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ON CAYLEY'S FORMULA FOR COUNTING TREES IN NESTED INTERVAL GRAPHS*

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Abstract. In this paper it is shown that the spectrum of a nested interval graph has a very simple structure. From this result a formula is derived to the number of spanning trees in a nested interval graph; this is a generalization of the Cayley formula.

Key words. Spectrum, Interval graph, Number of spanning trees, Cayley formula.

AMS subject classifications. 15A15, 15F10.

1. Introduction. Given a graph $G = (V, E)$ with vertices $V = V(G)$ and edges $E = E(G)$, a *spanning tree* $T = (V, E')$ of G is a connected subgraph of G having no cycles. That is, T is a connected graph with $V(T) = V(G)$, $E(T) \subseteq E(G)$, and $|E'| = |V| - 1$. One natural and very old problem is to determine the number $t(G)$ of labeled spanning trees for a fixed graph G, or better yet, give a formula of $t(G_i)$ for each graph G_i in a family $\mathcal{G} = \{G_1, G_2, \ldots\}$.

Related Work. The first result on counting the number of spanning trees in K_n (the complete graph) is due to Cayley [1], who proved that $t(K_n) = n^{n-2}$. This result was originally stated in terms of counting the number of labeled trees on n vertices, his motivation coming from the enumeration of certain chemical isomers. Kirchoff [2] was later able to find a general algebraic method to find $t(G)$, known as the Matrix Tree Theorem. Let $A = A(G)$ be the adjacency matrix of G, i.e. $a_{i,j} = 1$ if vertex v_i is adjacent to vertex v_j and $a_{i,j} = 0$ otherwise. The *Laplacian* matrix of graph G is $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix where $d_{i,i}$ is equal to the degree d_i of vertex v_i of the graph. We sometimes write L instead of $L(G)$. We denote row i of $L(G)$ by $\overrightarrow{L(G)}_i$. One property of $L(G)$ is that its smallest eigenvalue is 0 and the corresponding eigenvector is $(1, 1, \ldots, 1)$. If G is connected, all other eigenvalues are greater than 0. Let $L(G)[u]$ denote the submatrix of $L(G)$ obtained by deleting row u and column u. Then, by the Matrix Tree Theorem, for each vertex $u \in V$ we have $t(G) = |\det(L(G)[u])|$. One can phrase the Matrix Tree Theorem in terms of the spectrum of the Laplacian matrix. The next theorem appears in [4, p. 284]; it connects the eigenvalues of the Laplacian of G and $t(G)$.

THEOREM 1.1. *Let G be a graph on n vertices, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of the Laplacian of G. Then the number of labeled spanning trees in G is $\frac{1}{n}\prod_{i=2}^{n}\lambda_i$.*

Finding the Laplacian spectrum of infinite families of graphs is a well explored problem; for instance, the spectrum of K_n is $\{0^1, n^{n-1}\}$. For more families of graphs

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ELA

see the survey [3].

2. Interval Graphs. A graph G is called an *interval graph* if its vertices can be assigned to intervals on the real line so that two vertices are adjacent in G if and only if their assigned intervals intersect. The set of intervals assigned to the vertices of G is called a realization of G. If the set of intervals can be chosen to be nested (i.e. if the intersection of two intervals is not empty then one of the intervals is nested in the other), then G is called a *nested interval* graph.

Let $G = (V, E)$ be a nested interval graph with $I_i = [a_i, b_i]$, $i = 1, 2, \ldots, n$ intervals. For the sake of simplicity, we denote the node that represents the interval I_i by i. We denote the neighborhood of vertex i by $N(i) = \{j : I_i \cap I_j \neq \emptyset\}$; note that then $i \in N(i)$. Without loss of generality, we assume that all the vertices that have the same neighborhood have the same interval; i.e. if $N(i) = N(j)$ then $I_i = I_j$. Vertices i, j are *similar* if $N(i) = N(j)$. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_t)$ be the partition of the vertices generated by the similar equivalence relation and let $r_1, \ldots, r_t \in V$ be the representatives of the partition. Let $\Pi(i)$ be the set of all vertices that have the same neighborhood as the vertex i, $\Pi(i) = \{j : N(j) = N(i)\}$. Since all the vertices in π_i have the same neighborhood, we can define the degree of π_i to be $d_{\pi_i} = |N(v_i)| - 1$.

For the vertex i we define three sets and one function. Let $D^{+}(i)$ be the set of those vertices whose intervals properly contain I_i, formally $D^{+}(i) = \{j : I_i \subset I_j\}$; note that $i \notin D^{+}(i)$. Let $D^{-}(i)$ be the set of all vertices whose intervals are properly contained in the interval I_i; i.e. $D^{-}(i) = \{j : I_j \subset I_i\}$. Let $\Delta(i)$ be the set of vertices that are not in the neighborhood of vertex i; i.e. $\Delta(i) = \{j : I_j \cap I_i = \emptyset\}$. Lastly, let $\gamma(i)$ denote the number of disjoint intervals in

$$\bigcup_{k \in D^{-}(i)} I_k.$$

Again we can extend the definition to equivalence classes by setting $\gamma(\pi_i) = \gamma(r_i)$. Let $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$ be the standard base of $\mathbb{R}^n$, and let $\langle \overrightarrow{\bullet}, \overrightarrow{\bullet} \rangle$ be the standard inner product in $\mathbb{R}^n$. Finally we define the closure of a set to be the set union with the relevant equivalence class; for example, $\overline{D^{+}(i)} = D^{+}(i) \cup \Pi(i)$, $\overline{D^{-}(i)} = D^{-}(i) \cup \Pi(i)$. In the next three lemmas we specify all the eigenvalues and eigenvectors of L.

LEMMA 2.1. *Let G be a nested interval graph. Let $1 \leq i < j \leq n$. If $N(i) = N(j)$, then the vector $\overrightarrow{v} = \overrightarrow{e_i} - \overrightarrow{e_j}$ is an eigenvector with the eigenvalue $d_i + 1$.*

Proof. Since $N(i) = N(j)$ it follows that row i is equal to row j in the Laplacian matrix L, except for the diagonal entries; therefore

$$\begin{aligned} L \cdot \overrightarrow{v} &= L \cdot \overrightarrow{e_i} - L \cdot \overrightarrow{e_j} = \sum_{k=1}^{n} (L_{k,i} - L_{k,j}) \cdot \overrightarrow{e_k} \\ &= (d_i + 1) \cdot \overrightarrow{e_i} - (d_j + 1) \cdot \overrightarrow{e_j} = (d_j + 1) \cdot \overrightarrow{v}. \qquad \square \end{aligned}$$

COROLLARY 2.2. *Let G be a nested interval graph. Let π be the similar partition. Then,*

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ELA

1. *For each $\pi_i \in \pi$ we can use Lemma 2.1 and find $|\pi_i| - 1$ linear independent eigenvectors.*
2. *The total number of eigenvectors we get from Lemma 2.1 is $n - t$. Denote those eigenvectors by $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_{n-t}$ and $X = \{\overrightarrow{x}_1, \ldots, \overrightarrow{x}_{n-t}\}$.*
3. *The product of all those eigenvalues is $\prod_{k=1}^{t}(d_{\pi_k} + 1)^{|\pi_k|-1}$.*

LEMMA 2.3. *Let $1 \leq i < j \leq n$. Assume that $N(i) \neq N(j)$ and that $D^+(i) = D^+(j)$. Then,*

$$\overrightarrow{v} = |\overline{D^-(j)}| \cdot \left(\sum_{k \in \overline{D^-(i)}} \overrightarrow{e_k} \right) - |\overline{D^-(i)}| \cdot \left(\sum_{k \in \overline{D^-(j)}} \overrightarrow{e_k} \right)$$

is an eigenvector with the eigenvalue $|D^+(i)|$.

Proof. First note that for all $k \in \Delta(i) \cap \Delta(j)$ we have $I_k \bigcap \bigcup_{l \in D^-(j)} I_l = \emptyset$. Therefore $\langle \overrightarrow{L_k}, \overrightarrow{v} \rangle \cdot \overrightarrow{e_k} = 0$. Next, since $D^+(i) = D^+(j)$ it follows that for all $k \in D^+(i)$ we have

$$\begin{aligned} \langle \overrightarrow{L}_k, \overrightarrow{v} \rangle \cdot \overrightarrow{e}_k &= \left(\sum_{k \in \overline{D^-(i)}} -|\overline{D^-(j)}| + \sum_{k \in \overline{D^-(j)}} |\overline{D^-(i)}| \right) \cdot \overrightarrow{e}_k \\ &= (-|\overline{D^-(i)}| \cdot |\overline{D^-(j)}| + |\overline{D^-(j)}| \cdot |\overline{D^-(i)}|) \cdot \overrightarrow{e}_k \\ &= 0. \end{aligned}$$

Now, by a simple matrix multiplication we get that

$$\begin{aligned} L \cdot \overrightarrow{v} &= \sum_{k \in \overline{D^-(i)}} \langle \overrightarrow{L}_k, \overrightarrow{v} \rangle \cdot \overrightarrow{e}_k + \sum_{k \in \overline{D^-(j)}} \langle \overrightarrow{L}_k, \overrightarrow{v} \rangle \cdot \overrightarrow{e}_k \\ &= |\overline{D^-(j)}| \sum_{k \in \overline{D^-(i)}} \Big(d_k - |\overline{D^-(i)} \cap N(k)| + 1 \Big) \cdot \overrightarrow{e_k} + \\ &\quad -|\overline{D^-(i)}| \sum_{k \in \overline{D^-(j)}} \Big(d_k - |\overline{D^-(i)} \cap N(k)| + 1 \Big) \cdot \overrightarrow{e_k}. \end{aligned}$$

Since $d_i = |D^+(i)| + |\overline{D^-(i)}| - 1$, it follows that

$$\begin{aligned} L \cdot \overrightarrow{v} &= |\overline{D^-(j)}| \left(\sum_{k \in \overline{D^-(i)}} |D^+(i)| \cdot \overrightarrow{e_k} \right) - |\overline{D^-(i)}| \left(\sum_{k \in \overline{D^-(j)}} |D^+(j)| \cdot \overrightarrow{e_k} \right) \\ &= |D^+(i)| \cdot \overrightarrow{v}. \quad \square \end{aligned}$$

In order to describe all the eigenvectors coming out from Lemma 2.3 we define $F = \{\pi_j \in \pi | D^-(r_j) = \emptyset\}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Corollary 2.4. *Let G be a nested interval graph. Let π be the similar partition. Then,*

1. *For each $\pi_i \in \pi \setminus F$ we can use Lemma 2.3 and find $\gamma(\pi_i) - 1$ linearly independent eigenvectors.*
2. *The total number of eigenvectors we get from Lemma 2.3 is $|F| - 1$. Denote those eigenvectors by $\overrightarrow{y}_1, \dots, \overrightarrow{y}_{|F|-1}$, and $Y = \{\overrightarrow{y}_1, \dots, \overrightarrow{y}_{|F|-1}\}$.*
3. *The product of all those eigenvalues is $\prod_{\pi_k \notin F} \overline{|D^+(\pi_k)|}^{\gamma(\pi_k)-1}$.*

Lemma 2.5. *Let $\pi_i \in \pi$, r_i be the representative of π_i. Assume that $i \notin F$. Then,*

$$\overrightarrow{v} = |D^-(r_i)| \cdot \overrightarrow{e_{r_i}} - \sum_{k \in D^-(r_i)} \overrightarrow{e_k}$$

is an eigenvector with the eigenvalue $d_{r_i} + 1$.

Proof. First note that for all $k \in \Delta(r_i)$ we have $\langle \overrightarrow{L_k}, \overrightarrow{v} \rangle \cdot \overrightarrow{e_k} = 0$. Next, assume that $k \in D^+(r_i) \cup \pi_i \setminus \{r_i\}$. In this case we get that

$$\langle \overrightarrow{L_k}, \overrightarrow{v} \rangle \cdot \overrightarrow{e_k} = -|D^-(r_i)| \cdot \overrightarrow{e_k} + \sum_{k \in D^-(r_i)} \overrightarrow{e_k} = 0$$

Now, by matrix multiplication we get that

$$\begin{aligned} L \cdot \overrightarrow{v} &= \langle \overrightarrow{L}_{r_i}, \overrightarrow{v} \rangle \cdot \overrightarrow{e}_{r_i} + \sum_{k \in D^-(\pi_i)} \langle \overrightarrow{L}_k, \overrightarrow{v} \rangle \cdot \overrightarrow{e}_k \\ &= \Big(D^-(r_i) \cdot (d_{r_i} + 1)\Big) \cdot \overrightarrow{e}_{r_i} + \sum_{k \in D^-(\pi_i)} \Big(-(|D^-(r_i)|) - D^+(r_i)\Big) \cdot \overrightarrow{e}_k \\ &= (d_{r_i} + 1) \cdot \overrightarrow{v}. \quad \square \end{aligned}$$

Corollary 2.6. *Let G be a nested interval graph. Let π be the similar partition. Then,*

1. *For each $\pi_i \in \pi \setminus F$ we can use Lemma 2.5 and find one eigenvector.*
2. *The total number of eigenvectors we get from Lemma 2.5 is $t - |F|$. Denote those eigenvectors by $\overrightarrow{z}_1, \dots, \overrightarrow{z}_{t-|F|}$, and $Z = \{\overrightarrow{z}_1, \dots, \overrightarrow{z}_{t-|F|}\}$.*
3. *The product of all those eigenvalues is $\prod_{\pi_k \notin F} (d_{\pi_k} + 1)$.*

2.1. Vector Independence. In this subsection we show that indeed we find all the eigenvectors and eigenvalues. We do this using a dimension argument.

Lemma 2.7. *$Z \perp Y$.*

Proof. Assume that $\overrightarrow{v} \in Z$, i.e., $\overrightarrow{v} = (|D^-(r_i)| - |\pi_i|) \cdot \overrightarrow{e_{r_i}} - \sum_{k \in D^-(r_i) \setminus \pi_i} \overrightarrow{e_k}$ and that $\overrightarrow{u} \in Y$, i.e., $\overrightarrow{u} = |D^-(j')| \cdot (\sum_{k \in D^-(i')} \overrightarrow{e_k}) - |D^-(i')| \cdot (\sum_{k \in D^-(j')} \overrightarrow{e_k})$. It is enough to show that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = 0$. Suppose first that $I_{r_i} \subseteq I_{i'}$. Clearly, $I_{r_i} \cap I_{j'} = \emptyset$. It follows that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = |D^-(j')|(|D^-(r_i)| - |\pi_i| - |D^-(r_i)| + |\pi_i|) = 0$. Now we assume that $I_{j'} \subset I_{v_i}$. Since $D^+(i') = D^+(j')$, it follows that $I_{j'} \subset I_{r_i}$. And therefore, we get $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = |D^-(j')| \cdot |D^-(i')| - |D^-(i')| \cdot |D^-(j')| = 0$. We end the proof by assuming that $I_{v_i} \cap (I_{i'} \cup I_{j'}) = \emptyset$; in this case we get that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = 0$. $\square$

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ELA

LEMMA 2.8. $X \perp Y$.

Proof. Assume that $\overrightarrow{v} \in X$, i.e., $\overrightarrow{v} = \overrightarrow{e_i} - \overrightarrow{e_j}$ and that $\overrightarrow{u} \in Y$, i.e., $\overrightarrow{u} = |D^{^-}(j')| \cdot (\sum_{k \in D^-(i')} \overrightarrow{e_k}) - |D^{^-}(i')| \cdot (\sum_{k \in D^-(j')} \overrightarrow{e_k})$. We will show that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = 0$. Now if $i, j \in D^{^-}(i')$ then we get that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = |D^{^-}(j)|(\langle \overrightarrow{e_i}, \overrightarrow{e_i} \rangle - \langle \overrightarrow{e_j}, \overrightarrow{e_j} \rangle) = 0$. We can use the same argument for the case $i, j \in D^{^-}(j')$. Finally if $i, j \notin D^{^-}(i') \cup D^{^-}(j')$ then since $\langle \overrightarrow{e_i}, \overrightarrow{u} \rangle = \langle \overrightarrow{e_j}, \overrightarrow{u} \rangle = 0$, we get that $\langle \overrightarrow{v}, \overrightarrow{u} \rangle = 0$. ☐

LEMMA 2.9. $\dim(span(X \bigcup Z)) = n - |F|$.

Proof. In order to prove this lemma we have to define an order on the nodes. Let $1 \leq i \leq j \leq n$. Since the intervals are nested we can assume, without loss of generality, that one of the next two condition holds:

1. $[a_i, b_i] \subseteq [a_j, b_j]$.
2. $b_i < a_j$.

Finally we assume that the representative v_i of partition π_i is the first node in π_i. That is, for all $j \in \pi_i$, $i \leq j$. Let $\overrightarrow{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$; we define $\mathcal{P}(\overrightarrow{w}) = \{k | w_k \neq 0\}$. Now we define an order on the eigenvectors. Let $\overrightarrow{p}, \overrightarrow{q} \in \mathbb{R}^n$ be two eigenvectors. We say that $\overrightarrow{p} \prec \overrightarrow{q}$ if $\sum_{k \in \mathcal{P}(\overrightarrow{p})} 2^k < \sum_{k \in \mathcal{P}(\overrightarrow{q})} 2^k$. Now by writing the eigenvectors according the order $\prec$ we get that the column rank is $n - |F|$. ☐

From Corollaries 2.2, 2.4, 2.6 and Lemmas 2.7, 2.8, 2.9 we get that we find $n-1$ independent eigenvectors. The last eigenvector is $(1, 1, \ldots., 1)$ with a 0 eigenvalue. Since L has n eigenvectors we have found all of them.

THEOREM 2.10. *The number of spanning trees in a nested interval graph G is*

$$\frac{\left(\prod_{k=1}^{t} (d_{\pi_k} + 1)^{|\pi_k|}\right) \cdot \left(\prod_{\pi_k \notin F} |\overline{D^+(\pi_k)}|^{\gamma(\pi_k)-1}\right)}{n \cdot \prod_{\pi_i \in F} (d_{\pi_i} + 1)}.$$

3. Conclusion. In this paper we count the number of spanning trees in nested interval graphs. Our proof is based on the spectral decomposition of the Laplacian matrix. It is interesting to think of a combinatorial proof of this result. Another natural problem is to extend this result to interval graphs in general.

REFERENCES

[1] A. Cayley. A theorem on trees. *Quart. J. Math.*, 23:376–378, 1889. (Collected papers, Cambridge, 13:26–28, 1897.)

[2] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme gefuhrt wird. *Ann. Phys. Chem.*, 72:497–508, 1847. Gesammelte Abhandlungen, pp. 22-33, Leipzig, 1882.

[3] Bojan Mohar. The Laplacian spectrum of graphs. *Graph Theory, Combinatorics, and Applications*, Vol. 2, pp. 871–898. Wiley, New York, 1991.

[4] Chris D. Godsil and Gordon F. Royle. *Algebraic Graph Theory.* Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.