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A NEW FAMILY OF COMPANION FORMS OF POLYNOMIAL MATRICES*

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Abstract. In this paper a new family of companion forms associated to a regular polynomial matrix is presented. Similar results have been presented in a recent paper by M. Fiedler, where the scalar case is considered. It is shown that the new family of companion forms preserves both the finite and infinite elementary divisors structure of the original polynomial matrix, thus all its members can be seen as linearizations of the corresponding polynomial matrix. Furthermore, for the special class of self-adjoint polynomial matrices a particular member is shown to be self-adjoint itself.

Key words. Polynomial matrix, Companion form, Linearization, Self-adjoint polynomial matrix.

AMS subject classifications. 15A21, 15A22, 15A23, 15A57.

1. Preliminaries. We consider polynomial matrices of the form

$$T(s) = T_0 s^n + T_1 s^{n-1} + ... + T_n, \tag{1.1}$$

with $T_i \in \mathbb{C}^{p\times p}$. A polynomial matrix $T(s)$ is said to be *regular* iff $\det T(s) \neq 0$ for almost every $s \in \mathbb{C}$. The associated with $T(s)$ matrix pencil

$$P(s) = sP_0 - P_1,$$

where

$$P_0 = \begin{bmatrix} T_0 & 0 & \cdots & 0 \\ 0 & I_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_p \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} -T_1 & -T_2 & \cdots & -T_n \\ I_p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_p & 0 \end{bmatrix}, \tag{1.2}$$

is known as the *first companion form* of $T(s)$. The first companion form is well known to be a linearization of the polynomial matrix $T(s)$ (see [4]), that is there exist unimodular polynomial matrices $U(s)$ and $V(s)$ such that

$$P(s) = U(s) \operatorname{diag}\left\{T(s), I_{p(n-1)}\right\} V(s).$$

An immediate consequence of the above relation is that the first companion form has the same finite elementary divisors structure with $T(s)$. However, in [11], [8], this

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important property of the first companion form of $T(s)$ has been shown to hold also for the infinite elementary divisors structures of $P(s)$ and $T(s)$.

Motivated by the preservation of both finite and infinite elementary divisors structure, a notion of strict equivalence between a polynomial matrix and a pencil has been proposed in [11]. According to this definition, a polynomial matrix is said to be strictly equivalent to a matrix pencil iff they possess identical finite and infinite elementary divisors structure, which in the special case where both matrices are of degree one (i.e., pencils) reduces to the standard definition of [2].

Similar results hold for the *second companion form* of $T(s)$ defined by

$$\hat{P}(s) = sP_0 - \hat{P}_1,$$

where P_0 is defined in (1.2) and

$$\hat{P}_1 = \begin{bmatrix} -T_1 & I_p & \cdots & 0 \\ -T_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & I_p \\ -T_n & 0 & \cdots & 0 \end{bmatrix}.$$

It can be easily seen that $\det T(s) = \det P(s) = \det \hat{P}(s)$, so the matrix pencils $P(s)$, $\hat{P}(s)$ are regular iff $T(s)$ is regular.

The aim of the present note is to introduce a new family of companion forms for a given regular polynomial matrix, which can be parametrized by products of elementary constant matrices, an idea appeared recently in [1] for the scalar case. Surprisingly, this new family contains apart form the first and second companion forms, many new ones, unnoticed in the subject's bibliography. Companion forms of polynomial matrices (or even scalar polynomials) are of particular interest in many research fields as a theoretical or computational tool. First order representations are in general easier to manipulate and provide better insight on the underlying problem. In view of the variety of forms arising from the proposed family of linearizations, one may choose particular ones that are better suited for specific applications (for instance when dealing with self-adjoint polynomial matrices [3], [4], [7], [5], [6] or the quadratic eigenvalue problem [10]).

The content is organized as follows: in Section 2, we present the new family of companion forms and prove some basic properties of its members. In Section 3, the class of self-adjoint polynomial matrices is considered and a new linearization with the appealing property of being self-adjoint is proposed and some spectral properties are discussed. Finally in Section 4, we summarize our results and briefly discuss subjects for further research and applications.

2. Main Results. In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively and $\mathbb{K}^{p\times m}$ where $\mathbb{K}$ is a field, stands for the set of $p \times m$ matrices with elements in $\mathbb{K}$. The transpose (resp. conjugate transpose) of a matrix A will be denoted by $A^\top$ (resp. A^*), $\det A$ is the determinant and $\ker A$ is the right null-space or kernel of the matrix A. A standard assumption throughout the paper is the regularity of the polynomial matrix $T(s)$, i.e., $\det T(s) \neq 0$ for almost every $s \in \mathbb{C}$.

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Following similar lines with [1] we define the matrices

$$A_0 = \operatorname{diag}\{T_0, I_{p(n-1)}\}, \tag{2.1}$$

$$A_k = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix},\ k = 1, 2, \ldots, n-1, \tag{2.2}$$

$$A_n = \operatorname{diag}\{I_{p(n-1)}, -T_n\}, \tag{2.3}$$

where

$$C_k = \begin{bmatrix} -T_k & I_p \\ I_p & 0 \end{bmatrix}. \tag{2.4}$$

The above defined sequence of matrices $A_i,\ i = 0, 1, \ldots, n$ can be easily shown to provide an easy way to derive the first and second companion forms of the polynomial matrix $T(s)$. The following lemma is a generalization of Lemma 2.1 in [1] where the monic, scalar case is considered.

Lemma 2.1. *The first and second companion forms of $T(s)$ are given respectively by*

$$P(s) = sA_0 - A_1 A_2 \ldots A_n, \tag{2.5}$$

$$\hat{P}(s) = sA_0 - A_n A_{n-1} \ldots A_1. \tag{2.6}$$

Proof. The matrix product $A_1 A_2 \ldots A_n$ can be easily seen to be equal to P_1 in (1.2) by following similar inductive steps as in the proof of Lemma 2.1 in [1], while obviously $A_0 = P_0$, so (2.5) holds. Equation (2.6) can be shown using similar arguments. □

Before we proceed to the main result of this section, we state and prove the following lemma which will be instrumental in the sequel.

Lemma 2.2. *Let E, F, G be a triple of square matrices, with F non-singular and $GE = EG$. Then,*

a) the following equation holds:

$$(\lambda E - GF)F^{-1}(sE - FG) = (sE - GF)F^{-1}(\lambda E - FG), \tag{2.7}$$

for every $(s, \lambda) \in \mathbb{C}^2$.

b) If $\ker E^\top \cap \ker G^\top = \{0\}$ and $\lambda E - FG$ is non-singular for some $\lambda \in \mathbb{C}$, then $\lambda E - GF$ is non-singular as well.

Proof. a) Equation (2.7) can be verified by straightforward computation using the assumption $GE = EG$.

b) Assume that $\lambda E - GF$ is singular. Thus there exists a row vector $x^\top \neq 0$ such that $x^\top(\lambda E - GF) = 0$. Premultiplying (2.7) by $x^\top$ and using the fact that $F^{-1}(\lambda E - FG)$ is non-singular, we obtain

$$x^\top(sE - GF) = 0,$$

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for every $s \in \mathbb{C}$, or equivalently $x^\top E = 0$ and $x^\top GF = 0$. Taking into account the invertibility of F, $x^\top GF = 0$ reduces to $x^\top G = 0$. Thus $x \in \ker E^\top \cap \ker G^\top = \{0\}$ which contradicts the assumption $x^\top \neq 0$. Hence $\lambda E - GF$ is non-singular. ∎

The above lemma will be very useful in the sequel, since the proof of our main result, in the polynomial matrix case, requires a slightly different treatment than the one in [1], where the scalar case is considered. The following theorem will serve as the main tool for the construction of the new family of companion forms of $T(s)$.

THEOREM 2.3. *Let $P(s)$ be the first companion form of a regular polynomial matrix $T(s)$. Then for every possible permutation $(i_1, i_2, \ldots, i_n)$ of the n-tuple $(1, 2, \ldots, n)$ the matrix pencil $Q(s) = sA_0 - A_{i_1}A_{i_2}\ldots A_{i_n}$ is strictly equivalent to $P(s)$, i.e., there exist non-singular constant matrices M and N such that*

$$P(s) = MQ(s)N, \tag{2.8}$$

where A_i, $i = 0, 1, 2, \ldots, n$ are defined in (2.1), (2.2) and (2.3).

Proof. If $(i_1, i_2, \ldots, i_n) = (1, 2, \ldots, n)$ then (2.8) holds trivially, so let

$$(i_1, i_2, \ldots, i_n) \neq (1, 2, \ldots, n).$$

Notice that due to the special structure of A_i's, $A_iA_j = A_jA_i$ for $|i - j| > 1$. This allows us to rearrange A_i's so that

$$A_{i_1}A_{i_2}\ldots A_{i_n} = (A_{j_v+1}A_{j_v+2}\ldots A_n)\ldots(A_{j_1+1}A_{j_1+2}\ldots A_{j_2})(A_1A_2\ldots A_{j_1}), \tag{2.9}$$

where $0 < j_1 < j_2 < \ldots < j_v < n$. In order to avoid writing long products we shall adopt the notation $A_{k,l} = A_kA_{k+1}\ldots A_l$, for $k \leq l$. With this notation we can write

$$Q(s) = sA_0 - A_{j_v+1,n}\ldots A_{j_1+1,j_2}A_{1,j_1}.$$

Note that A_i is non-singular for $i = 1, 2, 3, \ldots, n-1$, so $A_{k,l}$ is non-singular for $0 < k \leq l < n$. Thus A_{1,j_1} is invertible. Moreover the product $(A_{j_v+1,n}\ldots A_{j_1+1,j_2})$ commutes with A_0 since it does not contain A_1, so A_0, A_{1,j_1}, $(A_{j_v+1,n}\ldots A_{j_1+1,j_2})$ satisfy the assumptions of Lemma 2.2a) so

$$Q_0(\lambda)A_{1,j_1}^{-1}Q_1(s) = Q_0(s)A_{1,j_1}^{-1}Q_1(\lambda),\ \forall(s,\lambda) \in \mathbb{C}^2, \tag{2.10}$$

where $Q_0(s) = Q(s)$ and $Q_1(s) = sA_0 - A_{1,j_1}(A_{j_v+1,n}\ldots A_{j_2+1,j_3})A_{j_1+1,j_2} = sA_0 - (A_{j_v+1,n}\ldots A_{j_2+1,j_3})A_{1,j_2}$.

Using similar arguments as above, Lemma 2.2a) can be now applied for A_0, A_{1,j_2}, $(A_{j_v+1,n}\ldots A_{j_2+1,j_3})$ to obtain

$$Q_1(\lambda)A_{1,j_2}^{-1}Q_2(s) = Q_1(s)A_{1,j_2}^{-1}Q_2(\lambda),\ \forall(s,\lambda) \in \mathbb{C}^2, \tag{2.11}$$

where

$$Q_2(s) = sA_0 - A_{1,j_2}(A_{j_v+1,n}\ldots A_{j_3+1,j_4})A_{j_2+1,j3} = sA_0 - (A_{j_v+1,n}\ldots A_{j_3+1,j_4})A_{1,j3}.$$

Proceeding this way, after v cyclic permutations we obtain

$$Q_{v-1}(\lambda)A_{1,j_v}^{-1}Q_v(s) = Q_{v-1}(s)A_{1,j_v}^{-1}Q_v(\lambda),\ \forall(s,\lambda) \in \mathbb{C}^2, \tag{2.12}$$

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where $Q_{v-1}(s) = sA_0 - A_{j_v+1,n}A_{1,j_v}$, $Q_v(s) = sA_0 - A_{1,j_v}A_{j_v+1,n} = sA_0 - A_{1,n} = P(s)$.

In order to establish (2.8) it remains to show that there exists $\lambda \in \mathbb{C}$ such that $Q_i(\lambda)$, $i = 0, 1, \ldots, v$ is non-singular. Indeed, starting from $Q_v(s) = P(s)$ which is a regular matrix pencil since $T(s)$ is regular, there exists $\lambda \in \mathbb{C}$ such that $Q_v(\lambda)$ is non-singular. In view of the special structure of A_i's it is easy to see that $\ker A_0^\top \cap \ker A_{j_v+1,n}^\top = \{0\}$[1], so Lemma 2.2b) applies for A_0, A_{1,j_v}, $A_{j_v+1,n}$. Thus $Q_{v-1}(\lambda)$ is non-singular. Proceeding, inductively and applying similar arguments one can show that $Q_i(\lambda)$ is non-singular for every $i = v-2, v-3, \ldots, 0$. $\square$

The above theorem states that any matrix pencil of the form $Q(s) = sA_0 - A_{i_1}A_{i_2}\ldots A_{i_n}$ has identical finite and infinite elementary divisor structure with $T(s)$. Thus for any permutation $(i_1, i_2, \ldots, i_n)$ of the n-tuple $(1, 2, \ldots, n)$ the resulting companion matrices are by transitivity strictly equivalent amongst each other. Furthermore the companion forms arising from Theorem 2.3 can be considered to be strictly equivalent to the polynomial matrix $T(s)$ in the sense of [11]. Notice, that the members of the new family of companion forms cannot in general be produced by permutational similarity transformations of $P(s)$ not even in the scalar case (see [1]).

In view of the asymmetry in the distribution of A_i's in the constant and first order terms of $Q(s)$, it is natural to expect more freedom in the construction of companion forms. In this sense the following corollary is an improvement of Theorem 2.3.

Corollary 2.4. *Let $P(s)$ be the first companion form of a regular polynomial matrix $T(s)$. For any four ordered sets of indices $I_k = (i_{k,1}, i_{k,2}, \ldots, i_{k,n_k})$, $k = 1, 2, 3, 4$ such that $I_i \cap I_j = \varnothing$ for $i \neq j$ and $\bigcup_{k=1}^{4} I_k = \{1, 2, \ldots, n-1\}$ the matrix pencil*

$$R(s) = sA_{I_1}^{-1}A_0A_{I_2}^{-1} - A_{I_3}A_nA_{I_4},$$

is strictly equivalent to $P(s)$, where $A_{I_k} = A_{i_{k,1}}A_{i_{k,2}}\ldots A_{i_{k,n_k}}$ for $I_k \neq \varnothing$ and $A_{I_k} = I$ for $I_k = \varnothing$.

Proof. We first notice that A_{I_k} are invertible as a product of matrices A_i with $0 < i < n$. Then $R(s)$ can be written as

$$R(s) = A_{I_1}^{-1}(sA_0 - A_{I_1}A_{I_3}A_nA_{I_4}A_{I_2})A_{I_2}^{-1}.$$

Obviously the pencil $sA_0 - A_{I_1}A_{I_3}A_nA_{I_4}A_{I_2}$ belongs to the family of companion matrices $Q(s)$ of Theorem 2.3 which is strictly equivalent to $P(s)$. Thus

$$R(s) = A_{I_1}^{-1}Q(s)A_{I_2}^{-1},$$

which establishes our result. $\square$

Notice that the inverses of A_k, $k = 1, 2, \ldots, n-1$ have a particularly simple form, that is

$$A_k^{-1} = \begin{bmatrix} I_{p(k-1)} & 0 & \cdots \\ 0 & C_k^{-1} & \ddots \\ \vdots & \ddots & I_{p(n-k-1)} \end{bmatrix},\ k = 1, 2, \ldots, n-1,$$

[1] In fact $\ker A_0^\top \cap \ker B^\top = \{0\}$, for B being any product of A_i's for $1 < i \leq n$.

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with

$$C_k^{-1} = \begin{bmatrix} 0 & I_p \\ I_p & T_k \end{bmatrix}.$$

In view of this simple inversion formula, Corollary 2.4 produces a broader class of companion forms than the one derived from Theorem 2.3, which are strictly equivalent (in the sense of [11]) to the polynomial matrix $T(s)$.This is justified by the fact that the"middle" coefficients of $T(s)$ can be chosen to appear either on the constant or first-order term of the companion pencil $R(s)$.

The following example illustrates such a case.

EXAMPLE 2.5. Let $T(s) = T_0s^3 + T_1s^2 + T_2s + T_3$. We can choose to move the coefficients T_1, T_2 on any term of the companion matrix $R(s)$. For instance we can have T_1 on the first order term and T_2 on the constant term of $R(s)$, i.e.,

$$R(s) = sA_0A_1^{-1} - A_2A_3,$$

or

$$R(s) = s\begin{bmatrix} 0 & T_0 & 0 \\ I & T_1 & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & -T_2 & -T_3 \\ 0 & I & 0 \end{bmatrix}.$$

3. Self-adjoint Polynomial Matrix Linearizations. In this section we consider self-adjoint polynomial matrices of the form (1.1). A polynomial matrix $T(s)$ is said to be self-adjoint if and only if $T^*(s) = T(s)$, that is $T_i^* = T_i$ (see [3], [4]). We shall further assume that the leading coefficient matrix of $T(s)$ is non-singular, i.e., $\det(T_0) \neq 0$.

Self-adjoint polynomial matrices are of particular importance in the study of damped oscillatory systems with finite number of degrees of freedom, which can be described by second-order differential equations with self-adjoint (Hermitian) matrix coefficients. The algebraic structure of such systems and the associated polynomial matrices has been extensively studied by many authors and a very rich literature on the subject is available (see for instance [3], [4], [7], [10] and references therein).

A question arising naturally is whether there exist linearizations of a self-adjoint polynomial matrix, which are self-adjoint themselves. Although this seems to be a desirable property for linearizations of self-adjoint polynomial matrices (see [10]) and the structure of self-adjoint matrix pencils has been studied in [5], [6], [9], the above question has not been discussed in the general context of self-adjoint polynomial matrices. Having introduced the new family of companion forms in the previous section, we prove the existence of a particular member possessing this appealing property.

THEOREM 3.1. *Let $T(s)$ be a self-adjoint polynomial matrix of degree n, with $\det T_0 \neq 0$. Then the companion form of $T(s)$*

$$R_s(s) = \begin{cases} sA_{odd}^{-1} - A_{even} & \text{for } n \text{ even} \\ sA_{even}^{-1} - A_{odd} & \text{for } n \text{ odd} \end{cases}, \tag{3.1}$$

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where $A_{even} = A_0^{-1}A_2A_4A_6\ldots$, $A_{odd} = A_1A_3A_5\ldots$, *is a self adjoint polynomial matrix pencil.*

Proof. Notice that the matrices A_{2k} (resp. A_{2k+1}) in A_{even} (resp. A_{odd}) for $k = 0, 1, \ldots$ commute, so the particular order of indices in A_{even} (resp. A_{odd}) is not significant.

For n even, applying Corollary 2.4 for the ordered sets of indices

$$I_1 = \varnothing, I_2 = (1, 3, \ldots, n-1), I_3 = (2, 4, \ldots, n-2), I_4 = \varnothing,$$

we take the companion pencil

$$R(s) = sA_0A_{odd}^{-1} - A_{I_3}A_n.$$

Since T_0 is invertible, we easily see that $R_s(s) = A_0^{-1}R(s)$, by noting that $A_{even} = A_0^{-1}A_{I_3}A_n$. Thus $R_s(s)$ is a companion form of $T(s)$.

Similarly, for n odd, we choose

$$I_1 = (2, 4, \ldots, n-1), I_2 = \varnothing, I_3 = (1, 3, \ldots, n-2), I_4 = \varnothing$$

and we get the odd case of $R_s(s)$.

Now, since $T_i^* = T_i$, in view of the special structure of A_i's, $A_i^* = A_i$ for $i = 0, 1, \ldots, n$. It is easy to verify that

$$A_{odd}^* = \ldots A_5^*A_3^*A_1^* = \ldots A_5A_3A_1 = A_1A_3A_5\ldots = A_{odd},$$

where again we used the fact that $A_iA_j = A_jA_i$ for $|i-j| > 1$. Similar arguments apply for A_{even}, i.e.,

$$A_{even}^* = A_{even}.$$

Hence $R_s^*(s) = R_s(s)$, which completes the proof. □

It easy to see that the proposed self-adjoint linearization of $T(s)$ has a particularly simple form as shown in the following example:

EXAMPLE 3.2. We illustrate the form of $R_s(s)$ for $n = 4$ and $n = 5$ respectively. For $n = 4$,

$$R_s(s) = s\begin{bmatrix} 0 & I_p & & \\ I_p & T_1 & & \\ & & 0 & I_p \\ & & I_p & T_3 \end{bmatrix} - \begin{bmatrix} T_0^{-1} & & & \\ & -T_2 & I_p & \\ & I_p & 0 & \\ & & & -T_4 \end{bmatrix}.$$

For $n = 5$,

$$R_s(s) = s\begin{bmatrix} T_0 & & & & \\ & 0 & I_p & & \\ & I_p & T_2 & & \\ & & & 0 & I_p \\ & & & I_p & T_4 \end{bmatrix} - \begin{bmatrix} -T_1 & I_p & & & \\ I_p & 0 & & & \\ & & -T_3 & I_p & \\ & & I_p & 0 & \\ & & & & -T_5 \end{bmatrix}.$$

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In order to investigate the spectral properties of $R_s(s)$ it is necessary to introduce some additional notation borrowed from [4, Chapter S.5]. Let $A \in \mathbb{C}^{r\times r}$ be a constant square matrix. Select a maximal set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_a\}$ containing no conjugate pair and let $\{\lambda_{a+1}, \lambda_{a+2}, \ldots, \lambda_b\}$ be the distinct real eigenvalues of A. Set $\lambda_{a+b+j} = \bar{\lambda}_j$, $j = 1, 2, ..., a$ and let

$$J = \operatorname{diag}[J_i]_{i=1}^{2a+b},$$

where $J_i = \operatorname{diag}[J_{ij}]_{j=1}^{k_i}$ is a Jordan form corresponding to λ_i and J_{ij} of sizes $a_{i1} \geq \ldots \geq a_{i,k_i}$ respectively. Define the matrix

$$P_{\varepsilon,J} = \begin{bmatrix} 0 & 0 & P_c \\ 0 & P_r & 0 \\ P_c & 0 & 0 \end{bmatrix}, \tag{3.2}$$

where $P_c = \operatorname{diag}[\operatorname{diag}[P_{ij}]_{j=1}^{k_i}]_{i=1}^{a}$ and $P_r = \operatorname{diag}[\operatorname{diag}[\varepsilon_{ij}P_{ij}]_{j=1}^{k_i}]_{i=a+1}^{a+b}$, with P_{st} being standard involuntary permutation (sip) matrices of sizes $a_{st} \times a_{st}$, $\varepsilon = \{\varepsilon_{ij}\}$ is the ordered set of signs for $i = a+1, \ldots, a+b$, $j = 1, \ldots, k_i$ for some $\varepsilon_{ij} = \pm 1$.

With the above notation we state the following corollary:

Corollary 3.3. *Let $T(s)$ be a self-adjoint polynomial matrix of degree n, with $\det T_0 \neq 0$. Then there exists a non-singular matrix X such that*

$$R_s(s) = XP_{\varepsilon,J}(sI - J)X^*, \tag{3.3}$$

where $R_s(s)$ is the self-adjoint linearization of $T(s)$ defined in (3.1), J is the Jordan canonical form of $A_{odd}A_{even}$ for n even (resp. $A_{even}A_{odd}$ for n odd) with the structure described above and $P_{\varepsilon,J}$ the corresponding matrix defined in (3.2) for some ordered set of signs ε.

Proof. The proof is a straightforward application of [4, Corollary S5.3, p.378]. ☐

Using the above result we can obtain a factorization of the underlying polynomial matrix $T(s)$.

Corollary 3.4. *Let $T(s)$ be a self-adjoint polynomial matrix of degree n, with $\det T_0 \neq 0$ and $R_s(s)$ be the companion matrix described in Theorem 3.1. Then $T(s)$ can be factored as*

$$T(s) = U(s)P_{\varepsilon,J}(sI - J)U^*(s), \tag{3.4}$$

where $U(s)$ is left unimodular (i.e., rank$U(s) = p$, for every $s \in \mathbb{C}$) polynomial matrix and J, $P_{\varepsilon.J}$, ε as in Corollary 3.3.

Proof. Let $k = \left[\frac{n}{2}\right]$. Define the polynomial matrix $U_n(s)$ as follows

$$U_n(s) = \begin{cases} \begin{bmatrix} s^k I_p & Q_1 & s^{k-1}I_p & Q_2 & ... & sI_p & Q_k & I_p \end{bmatrix}, \text{ for } n \text{ odd} \\ \begin{bmatrix} T_0 s^k & U_{n-1} \end{bmatrix}, \text{ for } n \text{ even,} \end{cases}$$

where

$$Q_i = \sum_{j=0}^{2i-1} T_j s^{k+i-j},\ i = 1, \ldots, k.$$

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Then it can be seen by straightforward computations that

$$U_n(s)R_s(s)U_n^*(s) = T(s), \tag{3.5}$$

while $U_n(s)$ is obviously left unimodular. Setting $U(s) = U_nX$, where X is as in (3.3), establishes (3.4). $\square$

EXAMPLE 3.5. Consider the self-adjoint polynomial matrix (see [4, Example 10.4])

$$T(s) = \begin{bmatrix} s^4+1 & 2s^2 \\ 2s^2 & s^4+1 \end{bmatrix}.$$

The self-adjoint linearization of $T(s)$ is given by

$$R_s(s) = s\begin{bmatrix} 0 & I_2 & & \\ I_2 & 0 & & \\ & & 0 & I_2 \\ & & I_2 & 0 \end{bmatrix} - \begin{bmatrix} I_2 & & & \\ & -2S & I_2 & \\ & I_2 & 0 & \\ & & & -I_2 \end{bmatrix},$$

where $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then

$$R_s(s) = XP_{\varepsilon,J}(sI-J)X^*,$$

where

$$X = \frac{1}{4\sqrt{2}}\begin{bmatrix} -i & -2 & -1 & -2 & -1 & 2 & i & -2 \\ -i & -2 & 1 & 2 & 1 & -2 & i & -2 \\ -3 & 2i & -3 & -2 & 3 & -2 & -3 & -2i \\ -3 & 2i & 3 & 2 & -3 & 2 & -3 & -2i \\ -1 & -2i & 1 & -2 & -1 & -2 & -1 & 2i \\ -1 & -2i & -1 & 2 & 1 & 2 & -1 & 2i \\ 3i & -2 & -3 & 2 & -3 & -2 & -3i & -2 \\ 3i & -2 & 3 & -2 & 3 & 2 & -3i & -2 \end{bmatrix}, \quad P_{\varepsilon,J} = \begin{bmatrix} & & & S \\ & S & 0 & \\ & 0 & S & \\ S & & & \end{bmatrix},$$

$$J = \operatorname{diag}\left\{\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}\right\},$$

is the Jordan canonical form of $A_{odd}A_{even}$. Using equation (3.5) we compute

$$U_4(s) = \begin{bmatrix} s^2 & 0 & s & 0 & s^2 & 0 & 1 & 0 \\ 0 & s^2 & 0 & s & 0 & s^2 & 0 & 1 \end{bmatrix},$$

and from Corollary 3.4 we get the factorization

$$T(s) = U(s)P_{\varepsilon,J}(sI-J)U^*(s),$$

where $U(s) = U_4(s)X$.

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4. Conclusions. In this paper we present a new family of companion forms associated to a regular polynomial matrix. This study was motivated by recent results presented in [1] concerning the scalar case. The members of the proposed family of companion matrix pencils is shown to be strictly equivalent to the polynomial matrix, preserving both finite and infinite elementary divisors structure. The new companion forms are parametrized as combination of products of elementary constant matrices and the well known first and second companion forms are shown to correspond to specific permutations of the products.

Throughout the variety of forms arising from this new family, a particular one seems to be of special interest, since in the case of self-adjoint polynomial matrix, the resulting linearization is self-adjoint as well. In the sequel using a known result for self-adjoint polynomial matrix pencils, we present a new factorization of the polynomial matrix.

The present note aims to present only preliminary results regarding this new family of companion forms, leaving many theoretical and computational aspects to be the subject of further research.

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