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SOLUTION OF LINEAR MATRIX EQUATIONS IN A *CONGRUENCE CLASS [§]

ROGER A. HORN*, VLADIMIR V. SERGEICHUK[†], AND NAOMI SHAKED-MONDERER[‡]

Abstract. The possible *congruence classes of a square solution to the real or complex linear matrix equation $AX = B$ are determined. The solution is elementary and self contained, and includes several known results as special cases, e.g., X is Hermitian or positive semidefinite, and X is real with positive definite symmetric part.

Key words. Linear matrix equations, *Congruence, Positive definite matrix, Positive semidefinite matrix, Hermitian part, Symmetric part.

AMS subject classifications. 15A04, 15A06, 15A21, 15A57, 15A63.

1. Introduction. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, let $\mathbb{F}^{p\times q}$ denote the vector space (over $\mathbb{F}$) of p-by-q matrices with entries in $\mathbb{F}$, and let $A, B \in \mathbb{F}^{k\times n}$ be given. We are interested in the linear matrix equation $AX = B$, which we assume to be *consistent*: $\operatorname{rank} A = \operatorname{rank} [A\ B]$.

For a given $S \in \mathbb{F}^{n\times n}$ let $S^* \equiv \bar{S}^T$ denote the conjugate transpose, so $S^* = S^T$ if $\mathbb{F} = \mathbb{R}$. Matrices $X, Y \in \mathbb{F}^{n\times n}$ are in the same **congruence class* if there is a nonsingular $S \in \mathbb{F}^{n\times n}$ such that $X = S^*YS$. The *Hermitian part* of $X \in \mathbb{F}^{n\times n}$ is $H(X) \equiv (X + X^*)/2$; when $\mathbb{F} = \mathbb{R}$, $H(X)$ is also called the *symmetric part* of X. Let I_p (respectively, 0_p) denote the p-by-p identity (respectively, zero) matrix.

When does $AX = B$ have a solution X in a given *congruence class? Special cases of this question involving positive semidefinite or Hermitian solutions were investigated in [1]; [2] asked an equivalent question: If $\{\xi_1, \dots, \xi_k\}$ and $\{\eta_1, \dots, \eta_k\}$ are given sets of real or complex vectors of the same size, when is there a Hermitian or positive definite matrix K such that $K\xi_i = \eta_i$ for $i = 1, \dots, k$?

2. Solution of $AX = B$ in a given *congruence class. Our main result is the following theorem.

THEOREM 1. *Let $A, B \in \mathbb{F}^{k\times n}$ be given, and suppose the linear matrix equation $AX = B$ is consistent. Let $r = \operatorname{rank} A$, and let $M = BA^*$. Then there are matrices $N \in \mathbb{F}^{r\times r}$ and $E \in \mathbb{F}^{r\times(n-r)}$ such that:*
(a) *M is *congruent to $N \oplus 0_{k-r}$.*
(b) *For each given $F \in \mathbb{F}^{(n-r)\times r}$ and $G \in \mathbb{F}^{(n-r)\times(n-r)}$ there is an $X \in \mathbb{F}^{n\times n}$ such that $AX = B$ and X is *congruent to*

$$\begin{bmatrix} N & E \\ F & G \end{bmatrix}.$$

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*Department of Mathematics, University of Utah, Salt Lake City, Utah 84103, USA (rhorn@math.utah.edu).
[†]Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine (sergeich@imath.kiev.ua).
[‡]Emek Yezreel College, Emek Yezreel 19300, Israel (nomi@tx.technion.ac.il).

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(c) *If* $\operatorname{rank} M = \operatorname{rank} B$*, then for each given* $C \in \mathbb{F}^{(n-r)\times(n-r)}$ *there is an* $X \in \mathbb{F}^{n\times n}$ *such that* $AX = B$ *and* X *is *congruent to* $N \oplus C$ *over* $\mathbb{F}$*.*

Proof. Using the singular value decomposition, one can construct a unitary $U \in \mathbb{F}^{n\times n}$ and a nonsingular $R \in \mathbb{F}^{k\times k}$ such that

$$RAU = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Consistency ensures that $B = AC$ for some $C \in \mathbb{F}^{n\times n}$, so

$$RBU = (RAU)(U^*CU) = \begin{bmatrix} N & E \\ 0 & 0 \end{bmatrix},$$

in which $N \in \mathbb{F}^{r\times r}$. A matrix $X = U\mathcal{X}U^*$ satisfies $AX = B$ if and only if $\mathcal{X} \in \mathbb{F}^{n\times n}$ has the property that $(RAU)\,\mathcal{X} = RBU$ if and only if it has the form

$$\mathcal{X} = \begin{bmatrix} N & E \\ F & G \end{bmatrix}, \qquad G \in \mathbb{F}^{(n-r)\times(n-r)}; \tag{1}$$

the entries of F and G may be any elements of $\mathbb{F}$. Since $RMR^* = RBU(RAU)^* = N \oplus 0_{k-r}$, M is *congruent to $N \oplus 0_{k-r}$.

We have

$$\operatorname{rank} M = \operatorname{rank} N \le \operatorname{rank}\,[N \;\, E] = \operatorname{rank} B,$$

so $\operatorname{rank} M = \operatorname{rank} B$ if and only if $\operatorname{rank} B = \operatorname{rank} N$ if and only if every column of E is in the range of N, that is, if and only if there is a matrix Z over $\mathbb{F}$ such that $E = NZ$. If $\operatorname{rank} M = \operatorname{rank} B$, we may take $X = U\mathcal{X}U^*$, in which

$$\begin{aligned}\mathcal{X} &= \begin{bmatrix} N & NZ \\ Z^*N & Z^*NZ + C \end{bmatrix} \\ &= \begin{bmatrix} I_r & Z \\ 0 & I_{n-r} \end{bmatrix}^* \begin{bmatrix} N & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I_r & Z \\ 0 & I_{n-r} \end{bmatrix}.\end{aligned}$$

Then $AX = B$ and X is *congruent to $N \oplus C$ over $\mathbb{F}$. ∎

Several known results follow easily from our theorem. In each of the following corollaries, we use the notation of the theorem and assume that $AX = B$ is consistent.

Corollary 2 ([2, Theorem 2.1]). *Suppose* $\operatorname{rank} A = k$*. There is a Hermitian positive definite matrix* X *over* $\mathbb{F}$ *such that* $AX = B$ *if and only if* M *is Hermitian positive definite.*

Proof. The rank condition implies that M is *congruent to N, so N is Hermitian positive definite if M is. The theorem ensures that there is a matrix X over $\mathbb{F}$ such that $AX = B$ and X is *congruent to $N \oplus I_{n-k}$ over $\mathbb{F}$, so this X is Hermitian positive definite. Conversely, if X is Hermitian positive definite and $AX = B$, then B and $AX^{1/2}$ have full row rank, so $M = BA^* = AXA^* = (AX^{1/2})(AX^{1/2})^*$ is Hermitian positive definite. ∎

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COROLLARY 3 ([1, Theorem 2.2]). *There is a Hermitian positive semidefinite matrix X over $\mathbb{F}$ such that $AX = B$ if and only if* $\operatorname{rank} M = \operatorname{rank} B$ *and M is Hermitian positive semidefinite.*

Proof. If M is Hermitian positive semidefinite, then so is N. For any Hermitian positive semidefinite $C \in \mathbb{F}^{(n-r)\times(n-r)}$, the theorem ensures that there is a matrix X over $\mathbb{F}$ such that $AX = B$ and X is *congruent to $N \oplus C$ over $\mathbb{F}$; such an X is Hermitian positive semidefinite. Conversely, if X is Hermitian positive semidefinite and $AX = B$, then $M = BA^* = AXA^*$ is Hermitian positive semidefinite, and $\operatorname{rank} M = \operatorname{rank}(AX^{1/2})(AX^{1/2})^* = \operatorname{rank}(AX^{1/2}) = \operatorname{rank} AX = \operatorname{rank} B$. ☐

The real case of part (b) in the following corollary was proved in [2, Theorem 2.1] with the restriction that A has full row rank.

COROLLARY 4. (a) *There is a square matrix X over $\mathbb{F}$ such that $AX = B$ and $H(X)$ is positive semidefinite if and only if $H(M)$ is positive semidefinite.*
(b) *There is a square matrix X over $\mathbb{F}$ such that $AX = B$ and $H(X)$ is positive definite if and only if $H(M)$ is positive semidefinite and* $\operatorname{rank} H(M) = \operatorname{rank} A$.

Proof. Necessity in both cases follows from observing that $H(M) = AH(X)A^* = (AH(X)^{1/2})(AH(X)^{1/2})^*$. Thus, $\operatorname{rank} H(M) = \operatorname{rank}(AH(X)^{1/2}) = \operatorname{rank} A$ if $H(X)$ is nonsingular.

Conversely, $H(M)$ is *congruent to $H(N) \oplus 0_{k-r}$ so $H(N)$ is positive semidefinite and $\operatorname{rank} H(N) = \operatorname{rank} H(M)$. Take $F = -E^*$ and $G = I_{n-r}$ in (1), so that $H(X)$ is *congruent to $H(\mathcal{X}) = H(N) \oplus I_{n-r}$. For this X, $AX = B$, $H(X)$ is positive semidefinite, and $H(X)$ is positive definite if $\operatorname{rank} H(M) = r$. ☐

Part (a) of the following corollary was proved in [1, Theorem 2.1].

COROLLARY 5. (a) *There is a square matrix X over $\mathbb{F}$ such that $AX = B$ and X is Hermitian if and only if M is Hermitian.*
(b) *There is a square matrix X over $\mathbb{F}$ such that $AX = B$ and X is skew-Hermitian if and only if M is skew-Hermitian.*

Proof. Necessity in both cases follows from observing that $M = AXA^*$. Conversely, choosing $G = 0$ and $F = \pm E^*$ in (1) proves sufficiency. ☐

The inertia of a Hermitian matrix H is $\operatorname{In} H = (\pi(H), \nu(H), \zeta(H))$, in which $\pi(H)$ is the number of positive eigenvalues of H, $\nu(H)$ is the number of negative eigenvalues, and $\zeta(H)$ is the nullity. Since we know the general parametric form (1), the preceding corollaries can be made more specific in the Hermitian cases by describing the inertias that are possible for X given the inertia of M. Our final corollary is an example of such a result.

COROLLARY 6. *Suppose M is Hermitian and* $\operatorname{rank} M = \operatorname{rank} B$. *Then X may be chosen to be Hermitian with inertia (α, β, γ) if and only if α, β, and γ are nonnegative integers such that $\alpha + \beta + \gamma = n$ and* $(\alpha, \beta, \gamma) \geq \operatorname{In} M - (0, 0, k - r)$.

Proof. Since $\operatorname{rank} M = \operatorname{rank} B$, the theorem ensures for any $C \in \mathbb{F}^{(n-r)\times(n-r)}$ the existence of an X that is *congruent over $\mathbb{F}$ to $N \oplus C$. Take C to be Hermitian, in which case $\operatorname{In} X = \operatorname{In} N + \operatorname{In} C \geq \operatorname{In} M - (0, 0, k - r)$, and all permitted inertias can be achieved by a suitable choice of C. ☐

If the rank condition in the preceding corollary is not satisfied, there may be

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further restrictions on the possible set of inertias of A. Consider the example $A = [1\ 0]$, $B = [0\ 1]$, $M = [0]$. Any Hermitian solution to $AX = B$ must have the form

$$X = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$$

for some real $t \in \mathbb{F}$, and any such matrix has inertia $(1, 1, 0) \not\geq (0, 0, 1)$.

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