SOLUTION OF LINEAR MATRIX EQUATIONS IN A
*CONGRUENCE CLASS

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Abstract. The possible *congruence classes of a square solution to the real or complex linear matrix equation \( AX = B \) are determined. The solution is elementary and self contained, and includes several known results as special cases, e.g., \( X \) is Hermitian or positive semidefinite, and \( X \) is real with positive definite symmetric part.

Key words. Linear matrix equations, *Congruence, Positive definite matrix, Positive semidefinite matrix, Hermitian part, Symmetric part.

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1. Introduction. Let \( F \) be either \( \mathbb{R} \) or \( \mathbb{C} \), let \( F^{p \times q} \) denote the vector space (over \( F \)) of \( p \)-by-\( q \) matrices with entries in \( F \), and let \( A, B \in F^{k \times n} \) be given. We are interested in the linear matrix equation \( AX = B \), which we assume to be consistent: \( \text{rank } A = \text{rank } [AB] \).

For a given \( S \in F^{n \times n} \) let \( S^* \equiv \bar{S}^T \) denote the conjugate transpose, so \( S^* = S^T \) if \( F = \mathbb{R} \). Matrices \( X, Y \in F^{n \times n} \) are in the same *congruence class if there is a nonsingular \( S \in F^{n \times n} \) such that \( X = S^*YS \). The Hermitian part of \( X \in F^{n \times n} \) is \( H(X) \equiv (X + X^*)/2 \); when \( F = \mathbb{R} \), \( H(X) \) is also called the symmetric part of \( X \). Let \( I_p \) (respectively, \( 0_p \)) denote the \( p \)-by-\( p \) identity (respectively, zero) matrix.

When does \( AX = B \) have a solution \( X \) in a given *congruence class? Special cases of this question involving positive semidefinite or Hermitian solutions were investigated in [1]; [2] asked an equivalent question: If \( \{\xi_1, \ldots, \xi_k\} \) and \( \{\eta_1, \ldots, \eta_k\} \) are given sets of real or complex vectors of the same size, when is there a Hermitian or positive definite matrix \( K \) such that \( K\xi_i = \eta_i \) for \( i = 1, \ldots, k \)?

2. Solution of \( AX = B \) in a given *congruence class. Our main result is the following theorem.

**Theorem 1.** Let \( A, B \in F^{k \times n} \) be given, and suppose the linear matrix equation \( AX = B \) is consistent. Let \( r = \text{rank } A \), and let \( M = BA^* \). Then there are matrices \( N \in F^{r \times r} \) and \( E \in F^{r \times (n-r)} \) such that:

(a) \( M \) is *congruent to \( N \oplus 0_{k-r} \).

(b) For each given \( F \in F^{(n-r) \times r} \) and \( G \in F^{(n-r) \times (n-r)} \) there is an \( X \in F^{n \times n} \) such that \( AX = B \) and \( X \) is *congruent to

\[
\begin{bmatrix}
N & E \\
F & G
\end{bmatrix}.
\]
(c) If rank \( M = \text{rank} \, B \), then for each given \( C \in \mathbb{F}^{(n-r)\times(n-r)} \) there is an \( X \in \mathbb{F}^{n\times n} \) such that \( AX = B \) and \( X \) is *congruent to \( N \oplus C \) over \( \mathbb{F} \).

Proof. Using the singular value decomposition, one can construct a unitary \( U \in \mathbb{F}^{n\times n} \) and a nonsingular \( R \in \mathbb{F}^{k\times k} \) such that
\[
RAU = \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}.
\]
Consistency ensures that \( B = AC \) for some \( C \in \mathbb{F}^{n\times n} \), so
\[
RBU = (RAU)(U^*CU) = \begin{bmatrix}
N & E \\
F & G
\end{bmatrix},
\]
in which \( N \in \mathbb{F}^{r\times r} \). A matrix \( X = UXU^* \) satisfies \( AX = B \) if and only if \( X \in \mathbb{F}^{n\times n} \) has the property that \( (RAU)X = RBU \) if and only if it has the form
\[
X = \begin{bmatrix}
N & E \\
F & G
\end{bmatrix}, \quad G \in \mathbb{F}^{(n-r)\times(n-r)};
\]
the entries of \( F \) and \( G \) may be any elements of \( \mathbb{F} \). Since \( RMR^* = RBU(RAU)^* = N \oplus 0_{k-r} \), \( M \) is *congruent to \( N \oplus 0_{k-r} \).

We have
\[
\text{rank } M = \text{rank } N \leq \text{rank } [N \ E] = \text{rank } B,
\]
so rank \( M = \text{rank } B \) if and only if rank \( B = \text{rank } N \) if and only if every column of \( E \) is in the range of \( N \), that is, if and only if there is a matrix \( Z \) over \( \mathbb{F} \) such that \( E = NZ \). If rank \( M = \text{rank } B \), we may take \( X = UXU^* \), in which
\[
X = \begin{bmatrix}
N & NZ \\
Z^*N & Z^*NZ + C
\end{bmatrix} = \begin{bmatrix}
I_r & Z \\
0 & I_{n-r}
\end{bmatrix}^* \begin{bmatrix}
N & 0 \\
0 & C
\end{bmatrix} \begin{bmatrix}
I_r & Z \\
0 & I_{n-r}
\end{bmatrix}.
\]
Then \( AX = B \) and \( X \) is *congruent to \( N \oplus C \) over \( \mathbb{F} \).

Several known results follow easily from our theorem. In each of the following corollaries, we use the notation of the theorem and assume that \( AX = B \) is consistent.

**Corollary 2 ([2, Theorem 2.1]).** Suppose \( \text{rank } A = k \). There is a Hermitian positive definite matrix \( X \) over \( \mathbb{F} \) such that \( AX = B \) if and only if \( M \) is Hermitian positive definite.

Proof. The rank condition implies that \( M \) is *congruent to \( N \), so \( N \) is Hermitian positive definite if \( M \) is. The theorem ensures that there is a matrix \( X \) over \( \mathbb{F} \) such that \( AX = B \) and \( X \) is *congruent to \( N \oplus I_{n-k} \) over \( \mathbb{F} \), so this \( X \) is Hermitian positive definite. Conversely, if \( X \) is Hermitian positive definite and \( AX = B \), then \( B \) and \( AX^{1/2} \) have full row rank, so \( M = BA^* = AXA^* = (AX^{1/2})(AX^{1/2})^* \) is Hermitian positive definite.
Corollary 3 ([1, Theorem 2.2]). There is a Hermitian positive semidefinite matrix $X$ over $\mathbb{F}$ such that $AX = B$ if and only if $\text{rank} M = \text{rank} B$ and $M$ is Hermitian positive semidefinite.

Proof. If $M$ is Hermitian positive semidefinite, then so is $N$. For any Hermitian positive semidefinite $C \in \mathbb{F}^{(n-r) \times (n-r)}$, the theorem ensures that there is a matrix $X$ over $\mathbb{F}$ such that $AX = B$ and $X$ is congruent to $N \oplus C$ over $\mathbb{F}$; such an $X$ is Hermitian positive semidefinite. Conversely, if $X$ is Hermitian positive semidefinite and $AX = B$, then $M = BA^* = AXA^*$ is Hermitian positive semidefinite, and $\text{rank} M = \text{rank} (AX^{1/2})(AX^{1/2})^* = \text{rank} (AX^{1/2}) = \text{rank} AX = \text{rank} B$. \[ \Box \]

The real case of part (b) in the following corollary was proved in [2, Theorem 2.1] with the restriction that $A$ has full row rank.

Corollary 4. (a) There is a square matrix $X$ over $\mathbb{F}$ such that $AX = B$ and $H(X)$ is positive semidefinite if and only if $H(M)$ is positive semidefinite.

(b) There is a square matrix $X$ over $\mathbb{F}$ such that $AX = B$ and $H(X)$ is positive definite if and only if $H(M)$ is positive semidefinite and $\text{rank} H(M) = \text{rank} A$.

Proof. Necessity in both cases follows from observing that $H(M) = AH(X)A^* = (AH(X)^{1/2})(AH(X)^{1/2})^*$. Thus, $\text{rank} H(M) = \text{rank} (AH(X)^{1/2}) = \text{rank} A$ if $H(X)$ is nonsingular.

Conversely, $H(M)$ is congruent to $H(N) \oplus 0_{k-r}$, so $H(N)$ is positive semidefinite and $\text{rank} H(N) = \text{rank} H(M)$. Take $F = -E^*$ and $G = I_{n-r}$ in (1), so that $H(X)$ is congruent to $H(X) = H(N) \oplus I_{n-r}$. For this $X$, $AX = B$, $H(X)$ is positive semidefinite, and $H(X)$ is positive definite if $\text{rank} H(M) = r$. \[ \Box \]

Part (a) of the following corollary was proved in [1, Theorem 2.1].

Corollary 5. (a) There is a square matrix $X$ over $\mathbb{F}$ such that $AX = B$ and $X$ is Hermitian if and only if $M$ is Hermitian.

(b) There is a square matrix $X$ over $\mathbb{F}$ such that $AX = B$ and $X$ is skew-Hermitian if and only if $M$ is skew-Hermitian.

Proof. Necessity in both cases follows from observing that $M = AXA^*$. Conversely, choosing $G = 0$ and $F = \pm E^*$ in (1) proves sufficiency. \[ \Box \]

The inertia of a Hermitian matrix $H$ is $\text{In} H = (\pi(H), \nu(H), \zeta(H))$, in which $\pi(H)$ is the number of positive eigenvalues of $H$, $\nu(H)$ is the number of negative eigenvalues, and $\zeta(H)$ is the nullity. Since we know the general parametric form (1), the preceding corollaries can be made more specific in the Hermitian cases by describing the inertias that are possible for $X$ given the inertia of $M$. Our final corollary is an example of such a result.

Corollary 6. Suppose $M$ is Hermitian and $\text{rank} M = \text{rank} B$. Then $X$ may be chosen to be Hermitian with inertia $(\alpha, \beta, \gamma)$ if and only if $\alpha, \beta$, and $\gamma$ are nonnegative integers such that $\alpha + \beta + \gamma = n$ and $(\alpha, \beta, \gamma) \geq \text{In} M - (0, 0, k - r)$.

Proof. Since $\text{rank} M = \text{rank} B$, the theorem ensures for any $C \in \mathbb{F}^{(n-r) \times (n-r)}$ the existence of an $X$ that is congruent over $\mathbb{F}$ to $N \oplus C$. Take $C$ to be Hermitian, in which case $\text{In} X = \text{In} N + \text{In} C \geq \text{In} M - (0, 0, k - r)$, and all permitted inertias can be achieved by a suitable choice of $C$. \[ \Box \]

If the rank condition in the preceding corollary is not satisfied, there may be
further restrictions on the possible set of inertias of $A$. Consider the example $A = [1 \ 0]$, $B = [0 \ 1]$, $M = [0]$. Any Hermitian solution to $AX = B$ must have the form

$$X = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$$

for some real $t \in F$, and any such matrix has inertia $(1, 1, 0) \triangleright (0, 0, 1)$.

REFERENCES