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THE CAUCHY DOUBLE ALTERNANT AND DIVIDED DIFFERENCES*

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Dedicated to my friend Pier Vittorio Ceccherini on the occasion of his 65th birthday

Abstract. As an extension of Cauchy's double alternant, a general determinant evaluation formula is established. Several interesting determinant identities are derived as consequences by means of divided differences.

Key words. Cauchy's double alternant, Divided differences, Symmetric functions.

AMS subject classifications. 15A15, 15A18.

1. Cauchy's Double Alternant and Extension. Cauchy's double alternant reads as

$$\Omega \;=\; \det_{0\le i,j\le n}\left[\frac{1}{x_i+y_j}\right] \;=\; \frac{\prod_{0\le i<j\le n}(x_i-x_j)(y_i-y_j)}{\prod_{0\le i,j\le n}(x_i+y_j)}.$$

For the subsequent use, we denote a variant of it by

$$\Omega' \;=\; \det_{1\le i,j\le n}\left[\frac{1}{x_i+y_j}\right] \;=\; \frac{\prod_{1\le i<j\le n}(x_i-x_j)(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i+y_j)}.$$

In this paper, we extend it to the following determinant identities:

Theorem 1 (Extension of Cauchy's double alternant).

$$M_n := \det_{0\le i,j\le n}\left[w_j+\frac{u_iv_j}{x_i+y_j}\right] \;=\; \Omega\big\{1+\Theta(x,y;u,v,w)\big\}\prod_{k=0}^{n}u_kv_k,$$

where $\Theta(x,y;u,v,w)$ *is given by the following double sum:*

$$\Theta(x,y;u,v,w) \;=\; \sum_{\imath,\jmath=0}^{n}\frac{w_\jmath}{u_\imath v_\jmath(x_\imath+y_\jmath)}\frac{\prod_{i=0}^{n}(x_i+y_\jmath)\prod_{j=0}^{n}(x_\imath+y_j)}{\prod_{i\ne\imath}(x_\imath-x_i)\prod_{j\ne\jmath}(y_\jmath-y_j)}.$$

When $u_0=0$, it reduces easily to the following interesting result.

Proposition 2 (Determinant identity).

$$\det_{0\le i,j\le n}\left[w_j+\frac{u_iv_j}{x_i+y_j}\right]_{u_0=0} = \Omega' v_0\prod_{\kappa=1}^{n}\frac{u_\kappa v_\kappa(y_0-y_\kappa)}{(y_0+x_\kappa)}\sum_{k=0}^{n}\frac{w_k}{v_k}\frac{\prod_{i=1}^{n}(x_i+y_k)}{\prod_{j\ne k}(y_k-y_j)}.$$

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Proof of Theorem 1. Consider the extended square matrix of order $(n+2)\times(2+n)$ given explicitly by

$$\left[\begin{array}{c|cc} 1 & w_j & {\scriptstyle (0\leq j\leq n)} \\ \hline 0 & w_j+\frac{u_iv_j}{x_i+y_j} & {\scriptstyle (0\leq i,j\leq n)} \end{array}\right],$$

whose determinant is obviously equal to the determinant stated in the theorem.

Now subtracting the first row from each other row, we transform the matrix into the following one:

$$\left[\begin{array}{c|cc} 1 & w_j & {\scriptstyle (0\leq j\leq n)} \\ \hline -1 & \frac{u_iv_j}{x_i+y_j} & {\scriptstyle (0\leq i,j\leq n)} \end{array}\right].$$

Then the Laplace expansion formula with respect to the first row gives

$$M_n = \det_{0\leq i,j\leq n}\left[\frac{u_iv_j}{x_i+y_j}\right]+\sum_{\jmath=0}^{n}(-1)^{3+\jmath}w_\jmath \det_{j\neq\jmath}\left[-1 \,\vdots\, \frac{u_iv_j}{x_i+y_j}\right].$$

Expanding further the last determinant with respect to the first column, we get

$$\det_{j\neq\jmath}\left[-1 \,\vdots\, \frac{u_iv_j}{x_i+y_j}\right] = \sum_{\imath=0}^{n}(-1)^{3+\imath}\det_{\substack{i\neq\imath\\ j\neq\jmath}}\left[\frac{u_iv_j}{x_i+y_j}\right],$$

which leads us to the following expression:

$$M_n = \det_{0\leq i,j\leq n}\left[\frac{u_iv_j}{x_i+y_j}\right]+\sum_{\imath,\jmath=0}^{n}(-1)^{\imath+\jmath}w_\jmath \det_{\substack{i\neq\imath\\ j\neq\jmath}}\left[\frac{u_iv_j}{x_i+y_j}\right].$$

Evaluating the last determinant by means of Cauchy's double alternant

$$\det_{\substack{i\neq\imath\\ j\neq\jmath}}\left[\frac{u_iv_j}{x_i+y_j}\right] = \frac{(-1)^{\imath+\jmath}\,\Omega}{u_\imath v_\jmath(x_\imath+y_\jmath)}\frac{\prod_{i=0}^{n}(x_i+y_\jmath)\prod_{j=0}^{n}(x_\imath+y_j)}{\prod_{i\neq\imath}(x_\imath-x_i)\prod_{j\neq\jmath}(y_\jmath-y_j)}\prod_{k=0}^{n}u_kv_k,$$

we find the determinant identity stated in the theorem. □

2. Divided Differences. In order to make the paper self-contained, we review some basic facts about divided differences. The details can be found in Lascoux [4, Chapter 7], where different notation has been introduced. For a complex function $f(y)$ and uneven spaced grid points $\{x_k\}_{k=0}^{n}$, the divided differences with respect to

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y are defined in succession as follows:

$$\begin{aligned}\Delta[x_0,x_1]f(y) &= \frac{f(x_0)-f(x_1)}{x_0-x_1},\\ \Delta[x_0,x_1,x_2]f(y) &= \frac{\Delta[x_0,x_1]f(y)-\Delta[x_1,x_2]f(y)}{x_0-x_2},\\ &\cdots \vdots \cdots\\ \Delta[x_0,x_1,\cdots,x_n]f(y) &= \frac{\Delta[x_0,x_1,\cdots,x_{n-1}]f(y)-\Delta[x_1,x_2,\cdots,x_n]f(y)}{x_0-x_n},\end{aligned}$$

which can also be expressed as

$$\Delta[x_0,x_1,\cdots,x_n]f(y) = \left\{\prod_{k=1}^{n}\Delta[x_k,y]f(y)\right\}\Big|_{y=x_0} \tag{2.1}$$

and the symmetric formula

$$\Delta[x_0,x_1,\cdots,x_n]f(y) \;=\; \sum_{k=0}^{n}\frac{f(x_k)}{\prod_{i\neq k}(x_k-x_i)}. \tag{2.2}$$

For variables $X=\{x_0,x_1,\cdots,x_n\}$, the elementary and complete symmetric functions in X are defined (cf. Macdonald [5, §1.2]), respectively, by

$$\begin{aligned}\mathbf{e}_0(X) \;=\; 1 \text{ and } \mathbf{e}_m(X) \;&=\; \sum_{0\le k_1<k_2<\cdots<k_m\le n} x_{k_1}x_{k_2}\cdots x_{k_m} \quad\text{for } m=1,2,\cdots;\\ \mathbf{h}_0(X) \;=\; 1 \text{ and } \mathbf{h}_m(X) \;&=\; \sum_{0\le k_1\le k_2\le\cdots\le k_m\le n} x_{k_1}x_{k_2}\cdots x_{k_m} \quad\text{for } m=1,2,\cdots.\end{aligned}$$

Then the divided differences on monomials result in complete symmetric functions.

Lemma 3 (Sylvester (1839), cf. Bhatnagar [1] and Chu [2]).

$$\begin{aligned}\Delta[x_0,x_1,\cdots,x_n]y^m \;&=\; \sum_{k=0}^{n}\frac{x_k^m}{\prod_{i\neq k}(x_k-x_i)}\\ &= \begin{cases}0, & m=0,1,\cdots,n-1;\\ \mathbf{h}_{m-n}(x_0,x_1,\cdots,x_n), & m=\;n,\;n+1,\;\cdots;\\ \frac{(-1)^n}{x_0x_1\cdots x_n}\mathbf{h}_{-1-m}\Big(\frac{1}{x_0},\frac{1}{x_1},\cdots,\frac{1}{x_n}\Big), & m=-1,-2,-3,\cdots.\end{cases}\end{aligned}$$

From this lemma, we display a short list of the divided differences for rational

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functions, which will be used in the next section for determinant evaluation.

$$\Delta[X]\prod_{k=1}^{n}(y+u_k) \;=\; 1, \tag{2.3}$$

$$\Delta[X]\prod_{k=0}^{n}(y+u_k) \;=\; \sum_{k=0}^{n}(x_k+u_k), \tag{2.4}$$

$$\Delta[X]\prod_{k=0}^{n+1}(y+u_k) \;=\; \sum_{k=0}^{n}x_k^2+\mathbf{e}_2(X,U), \tag{2.5}$$

$$\Delta[X]\frac{1}{y+v} \;=\; \frac{(-1)^n}{\prod_{k=0}^{n}(v+x_k)}, \tag{2.6}$$

$$\Delta[X]\frac{\prod_{k=1}^{n}(y+u_k)}{y+v} \;=\; \frac{\prod_{k=1}^{n}(v-u_k)}{\prod_{k=0}^{n}(v+x_k)}, \tag{2.7}$$

$$\Delta[X]\frac{\prod_{k=0}^{n}(y+u_k)}{y+v} \;=\; 1-\frac{\prod_{k=0}^{n}(v-u_k)}{\prod_{k=0}^{n}(v+x_k)}, \tag{2.8}$$

$$\Delta[X]\frac{\prod_{k=0}^{n+1}(y+u_k)}{y+v} \;=\; \mathbf{e}_1(U,X)-v+\frac{\prod_{i=0}^{n+1}(v-u_i)}{\prod_{j=0}^{n}(v+x_j)}. \tag{2.9}$$

3. Determinant Identities. Suppose that u_i, v_j and w_j are the three functions given by

$$u_i := u(x_i), \quad v_j := v(y_j) \quad \text{and} \quad w_j := w(y_j).$$

Then we can express the double Θ-sum in Theorem 1 and the sum with respect to k in Proposition 2 in terms of divided differences:

$$\begin{aligned}\Theta(x,y;u,v,w) \;&=\; \Delta_x[x_0,x_1,\cdots,x_n]\Delta_y[y_0,y_1,\cdots,y_n] \\ &\quad\times\;\Big\{\frac{w(y)}{u(x)v(y)}\frac{\prod_{k=0}^{n}(x_k+y)(x+y_k)}{x+y}\Big\},\end{aligned} \tag{3.1}$$

$$\sum_{k=0}^{n}\frac{w_k}{v_k}\frac{\prod_{i=1}^{n}(x_i+y_k)}{\prod_{j\neq k}(y_k-y_j)} \;=\; \Delta_y[y_0,y_1,\cdots,y_n]\Big\{\frac{w(y)}{v(y)}\prod_{k=1}^{n}(x_k+y)\Big\}. \tag{3.2}$$

Applying the divided difference formulae displayed in the last section, we can derive without difficulty the following determinant identities.

Example 1 ($w_k = v_k = 1$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[\frac{u_i+x_i+y_j}{x_i+y_j}\right]_{u_0=0} \;=\; \Omega'\prod_{k=1}^{n}\frac{u_k(y_0-y_k)}{(y_0+x_k)}.$$

Example 2 ($w_k = 1$ and $v_k = v+y_k$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[1+\frac{u_i(v+y_j)}{x_i+y_j}\right]_{u_0=0} \;=\; \Omega'\prod_{k=1}^{n}\frac{u_k(y_0-y_k)(v-x_k)}{(y_0+x_k)}.$$

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EXAMPLE 3 ($w_k = w + y_k$ and $v_k = 1$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[w + y_j + \frac{u_i}{x_i + y_j}\right]_{u_0=0} = \Omega'\Big\{w - x_0 + \sum_{k=0}^{n}(x_k + y_k)\Big\}\prod_{k=1}^{n}\frac{u_k(y_0 - y_k)}{(y_0 + x_k)}.$$

EXAMPLE 4 ($w_k = w + y_k$ and $v_k = v + y_k$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[w + y_j + \frac{u_i(v + y_j)}{x_i + y_j}\right]_{u_0=0} = \Omega'\prod_{k=1}^{n}\frac{u_k(y_0 - y_k)(v - x_k)}{(y_0 + x_k)} \times\left\{(w - v) + \frac{\prod_{i=0}^{n}(v + y_i)}{\prod_{j=1}^{n}(v - x_j)}\right\}.$$

EXAMPLE 5 ($w_k = (w + y_k)(v + y_k)$ and $v_k = 1$ in Proposition 2).

$$\det_{0\le i,j\le n}\left[(w + y_j)(v + y_j) + \frac{u_i}{x_i + y_j}\right]_{u_0=0} = \Omega'\prod_{k=1}^{n}\frac{u_k(y_0 - y_k)}{(y_0 + x_k)} \times\Big\{wv + (w + v)\mathbf{e}_1(X, Y) + \mathbf{e}_2(X, Y) + \sum_{\kappa=0}^{n}y_\kappa^2\Big\}.$$

EXAMPLE 6 ($u_k = v_k = 1$ and $w_k = 1/w$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[\frac{w + x_i + y_j}{x_i + y_j}\right] = \Omega w^n\Big\{w + \mathbf{e}_1(X, Y)\Big\}.$$

EXAMPLE 7 ($u_k = u + x_k$, $v_k = v$ and $w_k = w$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[w + \frac{(u + x_i)v}{x_i + y_j}\right] = \Omega v^n\Big\{v + w - w\prod_{\kappa=0}^{n}\frac{u - y_\kappa}{v + x_\kappa}\Big\}\prod_{\kappa=0}^{n}(u + x_\kappa).$$

EXAMPLE 8 ($u_k = u$, $v_k = v$ and $w_k = w + y_k$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[w + y_j + \frac{uv}{x_i + y_j}\right] = \Omega(uv)^n\Big\{uv + w\mathbf{e}_1(X, Y) + \mathbf{e}_2(X, Y) + \sum_{\kappa=0}^{n}y_\kappa^2\Big\}.$$

EXAMPLE 9 ($u_k = u + x_k$, $v_k = v$ and $w_k = u - y_k$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[u - y_j + \frac{(u + x_i)v}{x_i + y_j}\right] = \Omega v^n\Big\{v + \mathbf{e}_1(X, Y)\prod_{\kappa=0}^{n}\frac{u - y_\kappa}{u + x_\kappa}\Big\}\prod_{\kappa=0}^{n}(u + x_\kappa).$$

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EXAMPLE 10 ($u_k = 1/(u + x_k)$, $v_k = v$ and $w_k = w$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[w(u+x_i)+\frac{v}{x_i+y_j}\right] = \Omega v^n w\Big\{\frac{v}{w}+u\mathbf{e}_1(X,Y)+\mathbf{h}_2(X,Y)-\sum_{\kappa=0}^{n}y_\kappa^2\Big\}.$$

EXAMPLE 11 ($u_k = 1/(u + x_k)$, $v_k = u - y_k$ and $w_k = w$ in Theorem 1).

$$\det_{0\le i,j\le n}\left[(u+x_i)w+\frac{(u-y_j)}{x_i+y_j}\right] = \Omega\Big\{1+w\mathbf{e}_1(X,Y)\prod_{\kappa=0}^{n}\frac{u+x_\kappa}{u-y_\kappa}\Big\}\prod_{\kappa=0}^{n}(u-y_\kappa).$$

EXAMPLE 12 ($u_k = u$, $v_k = v + y_k$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned}\det_{0\le i,j\le n}\left[w+y_j+\frac{u(v+y_j)}{x_i+y_j}\right] &= \Omega u^n\prod_{\kappa=0}^{n}(v+y_\kappa)\\ &\times\Big\{u-v+w+\mathbf{e}_1(X,Y)-(w-v)\prod_{\kappa=0}^{n}\frac{v-x_\kappa}{v+y_\kappa}\Big\}.\end{aligned}$$

EXAMPLE 13 ($u_k = u + x_k$, $v_k = v$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned}\det_{0\le i,j\le n}\left[w+y_j+\frac{(u+x_i)v}{x_i+y_j}\right] &= \Omega v^n(u+v+w)\prod_{\kappa=0}^{n}(u+x_\kappa)\\ &\times\Big\{1-\frac{u+w+\mathbf{e}_1(X,Y)}{u+v+w}\prod_{\kappa=0}^{n}\frac{u-y_\kappa}{u+x_\kappa}\Big\}.\end{aligned}$$

EXAMPLE 14 ($u_k = u + x_k$, $v_k = v$ and $w_k = (w + y_k)(u - y_k)$ in Theorem 1).

$$\begin{aligned}\det_{0\le i,j\le n}\left[(u-y_j)(w+y_j)+\frac{(u+x_i)v}{x_i+y_j}\right] &= \Omega v^n\prod_{\kappa=0}^{n}(u+x_\kappa)\\ &\times\Big\{v+\Big(\mathbf{e}_2(X,Y,w)+\sum_{\kappa=0}^{n}y_\kappa^2\Big)\prod_{\kappa=0}^{n}\frac{u-y_\kappa}{u+x_\kappa}\Big\}.\end{aligned}$$

EXAMPLE 15 ($u_k = 1/(u + x_k)$, $v_k = u - y_k$ and $w_k = w + y_k$ in Theorem 1).

$$\begin{aligned}\det_{0\le i,j\le n}\left[(u+x_i)(w+y_j)+\frac{u-y_j}{x_i+y_j}\right] &= \Omega\Big\{1-u\mathbf{e}_1(X,Y)-\mathbf{h}_2(X,Y)+\sum_{\kappa=0}^{n}y_\kappa^2\\ &+(w+u)\mathbf{e}_1(X,Y)\prod_{\kappa=0}^{n}\frac{u+x_\kappa}{u-y_\kappa}\Big\}\prod_{\kappa=0}^{n}(u-y_\kappa).\end{aligned}$$

In order to illustrate the method of proof, we prove the determinant identity displayed in Example 3 in detail. According to Proposition 2, we need only to evaluate

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(3.2) for $v(y) = 1$ and $w(y) = w+y$. In this case, the corresponding divided differences (3.2) can be evaluated by means of (2.4) as follows:

$$\begin{aligned}&\Delta_y[y_0, y_1, \cdots, y_n]\Big\{\frac{w(y)}{v(y)}\prod_{k=1}^{n}(x_k+y)\Big\}\\ &= \Delta_y[y_0, y_1, \cdots, y_n]\Big\{(w+y)\prod_{k=1}^{n}(x_k+y)\Big\}\\ &= w - x_0 + \sum_{k=0}^{n}(x_k+y_k).\end{aligned}$$

Then the determinant identity in Example 3 follows immediately.

In particular, Example 3 implies another interesting determinant identity. Reformulating the general entry of the matrix

$$\frac{(a+u_i+w_j)(c+v_i+w_j)}{c+u_i+v_i+w_j} \;=\; a+w_j+\frac{u_i(c-a+v_i)}{c+u_i+v_i+w_j},$$

we derive from Example 3 the following curious formula.

Proposition 4 (Determinant identity).

$$\begin{aligned}\det_{0\le i,j\le n}\left[\frac{(a+u_i+w_j)(c+v_i+w_j)}{c+u_i+v_i+w_j}\right]_{u_0=0} &= \frac{\prod_{1\le i<j\le n}(u_i+v_i-u_j-v_j)(w_i-w_j)}{\prod_{1\le i,j\le n}(c+u_i+v_i+w_j)}\\ &\times\Big\{a+cn+w_0+\sum_{k=1}^{n}(u_k+v_k+w_k)\Big\}\prod_{k=1}^{n}\frac{u_k(c-a+v_k)(w_0-w_k)}{c+u_k+v_k+w_0}.\end{aligned}$$

The very special case $u_k = ku$ of this identity reduces to the determinant evaluation.

Corollary 5 (Krattenthaler [3, Eq 5.3]).

$$\begin{aligned}\det_{0\le i,j\le n}\left[\frac{(a+ui+w_j)(c+v_i+w_j)}{c+ui+v_i+w_j}\right] &= \frac{\prod_{1\le i<j\le n}(ui-uj+v_i-v_j)(w_i-w_j)}{\prod_{1\le i,j\le n}(c+ui+v_i+w_j)}\\ &\times n!\,u^n\Big\{a+cn+u\tbinom{n+1}{2}+w_0+\sum_{k=1}^{n}(v_k+w_k)\Big\}\prod_{k=1}^{n}\frac{(c-a+v_k)(w_0-w_k)}{c+uk+v_k+w_0}.\end{aligned}$$

Krattenthaler [3, Eq 5.3] discovered this identity by means of the condensation method, which has been the author's primary motivation for this work.

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