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POSITIVE ENTRIES OF STABLE MATRICES*

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Abstract. The question of how many elements of a real positive stable matrix must be positive is investigated. It is shown that any real stable matrix of order greater than 1 has at least two positive entries. Furthermore, for every stable spectrum of cardinality greater than 1 there exists a real matrix with that spectrum with exactly two positive elements, where all other elements of the matrix can be chosen to be negative.

Key words. Stable matrix, Companion matrix, Positive elementary symmetric functions.

AMS subject classifications. 15A18, 15A29.

1. Introduction. For a square complex matrix A let $\sigma(A)$ be the spectrum of A, that is, the set of eigenvalues of A listed with their multiplicities. Recall that a (multi) set of complex numbers is called *(positive) stable* if all the elements of the set have positive real parts, and that a square complex matrix A is called *stable* if $\sigma(A)$ is stable. In this paper we investigate the question of how many elements of a real stable matrix must be positive.

We first show that a stable real matrix A has either positive diagonal elements or it has at least one positive diagonal element and one positive off-diagonal element. We then show that for any stable n-tuple ζ of complex numbers, $n > 1$, such that ζ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix A with exactly two positive entries such that $\sigma(A) = \zeta$.

The stable $n \times n$ matrix with exactly two positive entries, whose existence is proven in Section 2, has $(n-1)^2$ zeros in it. In Section 3 we prove that for any stable n-tuple ζ of complex numbers, $n > 1$, such that ζ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix A with two positive entries and all other entries negative such that $\sigma(A) = \zeta$.

In Section 4 we suggest some alternative approaches to obtain the results of Section 2.

2. Positive entries of stable matrices. Our aim in this section is to show that for any stable n-tuple ζ of complex numbers, $n > 1$, consisting of real numbers and conjugate pairs, there exists a real stable $n \times n$ matrix A with exactly two positive entries such that $\sigma(A) = \zeta$. We shall first show that every real stable matrix of order

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greater than 1 has at least two positive elements. In fact we show more than that, that is, that for a stable real matrix A either all diagonal elements of A are positive or A must have at least one positive entry on the main diagonal and one off the main diagonal.

NOTATION 2.1. For an n-tuple $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $s_1(\zeta), \ldots, s_n(\zeta)$ the elementary symmetric functions of ζ, that is,

$$s_k(\zeta) = \sum_{1 \le i_1 < \ldots < i_k \le n} \zeta_{i_1} \cdot \ldots \cdot \zeta_{i_k}, \qquad k = 1, \ldots, n.$$

Also, we let $s_0(\zeta) = 1$ and $s_k(\zeta) = 0$ whenever $k > n$ or $k < 0$. We say that ζ has positive elementary symmetric functions whenever $s_k(\zeta) > 0$, $k = 1, \ldots, n$.

LEMMA 2.2. *Let $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ be an n-tuple of complex numbers with positive elementary symmetric functions. Then ζ contains no nonpositive real numbers.*

Proof. Note that ζ has positive elementary symmetric functions if and only if the polynomial $p(x) = \prod_{i=1}^{n}(x + \zeta_i)$ has positive coefficients. It follows that $p(x)$ cannot have nonnegative roots, implying that none of the ζ_i's is a nonpositive real number. □

NOTATION 2.3. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, the fields of real and complex numbers respectively, we denote by $\mathrm{M_n}(\mathbb{F})$ the algebra of $n \times n$ matrices with entries in $\mathbb{F}$. For $A = (a_{ij})_1^n \in \mathrm{M_n}(\mathbb{F})$ we denote by $\mathrm{tr}\, A$ the trace of A, that is, the sum $\sum_{i=1}^{n} a_{ii}$.

PROPOSITION 2.4. *Let $A = (a_{ij})_1^n \in \mathrm{M_n}(\mathbb{R})$, and assume that $\sigma(A)$ has positive elementary symmetric functions. Then either all the diagonal elements of A are positive or A has at least one positive diagonal element and one positive off-diagonal element.*

Proof. As is well known, the trace of A is equal to $s_1(\sigma(A))$, and so we have $\sum_{i=1}^{n} a_{ii} > 0$, and it follows that at least one diagonal element of A is positive. Assume that all off-diagonal elements of A are nonpositive. Such a real matrix is called a *Z-matrix*. Since the elementary symmetric functions of $\sigma(A)$ are positive, it follows by Lemma 2.2 that A has no nonpositive real eigenvalues. Since a Z-matrix has no nonpositive real eigenvalues if and only if all its principal minors are positive, e.g., [1, Theorem (6.2.3), page 134], it follows that all the diagonal elements of A are positive. □

NOTATION 2.5. For an n-tuple $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $\overline{\zeta}$ be the n-tuple $\{\overline{\zeta}_1, \ldots, \overline{\zeta}_n\}$. We say that $\overline{\zeta} = \zeta$ whenever the two n-tuples $\overline{\zeta}$ and ζ are identical sets.

Note that $\overline{\zeta} = \zeta$ if and only if all elementary symmetric functions of ζ are real.

The following result is well known, and we provide a proof for the sake of completeness.

PROPOSITION 2.6. *Let ζ be a stable n-tuple of complex numbers such that $\overline{\zeta} = \zeta$. Then ζ has positive elementary symmetric functions.*

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Proof. We prove our claim by induction on n. For $n = 1, 2$ the result is trivial. Assume that the result holds for $n \leq m$ where $m \geq 2$, and let $n = m + 1$. Assume first that ζ contains a positive number λ, and let ζ' be the $(n-1)$-tuple obtained by eliminating λ from ζ. Note that ζ' is stable and $\overline{\zeta'} = \zeta'$. By the inductive assumption we have $s_k(\zeta') > 0$, $k = 1, \ldots, n-1$, and it follows that

$$s_k(\zeta) = s_k(\zeta') + \lambda s_{k-1}(\zeta') > 0, \quad k = 1, \ldots, n.$$

If ζ does not contain a positive number then it contains a conjugate pair $\{\lambda, \overline{\lambda}\}$, where $\mathrm{Re}(\lambda) > 0$. Let ζ'' be the $(n-2)$-tuple obtained by eliminating λ and $\overline{\lambda}$ from ζ. Note that the ζ'' is stable and $\overline{\zeta''} = \zeta''$. By the inductive assumption we have $s_k(\zeta'') > 0$, $k = 1, \ldots, n-2$, and it follows that

$$s_k(\zeta) = s_k(\zeta'') + 2\mathrm{Re}(\lambda) s_{k-1}(\zeta'') > 0 + |\lambda|^2 s_{k-2}(\zeta'') > 0, \quad k = 1, \ldots, n,$$

proving our claim. $\square$

It is easy to show that the converse of Proposition 2.6 holds when $n \leq 2$. However, the converse does not hold for a larger n, as is demonstrated by the nonstable triple $\zeta = \{\ 3\ ,\ -1 + 3i\ ,\ -1 - 3i\ \}$, whose elementary symmetric functions are positive.

As a corollary of Propositions 2.4 and 2.6 we obtain:

Corollary 2.7. *Let A be a stable real square matrix. Then either all the diagonal elements of A are positive or A has at least one positive diagonal element and one positive off-diagonal element.*

In order to prove the existence of a real stable $n \times n$ matrix A with exactly two positive entries, we introduce:

Notation 2.8. Let n be a positive integer. For an n-tuple ζ of complex numbers we denote by $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ the matrices

$$C_1(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-1} s_n(\zeta) \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-2} s_{n-1}(\zeta) \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-3} s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix},$$

$$C_2(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & s_n(\zeta) \\ -1 & 0 & 0 & \ldots & 0 & 0 & 0 & s_{n-1}(\zeta) \\ 0 & -1 & 0 & \ldots & 0 & 0 & 0 & s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 0 & s_2(\zeta) \\ 0 & 0 & 0 & \ldots & 0 & 0 & -1 & s_1(\zeta) \end{pmatrix},$$

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$$C_3(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & -s_n(\zeta) \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & -s_{n-1}(\zeta) \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & -s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix}.$$

Recall that $A \in \mathrm{M_n}(\mathbb{C})$ is called nonderogatory if for every eigenvalue λ of A the Jordan canonical form of A has exactly one Jordan block corresponding to λ. Equivalently, the minimal polynomial of A is equal to the characteristic polynomial of A.

LEMMA 2.9. *Let n be a positive integer, $n > 1$, and let $\zeta = \{\zeta_1, \dots, \zeta_n\}$ be an n-tuple of complex numbers. Then the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar, are nonderogatory and share the spectrum ζ.*

Proof. The matrix $C_1(\zeta)$ is the companion matrix of the polynomial $q(x) = \prod_{i=1}^n (x - \zeta_i)$. Hence $\sigma(C_1(\zeta)) = \zeta$ and $C_1(\zeta)$ is nonderogatory. Clearly

$$C_2(\zeta) = D_1 C_1(\zeta) D_1, \quad \text{where} \quad D_1 = \mathrm{diag}((-1)^1, (-1)^2, \dots, (-1)^n),$$

and

$$C_3(\zeta) = D_2 C_2(\zeta) D_2, \quad \text{where} \quad D_2 = \mathrm{diag}(1, 1, \dots, 1, -1).$$

Our claim follows. $\square$

In view of Lemma 2.9, the claim of Proposition 2.6 on $C_3(\zeta)$ yields the following main result of this section.

THEOREM 2.10. *Let n be a positive integer, $n > 1$, and let ζ be an n-tuple of complex numbers such that $\overline{\zeta} = \zeta$. If ζ has positive elementary symmetric functions then there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ such that $\sigma(A) = \zeta$ and A has one positive diagonal entry and one positive off-diagonal entry, while all other entries of A are nonpositive. In particular, every nonderogatory stable matrix $A \in \mathrm{M_n}(\mathbb{R})$ is similar to a real $n \times n$ matrix which has exactly two positive entries.*

3. Eliminating the zero entries. The proof of Theorem 2.10 uses the matrix $C_3(\zeta)$ which has $(n-1)^2$ zero entries. The aim of this section is to strengthen Theorem 2.10 by replacing $C_3(\zeta)$ with a real matrix A, having exactly two positive entries, all other entries being negative and $\sigma(A) = \zeta$.

We start with a weaker result, which one gets easily using perturbation techniques. Let $A \in \mathrm{M_n}(\mathbb{R})$ and let $\|\cdot\| : \mathrm{M_n}(\mathbb{R}) \to [0, \infty)$ be the l_2 operator norm. Since the eigenvalues of a A depend continuously on the entries of the A, it follows that if $\sigma(A)$ has positive elementary symmetric functions, then for $\varepsilon > 0$ sufficiently small, every matrix $\tilde{A} \in \mathrm{M_n}(\mathbb{R})$ with $\|\tilde{A} - A\| < \varepsilon$ has a spectrum $\sigma(\tilde{A})$ with positive elementary symmetric functions. Also, if A is stable then for $\varepsilon > 0$ sufficiently small, every matrix

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$\tilde{A} \in \mathrm{M_n}(\mathbb{R})$ with $\|\tilde{A} - A\| < \varepsilon$ is stable. Consequently, it follows immediately from Theorem 2.10 that

Corollary 3.1. *For a positive integer n, $n > 1$, there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ such that $\sigma(A)$ has positive elementary symmetric functions and A has one positive diagonal entry and one positive off-diagonal entry, while all other entries of A are negative. Furthermore, the matrix A can be chosen to be stable.*

In the rest of this section we prove that one can find such a matrix A with any prescribed stable spectrum.

Lemma 3.2. *Let n be a positive integer, $n > 1$, let ζ be an n-tuple of complex numbers such that $\overline{\zeta} = \zeta$, and assume that ζ has positive elementary symmetric functions. Suppose that there exists $X \in \mathrm{M_n}(\mathbb{R})$ such that*

$$(C_3(\zeta))_{ij} = 0 \implies (C_3(\zeta)X - XC_3(\zeta))_{ij} < 0, \qquad i, j = 1, \ldots, n.$$

Then there exists $A \in \mathrm{M_n}(\mathbb{R})$ similar to $C_3(\zeta)$ such that a_{nn}, $a_{n,n-1} > 0$ and all other entries of A are negative.

Proof. Assume the existence of such a matrix X. Define the matrix $T(t) = I - tX$, $t \in \mathbb{R}$. Let $r = ||X||^{-1}$. Using the Neumann series expansion, e.g., [2, page 7], for $|t| < r$ we have $T(t)^{-1} = \sum_{i=0}^{\infty} t^i X^i$. The matrix $A(t) = T(t)C_3(\zeta)T(t)^{-1}$ thus satisfies

$$A(t) = C_3(\zeta) + t(C_3(\zeta)X - XC_3(\zeta)) + O(t^2).$$

Therefore, there exists $\varepsilon \in (0, r)$ such that for $t \in (0, \varepsilon)$ the matrix $A(t)$ has positive entries in the $(n, n-1)$ and (n, n) positions, while all other entries of $A(t)$ are negative. □

The following lemma is well known, and we provide a proof for the sake of completeness.

Lemma 3.3. *Let $A, B \in \mathrm{M_n}(\mathbb{F})$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. The following are equivalent.*
(i) *The system $AX - XA = B$ is solvable over $\mathbb{F}$.*
(ii) *For every matrix $E \in \mathrm{M_n}(\mathbb{F})$ that commutes with A we have* $\operatorname{tr} BE = 0$.

Proof. (i)$\Longrightarrow$(ii). Let $E \in \mathrm{M_n}(\mathbb{F})$ commute with A. Then

$$\operatorname{tr} BE = \operatorname{tr}(AX - XA)E = \operatorname{tr} AXE - trXEA = \operatorname{tr} XEA - trXEA = 0.$$

(ii)$\Longrightarrow$(i). Consider the linear operator $L : \mathrm{M_n}(\mathbb{F}) \to \mathrm{M_n}(\mathbb{F})$ defined by $L(X) = AX - XA$. Its kernel consists of all matrices in $\mathrm{M_n}(\mathbb{F})$ commuting with A. By the previous implication, the image of L is contained in the subspace V of $\mathrm{M_n}(\mathbb{F})$ consisting of all matrices C such that $\operatorname{tr} CE = 0$ whenever $E \in \operatorname{kernel}(L)$. Since clearly $\dim(V) = n^2 - \dim(\operatorname{kernel}(L)) = \dim(\operatorname{image}(L))$, it follows that $\operatorname{image}(L) = V$. □

Theorem 3.4. *Let n be a positive integer, $n > 1$, and let ζ be an n-tuple of complex numbers. Let b_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be given complex numbers, and*

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let $C = C_k(\zeta)$ for some $k \in \{1, 2, 3\}$. Then there exists unique $b_{in} \in \mathbb{C}$, $i = 1, \ldots, n$, such that for the matrix $B = (b_{ij})_1^n \in \mathrm{M_n}(\mathbb{C})$ the system $CX - XC = B$ is solvable. Furthermore, if $\overline{\zeta} = \zeta$ and b_{ij} is real for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, then the matrix B is real, and the solution X can be chosen to be real.

Proof. Since $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar to $C_1(\zeta)$, where the corresponding diagonal matrices are real, it is enough to prove the theorem for $C = C_1(\zeta)$. So, let $C = C_1(\zeta)$ and consider the system

$$CX - XC = B. \tag{3.5}$$

As is well known, e.g., [3, Corollary 1, page 222], since $C = C_1(\zeta)$ is nonderogatory, every matrix that commutes with C is a polynomial in C. Therefore, it follows from Lemma 3.3 that the system (3.5) is solvable if and only if

$$\operatorname{tr} BC^k = 0, \quad k = 0, \ldots, n-1. \tag{3.6}$$

Denote $v_k = b_{n+1-k,n}$, $k = 1, \ldots, n$. Note that (3.6) is a system of n equations in the variables $v_1, \ldots, v_n$. Furthermore, it is easy to verify that the first nonzero element in the nth row of C^k is located at the position $(n, n-k)$ and its value is 1. It follows that if we write (3.6) as $Ev = f$, where $E \in \mathrm{M_n}(\mathbb{C})$ and $v = (v_1, \ldots, n)^T$, then E is a lower triangular matrix with 1's along the main diagonal. It follows that the matrix B is uniquely determined by (3.6).

If $\overline{\zeta} = \zeta$ and b_{ij} is real for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ then $C = C_1(\zeta)$ is real and hence the system (3.6) has real coefficients, and the uniquely determined B is real. It follows that the system (3.5) is real, and so it has a real solution X. □

If we choose the numbers b_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, to be negative, then Lemma 3.2 and Theorem 3.4 yield

Corollary 3.7. *Let n be a positive integer, $n > 1$, let ζ be an n-tuple of complex numbers, and assume that the elementary symmetric functions of ζ are positive. Then there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ with $\sigma(A) = \zeta$ such that A has one positive diagonal element, one positive off-diagonal element and all other entries of A are negative. In particular, the above holds for stable n-tuples ζ such that $\overline{\zeta} = \zeta$.*

4. Other types of companion matrices. Another way to prove some of the results of Section 2 is to parameterize the companion matrices in Notation 2.8. Consider

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_0 \\ \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1 \\ 0 & \beta_1 & 0 & \cdots & 0 & 0 & 0 & \gamma_2 \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_{n-3} & 0 & \gamma_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-2} & \gamma_{n-1} \end{pmatrix}.$$

Looking at the directed graph of C, which is

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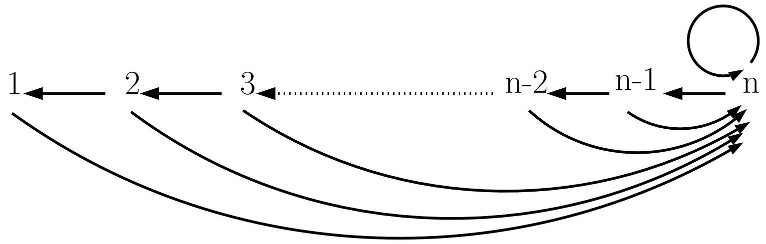


one can immediately see that there is exactly one simple cycle of length k for $1 \leq k \leq n$, that is, $(n, \ldots, n+1-k)$. Therefore, the only nonzero principal minors of C are those whose rows and columns are indexed by $\{k, \ldots, n\}$, $k = 1, \ldots, n$, and their respective values are $(-1)^{n-k}\gamma_{k-1}\beta_{k-1}\cdots\beta_{n-2}$ for $k < n$ and γ_{n-1} for $k = n$. It follows that the characteristic polynomial $\chi_C(x)$ of C is

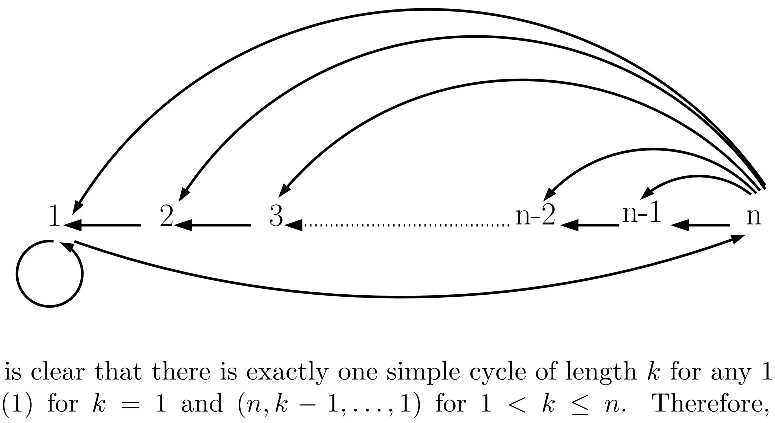
$$\begin{aligned} \chi_C(x) \quad &= \quad x^n - \gamma_{n-1}x^{n-1} - \gamma_{n-2}\beta_{n-2}x^{n-2} - \gamma_{n-3}\beta_{n-3}\beta_{n-2}x^{n-3} - \ldots \\ &\qquad \ldots - \gamma_1\beta_1\beta_2\cdots\beta_{n-2}x - \gamma_0\beta_0\beta_1\cdots\beta_{n-2}. \end{aligned} \tag{4.1}$$

Using this explicit formula, one can prove directly the claim contained in Lemma 2.9 that the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ share the spectrum ζ.

There are other possibilities to generate companion matrices. For example, consider the matrix

$$L = \begin{pmatrix} \gamma_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_{n-2} \\ \beta_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \beta_{n-4} & 0\cdots & 0 & 0 & 0 & 0 & \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_0 & 0 & 0 \\ \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 \end{pmatrix}.$$

The directed graph of L is



Again it is clear that there is exactly one simple cycle of length k for any $1 \leq k \leq n$, that is, (1) for $k = 1$ and $(n, k-1, \ldots, 1)$ for $1 < k \leq n$. Therefore, the only nonzero 1×1 principal minor of L is $l_{11} = \gamma_{n-1}$, and for $1 < k \leq n$ the only

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nonzero $k \times k$ principal minor of L is the one whose rows and columns are indexed by $\{1, \ldots, k-1, n\}$, and its value is $(-1)^{n-k}\gamma_{k-1}\beta_{k-1}\cdots\beta_{n-2}$. It follows that the characteristic polynomial $\chi_L(x)$ of L is identical to $\chi_C(x)$. Note that there is no permutation matrix P with $P^T C P = L$ or $P^T C^T P = L$.

Now, take the following specific choice of the parameters β and γ:

$$L_1 = \begin{pmatrix} -p_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -p_{n-2} & -p_{n-3} & -p_{n-4} & \cdots & -p_2 & -p_1 & -p_0 & 0 \end{pmatrix}.$$

By (4.1), the characteristic polynomial computes to

$$\chi_{L_1}(x) = \sum_{\nu=0}^{n} p_\nu x^\nu,$$

where $p_n = 1$.

So L_1 is another kind of companion matrix. Note that L_1 is almost lower triangular, with only one nonzero element above the main diagonal and one on the main diagonal.

Another specific choice of the parameters β and γ can be used to produce another direct proof of Theorem 2.10. For an n-tuple ζ of complex numbers with $\zeta = \overline{\zeta}$ and positive elementary symmetric functions, the polynomial $q(x) = \prod_{i=1}^{n}(x - \zeta_i) = \sum_{i=0}^{n} q_i x^i$ has coefficients q_i, $0 \le i \le n$ of alternating signs, where $q_n = 1$. By (4.1), the polynomial $q(x)$ is the characteristic polynomial of the matrix

$$\begin{pmatrix} -q_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \cdots\cdots\cdots & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ -q_{n-2} & q_{n-3} & -q_{n-4} & \cdots & (-1)^{n-3}q_2 & (-1)^{n-2}q_1 & (-1)^{n-1}q_0 & 0 \end{pmatrix},$$

which has exactly two positive entries, that is, $-q_{n-1}$ on the diagonal and 1 in the right upper corner.

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