$, $\zeta_1 = -\zeta_2 = 1$. We now use the standard map

$$\phi \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a + ib$$

that identifies a subalgebra of $\mathbb{R}^{2\times 2}$ with $\mathbb{C}$, and apply it 2×2 blockwise to real matrices of even size. We have

$$\phi\left(\begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix} \oplus - \begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix} \right) = \begin{bmatrix} i - \nu & 0 \\ 0 & -(i - \nu) \end{bmatrix}. \tag{5.6}$$

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By Theorem 2.6, the matrix on the right hand side of (5.6) is not stably bounded (in the sense of symmetry (I)). Thus, by Theorem 2.7 there exists $X \in \mathbb{C}^{2\times 2}$ as close as we wish to the zero matrix such that the system

$$\left(\begin{bmatrix} i-\nu & 0 \\ 0 & -(i-\nu) \end{bmatrix} + X\right)^* y_i + \left(\begin{bmatrix} i-\nu & 0 \\ 0 & -(i-\nu) \end{bmatrix} + X\right) y_{i+1} = 0, \quad y_i \in \mathbb{C}^2,$$

has a geometrically growing solution, in other words, the matrix

$$\left(\begin{bmatrix} i-\nu & 0 \\ 0 & -(i-\nu) \end{bmatrix} + X\right)^{-1} \left(\begin{bmatrix} i-\nu & 0 \\ 0 & -(i-\nu) \end{bmatrix} + X\right)^*$$

has nonunimodular eigenvalues. Since the map ϕ preserves eigenvalues (in the sense that $\lambda \pm i\mu$ being an eigenvalue of $Y \in \mathbb{R}^{2m\times 2m}$, where $\lambda, \mu \in \mathbb{R}$, is equivalent to at least one of the two numbers $\lambda + i\mu$ and $\lambda - i\mu$ being an eigenvalue of $\phi(Y) \in \mathbb{C}^{m\times m}$), the matrix

$$D := \left(\left(\begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix} \oplus - \begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix}\right) + \phi^{-1}(X)\right)^{-1}$$

$$\times \left(\left(\begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix} \oplus - \begin{bmatrix} -\nu & 1 \\ -1 & -\nu \end{bmatrix}\right) + \phi^{-1}(X)\right)^T$$

also has nonunimodular eigenvalues. Then the system

$$D^T x_i + D x_{i+1} = 0, \qquad i = 0, 1, 2, \ldots, \quad x_i \in \mathbb{R}^{4\times 1}$$

has geometrically growing solutions (by Theorem 2.7), a contradiction with the stable boundedness of (5.1). This proves the implication (1) $\Longrightarrow$ (3).

Proof of (3) $\Longrightarrow$ (4). Note that both conditions (3) and (4) are invariant under congruence transformation. Thus, without loss of generality we assume that

$$A = \pm I_r \oplus \bigoplus_{u=1}^{q} \zeta_u \begin{bmatrix} -\nu_u & 1 \\ -1 & -\nu_u \end{bmatrix},$$

where $\nu_u > 0$, $\zeta_u = \pm 1$, and condition (5.3) is satisfied. The invertibility of $A + A^T$ is obvious. Note that

$$A^{-1}A^T = I_r \oplus \bigoplus_{u=1}^{q} \frac{1}{\nu_u^2 + 1} \begin{bmatrix} \nu_u^2 - 1 & 2\nu_u \\ -2\nu_u & \nu_u^2 - 1 \end{bmatrix}.$$

Now it is clear that because of condition (5.3), the quadratic form $((A + A^T)x, x)$ is definite (positive or negative) on the subspace $\mathrm{Ker}_{\mathbb{R}}\,(A^{-1}A^T; \lambda)$, for every eigenvalue λ of $A^{-1}A^T$. Thus, (4) follows.

Proof of (4) $\Longrightarrow$ (2). Assume (4) holds; denote $U = A^{-1}A^T$ and $H = A + A^T$. Note the equality

$$U^T(A + A^T)U = A + A^T,$$

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so U is H-unitary. Observe that the quadratic form (Hx, x) is either positive definite or negative definite on

$$\mathrm{Ker}_{\mathbb{C}}\,(\lambda_0 I - U) := \{x \in \mathbb{C}^{n\times 1}\ :\ Ux = \lambda_0 x\},$$

for every $\lambda_0 \in \sigma(U)$. Indeed, since U is real, one can choose real bases in the complex subspaces $\mathrm{Ker}_{\mathbb{C}}\,(\lambda_0 I - U)$ (if λ_0 is real) and

$$\mathrm{Ker}_{\mathbb{C}}\,(\lambda_0 I - U)\dot{+}\mathrm{Ker}_{\mathbb{C}}\,(\overline{\lambda_0} I - U) = \mathrm{Ker}_{\mathbb{C}}\,(U^2 - (\lambda_0 + \overline{\lambda_0})U + |\lambda_0|^2 I).$$

Representing the quadratic form (Hx, x) with respect to this basis, and using condition (5.4), we see that (Hx, x) is definite on $\mathrm{Ker}_{\mathbb{C}}\,(\lambda_0 I - U)$. Now it follows from Theorem 2.6 that (2) holds.

Since the implication (2) $\Longrightarrow$ (1) is trivial, we are done. ◻

We have also the real analogue of Theorem 2.7:

Theorem 5.3. *Assume symmetry* (II). *If* (5.1) *is bounded but not stably bounded, then for every* $\varepsilon > 0$ *there exist a system*

$$\widetilde{A}^T x_i + \widetilde{A} x_{i+1} = 0, \qquad i = 0, 1, \ldots, x_i \in \mathbb{R}^{n\times 1}, \tag{5.7}$$

with $\widetilde{A} \in \mathbb{R}^{n\times n}$ *and* $\|\widetilde{A} - A\| < \varepsilon$ *and such that* (5.7) *has a geometrically growing solution.*

The proof of Theorem 5.3 is obtained by repeating the arguments in the proof of (1) $\Longrightarrow$ (3) in Theorem 5.2.

An interesting observation follows immediately from Theorem 5.2:

Corollary 5.4. *If* n *is even, then* $\det A > 0$ *for every stably bounded system* (5.1).

6. Connected components. We study in this section the connected components of bounded and stably bounded systems of linear difference equations with symmetry (II).

Theorem 6.1. *The set of bounded systems of the form* (5.1) *consists of exactly two arcwise connected components, one with* A*'s having positive determinants, the other with* A*'s having negative determinants.*

Proof. Let $\Omega_{\mathbb{R}}$ be the set of all matrices $A \in \mathbb{R}^{n\times n}$ that are $\mathbb{R}$-congruent to a matrix of the form (4.14), and let $\Omega_{\mathbb{R},0}$ be the set of $n \times n$ matrices of the form (4.14). Thus, $X \in \Omega_{\mathbb{R}}$ if and only if $X = S^T Y S$ for some $Y \in \Omega_{\mathbb{R},0}$ and some invertible $S \in \mathbb{R}^{n\times n}$. By Theorem 4.3, we need to show that the set $\Omega_{\mathbb{R}}$ has exactly two (arcwise) connected components, one with positive determinants, the other with negative determinants. We may assume that $n \geq 2$.

Step 1. We show that every matrix $X \in \Omega_{\mathbb{R},0}$ is connected within $\Omega_{\mathbb{R}}$ to a diagonal matrix with ± 1's on the diagonal. Indeed, for a block

$$\zeta_u \begin{bmatrix} -\nu_u & 1 \\ -1 & -\nu_u \end{bmatrix}$$

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A publication of the International Linear Algebra Society
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in X, by letting $\nu_u \longrightarrow \infty$ and scaling we see that it is connected to $-\zeta_u I_2$. The block $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is connected to $-I_2$ within $\Omega_{\mathbb{R}}$ via

$$\begin{bmatrix} -x & 1 \\ -1 & -x \end{bmatrix}, \qquad 0 \le x < \infty,$$

and scaling.

STEP 2. We show that every matrix $X \in \Omega_{\mathbb{R},0}$ is connected within $\Omega_{\mathbb{R}}$ to either I_n or $(-1) \oplus I_{n-1}$. In view of Step 1, we need only to show that $-I_2$ is connected within $\Omega_{\mathbb{R}}$ to I_2. This is easy: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is connected to $-I_2$ by Step 1, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is connected to I_2 by taking the negatives, and the matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are $\mathbb{R}$-congruent with the congruence matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

STEP 3. Let now $X \in \Omega_{\mathbb{R}}$, so that $X = S^T Y S$ for some $Y \in \Omega_{\mathbb{R},0}$ and some invertible $S \in \mathbb{R}^{n\times n}$. Using Step 2, we connect simultaneously Y within $\Omega_{\mathbb{R}}$ to either I_n or $(-1) \oplus I_{n-1}$, and S within the group of $n \times n$ real invertible matrices to either I_n or $(-1) \oplus I_{n-1}$ again (depending on the sign of the determinant of S). As a result, X is connected within $\Omega_{\mathbb{R}}$ either to I_n or to $(-1) \oplus I_{n-1}$. From here the result follows easily. □

We now pass to the stably bounded systems (5.1). As in the complex case, we need some preparation to state the results. For n fixed, consider the set $\Xi_{n,\mathbb{R}}$ of ordered tuples of integers of the form

(6.1)
$$\Xi_{n,\mathbb{R}} := \{(\xi; r; m_1, m_2, \ldots, m_\beta) \,:\, \xi = \pm 1, \quad r \ge 0; \quad m_j > 0 \ \text{ for } j = 1, 2, \ldots, \beta\}$$

subject to

$$r + 2\sum_{j=1}^{\beta} m_j = n.$$

Here $\beta \ge 0$; if $\beta = 0$, the terms m_j are absent in (6.1). For example, if $n = 6$, the set $\Xi_{6,\mathbb{R}}$ consists of 16 elements:

$$(\pm 1; 0; 1, 1, 1);\ (\pm 1; 0; 2, 1);\ (\pm 1; 0; 1, 2);\ (\pm 1; 0; 3);$$

$$(\pm 1; 2; 1, 1);\ (\pm 1; 2; 2);\ (\pm 1; 4; 1);\ (\pm 1; 6).$$

With each element

$$\omega := (\xi; r; m_1, m_2, \ldots, m_\beta) \in \Xi_{n,\mathbb{R}}$$

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A publication of the International Linear Algebra Society
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we associate a set $SBD_{n,\mathbb{R}}(\omega)$ of all $n \times n$ real matrices that are $\mathbb{R}$-congruent to a matrix of the form

$$\xi \left\{ I_s \oplus \left(-\bigoplus_{j=1}^{t} \begin{bmatrix} -\nu_{0,j} & 1 \\ -1 & -\nu_{0,j} \end{bmatrix} \right) \oplus \right.$$

$$\left(\bigoplus_{j=1}^{m_1} \begin{bmatrix} -\nu_{1,j} & 1 \\ -1 & -\nu_{1,j} \end{bmatrix} \right) \oplus \left(-\bigoplus_{j=1}^{m_2} \begin{bmatrix} -\nu_{2,j} & 1 \\ -1 & -\nu_{2,j} \end{bmatrix} \right) \oplus \cdots$$

$$\left. \oplus \left((-1)^{\beta-1} \bigoplus_{j=1}^{m_\beta} \begin{bmatrix} -\nu_{\beta,j} & 1 \\ -1 & -\nu_{\beta,j} \end{bmatrix} \right) \right\}, \tag{6.2}$$

where $r = s + 2t$, (s and t are nonnegative integers), and where

$$\nu_{0,1} \geq \cdots \geq \nu_{0,t} > \nu_{1,1} \geq \cdots \geq \nu_{1,m_1} > \nu_{2,1} \geq \cdots \geq \nu_{2,m_2} > \\ \cdots > \nu_{\beta,1} \geq \cdots \geq \nu_{\beta,m_\beta} > 0.$$

Finally, denote by $SBD_{n,\mathbb{R}}$ the set of all real $n \times n$ matrices A for which the system (5.1) is stably bounded, i.e., (in view of Theorem 5.2), $SBD_{n,\mathbb{R}}$ consists of all matrices $A \in \mathbb{R}^{n\times n}$ that are $\mathbb{R}$-congruent to a matrix of the form (5.2).

Proposition 6.2. (1) *The set $SBD_{n,\mathbb{R}}$ coincides with the union*

$$\bigcup_{\omega \in \Xi_{n,\mathbb{R}}} SBD_{n,\mathbb{R}}(\omega).$$

(2) *For every $\omega \in \Xi_{n,\mathbb{R}}$ with $r \geq 1$, the set $SBD_{n,\mathbb{R}}(\omega)$ is connected. For every $\omega \in \Xi_{n,\mathbb{R}}$ with $r = 0$ the set $SBD_{n,\mathbb{R}}(\omega)$ consists of two connected components, one component containing the matrix*

$$\xi \left\{ \bigoplus_{u=1}^{\beta} (-1)^{u-1} \left(\bigoplus_{j=1}^{m_u} \begin{bmatrix} -\nu_{u,j} & 1 \\ -1 & -\nu_{u,j} \end{bmatrix} \right) \right\} \tag{6.3}$$

and the other component containing the matrix

$$\operatorname{diag}(-1,1,1,\ldots,1)\xi \left\{ \bigoplus_{u=1}^{\beta} (-1)^{u-1} \left(\bigoplus_{j=1}^{m_u} \begin{bmatrix} -\nu_{u,j} & 1 \\ -1 & -\nu_{u,j} \end{bmatrix} \right) \right\} \operatorname{diag}(-1,1,1,\ldots,1). \tag{6.4}$$

(3) *For every $\omega \in \Xi_{n,\mathbb{R}}$, the set $SBD_{n,\mathbb{R}}(\omega)$ is open in $\mathbb{R}^{n\times n}$. If $r = 0$, then each of the two connected components of $SBD_{n,\mathbb{R}}(\omega)$ is open in $\mathbb{R}^{n\times n}$.*

(4) *If $\omega', \omega \in \Xi_{n,\mathbb{R}}$ and $\omega' \neq \omega$, then*

$$SBD_{n,\mathbb{R}}(\omega) \cap SBD_{n,\mathbb{R}}(\omega') = \emptyset.$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

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Proof. The statement (1) follows from Theorem 5.2.

Proof of (2): We fix

$$\omega = (\xi; r; m_1, m_2, \ldots, m_\beta) \in \Xi_{n,\mathbb{R}}.$$

First we verify that

$$\xi \left\{ I_r \oplus \bigoplus_{u=1}^{\beta} \left((-1)^{u-1} \bigoplus_{j=1}^{m_u} \begin{bmatrix} -\nu_{u,j} & 1 \\ -1 & -\nu_{u,j} \end{bmatrix} \right) \right\} \tag{6.5}$$

is connected within $SBD_{n,\mathbb{R}}(\omega)$ to

$$\xi \left\{ I_s \oplus \left(- \bigoplus_{j=1}^{t} \begin{bmatrix} -\nu_{0,j} & 1 \\ -1 & -\nu_{0,j} \end{bmatrix} \right) \oplus \bigoplus_{u=1}^{\beta} \left((-1)^{u-1} \bigoplus_{j=1}^{m_u} \begin{bmatrix} -\nu_{u,j} & 1 \\ -1 & -\nu_{u,j} \end{bmatrix} \right) \right\},$$

for all nonnegative integers s and t such that $s+2t = r$. Indeed, scaling if necessary the numbers $\nu_{u,j}$, we may assume that $\nu_{u,j} < 1$ for $j = 1, 2, \ldots, m_u$ and $u = 1, 2, \ldots, \beta$; then the continuous function

$$C(x) := I_s \oplus \left(- \bigoplus_{j=1}^{t} \begin{bmatrix} -\nu_{0,j} \frac{x+\nu_{0,j}-1}{\nu_{0,j}x} & 1/x \\ -1/x & -\nu_{0,j} \frac{x+\nu_{0,j}-1}{\nu_{0,j}x} \end{bmatrix} \right), \quad 1 \le x \le \infty,$$

realizes such a connection when substituted for I_r in (6.5). Now it is routine to see that the matrices of the form (6.2) are connected to each other within $SBD_{n,\mathbb{R}}(\omega)$ (cf. the proof of Proposition 3.4).

Next, let $A \in SBD_{n,\mathbb{R}}(\omega)$, so that $A = S^T A_0 S$ for some invertible $S \in \mathbb{R}^{n\times n}$ and some A_0 of the form (6.2). By the already proved part of (2), there exists a continuous function

$$D(x) \in SBD_{n,\mathbb{R}}(\omega), \quad 0 \le x \le 1,$$

such that $D(0) = A_0$ and

$$D(1) = \xi \left\{ I_r \oplus \left(\bigoplus_{j=1}^{m_1} \begin{bmatrix} -\nu_{1,j} & 1 \\ -1 & -\nu_{1,j} \end{bmatrix} \right) \oplus \left(- \bigoplus_{j=1}^{m_2} \begin{bmatrix} -\nu_{2,j} & 1 \\ -1 & -\nu_{2,j} \end{bmatrix} \right) \oplus \cdots \right.$$

$$\left. \oplus \left((-1)^{\beta-1} \bigoplus_{j=1}^{m_\beta} \begin{bmatrix} -\nu_{\beta,j} & 1 \\ -1 & -\nu_{\beta,j} \end{bmatrix} \right) \right\}. \tag{6.6}$$

On the other hand, there exists a continuous invertible function $S(x) \in \mathbb{R}^{n\times n}$ such that $S(0) = S$ and

$$S(1) = \mathrm{diag}\,(\pm 1, 1, 1, \ldots, 1).$$

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Then, if $r \geq 1$, the continuous function

$$A(x) = S(x)^T D(x) S(x), \quad 0 \leq x \leq 1,$$

connects A and $D(1)$ within $SBD_{n,\mathbb{R}}(\omega)$. This concludes the proof of (2) in the case $r \geq 1$. This also shows that in case $r = 0$, the set $SBD_{n,\mathbb{R}}(\omega)$ is a union of two connected sets, one containing the matrix (6.3), the other containing the matrix (6.4).

To conclude the proof of (2) in the case $r = 0$, it remains to show that the matrices (6.3) and (6.4) cannot be connected within $SBD_{n,\mathbb{R}}(\omega)$. Note that if $A \in SBD_{n,\mathbb{R}}(\omega)$, then the skew symmetric part $\frac{1}{2}(A - A^T)$ of A is invertible. Thus, if there existed a continuous path within $SBD_{n,\mathbb{R}}(\omega)$ between (6.3) and (6.4), then there would be a continuous path within the set of invertible skew symmetric real matrices between the skew symmetric parts of (6.3) and (6.4). However, this is impossible, since the set of invertible $n \times n$ real skew symmetric matrices consists of two connected components, one with positive pfaffians, the other with negative pfaffians (see, for example, [15], [20], and [2] for the basic properties of pfaffians), and the skew symmetric parts of (6.3) and (6.4) have pfaffians of opposite signs.

Proof of (3): The openness of $SBD_{n,\mathbb{R}}(\omega)$ follows in the same way as in the proof of Proposition 3.5(3), using [7, Theorem 9.8.1] and the $(A + A^T)$-unitary property of the matrix $U := A^{-1}A^T$. If $r = 0$, then the openness of each of the two components of $SBD_{n,\mathbb{R}}(\omega)$ follows from the openness of $SBD_{n,\mathbb{R}}(\omega)$ and the continuity of the pfaffian of real skew symmetric matrices. Thus, for every $A \in SBD_{n,\mathbb{R}}(\omega)$, there exists $\varepsilon > 0$ such that for all $B \in SBD_{n,\mathbb{R}}(\omega)$ with $\|B - A\| < \varepsilon$, the sign of the pfaffian of the skew symmetric part of B coincides with that of the skew symmetric part of A.

Finally, statement (4) follows from the uniqueness of the canonical form (4.1), (4.2) for a given matrix $A \in \mathbb{R}^{n \times n}$. $\square$

If $r = 0$, we denote the two components of $SBD_{n,\mathbb{R}}(\omega)$ by $SBD_{n,\mathbb{R}}(\omega)^{\pm}$.

Using Proposition 6.2, we now obtain, analogously to the proof of Theorem 3.2, the following characterization of the connected components of stably bounded systems (5.1).

THEOREM 6.3. *The classes $SBD_{n,\mathbb{R}}(\omega)$, $\omega \in \Xi_n$, for the elements ω with $r > 0$, and the classes $SBD_{n,\mathbb{R}}(\omega)^+$ and $SBD_{n,\mathbb{R}}(\omega)^-$, $\omega \in \Xi_n$, for the elements ω with $r = 0$, are the (arcwise) connected components of the set of stably bounded systems of difference equations*

$$A^T x_i + A x_{i+1} = 0, \qquad i = 0, 1, \ldots, \qquad x_i \in \mathbb{R}^{n \times 1}, \tag{6.7}$$

where $A \in \mathbb{R}^{n \times n}$ is invertible.

7. Symmetry (III). *In this section we assume that symmetry of type* (III) *holds*, i.e., the matrices are complex and $\star$ stands for the transpose.

We start with a canonical form for complex matrices under transposition. Matrices $A, B \in \mathbb{C}^{n \times n}$ are said to be T-*congruent* if there exists an invertible complex matrix S such that $A = S^T B S$.

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Introduce additional standard matrices:

$$\Upsilon^{(6a)}_{\ell}(\alpha) := \begin{bmatrix} 0 & (\alpha+1)F_{\ell/2}+G_{\ell/2} \\ (\alpha-1)F_{\ell/2}+G_{\ell/2} & 0 \end{bmatrix}.$$

The difference between $\Upsilon^{(6)}_k$ and $\Upsilon^{(6a)}_k$ is that α is required to be real and positive in $\Upsilon^{(6)}_k$, whereas in $\Upsilon^{(6a)}_k$, α is any nonzero complex number.

THEOREM 7.1. *Let $A \in \mathbb{C}^{n\times n}$. Then A is T-congruent to a matrix of the form*

$$0_{s\times s} \oplus \bigoplus_{c=1}^{x} \Upsilon^{(1)}_{2k'_c+1} \oplus \bigoplus_{j=1}^{r} \Upsilon^{(2)}_{k_j} \oplus \bigoplus_{d=1}^{y} \Upsilon^{(3)}_{k''_d} \oplus \bigoplus_{t=1}^{p} \Upsilon^{(4)}_{\ell_t} \oplus \bigoplus_{e=1}^{z} \Upsilon^{(5)}_{\ell'_e} \oplus \bigoplus_{f=1}^{w} \Upsilon^{(6a)}_{\ell''_f}(\alpha_f) \tag{7.1}$$

Some types of blocks may be absent in (7.1).

Moreover, the form (7.1) *under complex T-congruence is unique, for a given A, up to a permutation of blocks and up to a possible replacement of some of the parameters α_f with their negatives.*

Theorem 7.1 follows immediately from a well-known canonical form for matrix pencil $B + \lambda C$, where $B = B^T$ and $C = -C^T$ are complex matrices such that $A = B + C$. See, for example, [26, Theorem 1].

Analogously to Section 4, we now obtain:

THEOREM 7.2. *Let $A \in \mathbb{C}^{n\times n}$ be invertible. Then the matrix $U := A^{-1}A^T$ is similar to a matrix of the following form:*

$$\bigoplus_{j=1}^{r} J_{k_j}(1) \oplus \bigoplus_{d=1}^{y} (J_{k''_d/2}(1) \oplus J_{k''_d/2}(1)) \oplus \bigoplus_{t=1}^{p} J_{\ell_t}(-1) \tag{7.2}$$

$$\oplus \bigoplus_{e=1}^{z} (J_{\ell'_e/2}(-1) \oplus J_{\ell'_e/2}(-1)) \oplus \bigoplus_{f=1}^{w} (J_{\ell''_f/2}(\beta_f) \oplus J_{\ell''_f/2}(\beta_f^{-1})). \tag{7.3}$$

Here k_j's are odd integers, $k''_d/2$'s are even integers, ℓ_t's are even integers, $\ell'_e/2$'s are odd integers, $\ell''_f/2$'s are integers (even or odd), and the parameters β_f are nonzero complex numbers not equal to ± 1.

Conversely, if $U \in \mathbb{C}^{n\times n}$ is similar to a matrix in the form (7.2), (7.3), *then there exists a complex invertible matrix A such that $U = A^{-1}A^T$.*

THEOREM 7.3. *Let $A \in \mathbb{C}^{n\times n}$ be an invertible matrix. Then $U = A^{-1}A^T$ is diagonalizable with only unimodular eigenvalues if and only if the matrix A is T-congruent to a matrix of the following form:*

$$I_r \oplus \bigoplus_{i=1}^{z} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \bigoplus_{f=1}^{w} \begin{bmatrix} 0 & \alpha_f+1 \\ \alpha_f-1 & 0 \end{bmatrix}. \tag{7.4}$$

Here the parameters α_f are complex numbers with zero real parts and positive imaginary parts.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

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Combining Proposition 1.3 with Theorem 7.3, the following corollary is immediate:

COROLLARY 7.4. *Let $A \in \mathbb{C}^{n\times n}$ be invertible. Then all solutions of the system*

$$A^T x_i + Ax_{i+1} = 0, \quad i = 0, 1, \ldots, \qquad x_i \in \mathbb{C}^{n\times 1}, \tag{7.5}$$

are bounded if and only if A is T-congruent to a matrix of the form (7.4).

We now turn to the connectivity problem:

THEOREM 7.5. *The set of bounded equations of the form* (7.5) *is arcwise connected.*

Proof. By Corollary 7.4, we have to verify that the set of matrices $X \in \mathbb{C}^{n\times n}$ which are T-congruent to a matrix in the form (7.4) is arcwise connected. Indeed, by taking $\alpha_f = y_f i$ with $y_f > 0$ and $y_f \to 0$, we see that the block $\begin{bmatrix} 0 & \alpha_f + 1 \\ \alpha_f - 1 & 0 \end{bmatrix}$ is connected to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. On the other hand, taking $y_f \to \infty$, we obtain

$$(\sqrt{y_f^{-1}} I_2) \begin{bmatrix} 0 & \alpha_f + 1 \\ \alpha_f - 1 & 0 \end{bmatrix} (\sqrt{y_f^{-1}} I_2) \quad \longrightarrow \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

which is easily seen to be T-congruent to I_2. Finally, use the fact that the group of invertible complex matrices is connected. ∎

In contrast with symmetries (I) and (II), there are no stably bounded equations (7.5) (unless $n = 1$):

THEOREM 7.6. *Let $A \in \mathbb{C}^{n\times}$, $n > 1$, be such that all solutions of* (7.5) *are bounded. Then for every $\varepsilon > 0$ there exists a matrix $B \in \mathbb{C}^{n\times n}$ such that $\|B - A\| < \varepsilon$ and the corresponding system*

$$B^T x_i + Bx_{i+1} = 0, \quad i = 0, 1, \ldots, \qquad x_i \in \mathbb{C}^{n\times 1},$$

has a geometrically growing solution.

Proof. In view of Theorem 7.3 and Proposition 1.4, we need only to demonstrate the following fact: For each of the following two matrices

$$A_1 = I_2, \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0 & yi + 1 \\ yi - 1 & 0 \end{bmatrix}, \quad y \geq 0,$$

and for every $\varepsilon > 0$ there exists $B_j \in \mathbb{C}^{2\times 2}$, $j = 1, 2$, such that $\|B_j - A_j\| < \varepsilon$ and the matrix $B_j^{-1} B_j^T$ has an eigenvalue with absolute value larger than 1. Indeed, take

$$B_1 = \begin{bmatrix} 1 & xi \\ -xi & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & yi + x + 1 \\ yi + x - 1 & 0 \end{bmatrix}$$

where x is a small positive number. Then

$$B_1^{-1} B_1^T = \frac{1}{1 - x^2} \begin{bmatrix} 1 + x^2 & -2xi \\ 2xi & 1 + x^2 \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 22-49, January 2006

which is a Hermitian matrix close to I_2 with trace larger than 2. Therefore, it has an eigenvalue larger than 1. Next,

$$B_2^{-1}B_2 = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix},$$

where $q := (yi + x + 1)(yi + x - 1)^{-1}$ is not unimodular, and we are done. □

Acknowledgment. The author thanks T. Laffey for pointing out the references [1], [27], C.-K. Li for pointing out the reference [15], and the referee for several suggestions aimed at improving the exposition.

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