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ON RELATIONS OF INVARIANTS FOR VECTOR-VALUED FORMS*

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Abstract. An algorithm is given for computing explicit formulas for the generators of relations among the invariant rational functions for vector-valued bilinear forms. These formulas have applications in the geometry of Riemannian submanifolds and in CR geometry.

Key words. Vector-valued forms, Determinants, Relations.

AMS subject classifications. 15A72, 13A50, 14L24, 15A63.

1. Preliminaries.

1.1. Introduction. Vector-valued forms play a key role in the study of higher codimensional geometries. For example, they occur naturally in the study of Riemannian submanifolds (as the second fundamental form) and in CR geometry (as the Levi form). In each of these there are natural group actions acting on the vector-valued forms, taking care of different choices of local coordinates and the such. The algebraic invariants of these forms under these group actions provide invariants for the given geometries. In Riemannian geometry, for example, the scalar curvature can be expressed as an algebraic invariant of the second fundamental form. But before these invariants can be used, their algebraic structure must be known. In [5], an explicit list of the generators is given. In that paper, though, there is no hint as to the relations among these generators. In this paper, a method is given for producing a list of the generators for the relations of the invariants.

In [5], the problem of finding the rational invariants of bilinear maps from a complex vector space V of dimension n to a complex vector space W of dimension k, on which the group $GL(n,\mathbf{C}) \times GL(k,\mathbf{C})$ acts, is reduced to the problem of finding invariant one-dimensional subspaces of the vector spaces $(V \otimes V \otimes W^*)^{\otimes r}$, for each positive integer r. From this, it is shown that the invariants can be interpreted as being generated by

$$(\text{Invariants for } GL(n,\mathbf{C}) \text{ of } V \otimes V) \otimes (\text{Invariants for } GL(k,\mathbf{C}) \text{ of } W^*),$$

each component of which had been computed classically.

In this paper we extend this type of result, showing how to compute the relations of bilinear forms from knowledge of the relations for $V \otimes V$ under the action of $GL(n,\mathbf{C})$ and the relations for W^* under the action of $GL(k,\mathbf{C})$. In particular, in a way that will be made more precise later, we show that the relations can be interpreted as being generated by:

$$((\text{Relations for } GL(n,\mathbf{C}) \text{ of } V \otimes V) \otimes (\text{Generators for } GL(k,\mathbf{C}) \text{ of } W^*))$$

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$$\oplus((\text{Generators for } GL(n,\mathbf{C}) \text{ of } V\otimes V)\otimes(\text{Relations for } GL(k,\mathbf{C}) \text{ of } W^*)),$$

each component of which is known classically. While this paper concentrates on the case when $GL(n,\mathbf{C})$ acts on the vector space V and $GL(k,\mathbf{C})$ acts on W, the techniques that we use are applicable for when G and H are any completely reducible Lie groups acting on the vector spaces V and W, respectively. When the bilinear form is the second fundamental form of Riemannian geometry, then G is the orthogonal group $O(n)$ and H is the orthogonal group $O(k)$. In CR geometry, when the bilinear form is the Levi form, then $G = GL(n,\mathbf{C})$, but now $H = GL(k,\mathbf{R})$.

In sections 1.2 through 1.4, we set up our basic notation. Section 2 recalls how to compute the invariants of bilinear forms. Section 3 recalls the classically known relations for invariants of the general linear group. While well-known, we spend time on writing these relations both in the bracket notation for vectors and in the tensor language that we are interested in. Section 4 gives the relations among the invariant one dimensional subspaces of $(V\otimes V\otimes W^*)^{\otimes r}$, for each positive integer r. This is the key step in this paper. The key proof will be seen to be not hard, reflecting the fact that the difficulty in this paper is not the proofs but the finding of the correct statements and correct formulations of the theorems. Section 5 gives a concrete example of a relation from section 4. Section 6 finally deals with the finding the relations of the invariants for vector-valued bilinear forms. Section 7 gives concrete, if not painful, examples and discusses some geometric insights behind some of the computed invariants and relations. Section 8 closes with some further questions.

It appears that the closest earlier work to this paper is in the study of the invariants of $n\times n$ matrices (see [4] and [10]), but the group actions are different in this case and links are not apparent. For general background in invariant theory, see [3], [9], [11] or [13].

1.2. Vector-valued forms. For the rest of this paper, let V be a complex n-dimensional vector space and W be a complex k-dimensional vector space. We are concerned with the vector space $Bil(V,W)$, the space of bilinear maps from $V\times V$ to W. Each such bilinear map is an element of $V^*\otimes V^*\otimes W$, where V^* is the dual space of V. The group $Aut(V)\times Aut(W)$ acts on $Bil(V,W)$ by

$$gb(x,y) = pb(a^{-1}x, a^{-1}y)$$

for all $g=(a,p)\in Aut(V)\times Aut(W)$, $b\in Bil(V,W)$, and $x,y\in V$. Stated differently, we define gb so that the following diagram commutes:

$$\begin{array}{cccc} & V\times V & \xrightarrow{b} & W \\ a\times a & \downarrow & & \downarrow\ p \\ & V\times V & \xrightarrow[gb]{} & W \end{array}$$

As mentioned in the introduction, the results of this paper (and the results in [5]) also work for completely reducible subgroups of $Aut(V)$ and $Aut(W)$, though for simplicity, we restrict our attention to the full groups $Aut(V)$ and $Aut(W)$, which of course are isomorphic to $GL(n,\mathbf{C})$ and $GL(k,\mathbf{C})$.

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1.3. Invariants. Let G be a group that acts linearly on a complex vector space V. A function

$$f : V \to \mathbb{C}$$

is a (relative) invariant if for all $g \in G$ and all $v \in V$, we have

$$f(gv) = \chi(g)f(v),$$

where $\chi : G \to \mathbb{C} - \{0\}$ is a homomorphism (i.e. χ is an abelian character for the group G). We call χ the *weight* of the invariant. Note that the sum of two invariants of the same weight is another invariant. Thus the invariants of the same weight will form in natural way a vector space.

As seen in [3] on pp. 5-9, every rational invariant is the quotient of polynomial invariants, every polynomial invariant is the sum of homogeneous polynomial invariants, every degree r homogeneous polynomial corresponds to an invariant r-linear function on the Cartesian product $V^{\times r}$ and every invariant r-linear function on $V^{\times r}$ corresponds to an invariant linear function on the r-fold tensor product $V^{\otimes r}$. Thus to study rational invariants on V we can concentrate on understanding the invariant one-dimensional subspaces on $V^{*\otimes r}$.

Let $\mathbb{C}[V]$ be the algebra of polynomial functions on V and let $\mathbb{C}[V]^G$ denote the algebra of the polynomials invariant under the action of G. In general, the goal of invariant theory is to find a list of generators of algebra $\mathbb{C}[V]^G$ (a "First Fundamental Theorem"), a list of generators of the relations of these generators (a "Second Fundamental Theorem") and then relations of relations, etc. A full such description is the syzygy of $\mathbb{C}[V]^G$.

By the above, we need to find the homogeneous polynomials in $\mathbb{C}[V]^G$. Since the homogeneous polynomials of degree r in $\mathbb{C}[V]$ are isomorphic to symmetric tensors in $V^{*\odot r}$, we need to find the invariant one-dimensional subspaces of $V^{*\odot r}$. Now, if G acts on V, it will act on $V^{*\otimes r}$. Suppose we have all invariant one-dimensional subspaces on $V^{*\otimes r}$. Then we can easily recover all invariant one-dimensional subspaces on $V^{*\odot r}$ by the symmetrizing map from $V^{*\otimes r}$ to $V^{*\odot r}$. This is the procedure we will use.

We are interested in rational invariants for the vector space $Bil(V,W)$ under the group action of $Aut(V) \times Aut(W)$. Thus we are interested in rational invariants for the vector space $Bil(V,W)$. Hence the vector space we are interested in is $V^*\otimes V^*\otimes W$ under the group action of $Aut(V) \times Aut(W)$, which we will see means that we are interested initially in the invariant one-dimensional subspaces of $(V \otimes V \otimes W^*)^{\otimes r}$ for each r and finally in the invariant one-dimensional subspaces of $(V \otimes V \otimes W^*)^{\odot r}$.

1.4. Indicial notations. The following permutation notation will be used heavily throughout this paper. For any positive integer m, define the permutation symbol $\varepsilon^{i_1 i_2 \dots i_m}$ to be equal to 1 if $i_1 i_2 \dots i_m$ is an even permutation of $1, 2, \dots, m$, to be equal to (-1) if it is an odd permutation, and to be equal to 0 otherwise. To indicate the product of d (where d is a positive integer) such symbols for any permutation $\sigma \in S_{dm}$, we use the shorthand notation

$$\varepsilon^I(m, dm, \sigma) = \varepsilon^{i_{\sigma(1)} \cdots i_{\sigma(m)}} \varepsilon^{i_{\sigma(m+1)} \cdots i_{\sigma(2m)}} \dots \varepsilon^{i_{\sigma(dm-m+1)} \cdots i_{\sigma(dm)}}.$$

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The symbols $\varepsilon_{i_1 i_2 \ldots i_m}$ and $\varepsilon_I(m, dm, \sigma)$ are defined in a similar manner. The Einstein summation notation will be used. Thus whenever a superscript and a subscript appear in the same term, this means sum over that term.

As an example of the notation, let V be a two dimensional vector space with the basis $\{e_1, e_2\}$. Then $\varepsilon^I(2, 2, \text{identity}) e_{i_1} \otimes e_{i_2}$ denotes the following summation of two-tensors from $V \otimes V$:

$$\begin{aligned}
\varepsilon^I(2,2,\text{identity}) e_{i_1} \otimes e_{i_2} = {} & \varepsilon^{11} e_1 \otimes e_1 + \varepsilon^{12} e_1 \otimes e_2 \\
& + \varepsilon^{21} e_1 \otimes e_2 + \varepsilon^{22} e_2 \otimes e_2 \\
= {} & e_1 \otimes e_2 - e_2 \otimes e_1 \\
= {} & e_1 \wedge e_2
\end{aligned}$$

A slightly more complicated example is $\varepsilon^I(2, 4, \text{identity}) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$. In $\varepsilon^{i_1 i_2, i_3, i_4}$, each i_m can be either a 1 or a 2. Thus there are 2^4 terms being summed. But whenever at least three of the i_m are a 1 or 2, the corresponding term is zero. Hence there are really only six terms making up $\varepsilon^I(2, 4, \text{identity}) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$. We have

$$\begin{aligned}
\varepsilon^I(2,4,\text{identity}) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4} = {} & \varepsilon^{11}\varepsilon^{22} e_1 \otimes e_1 \otimes e_2 \otimes e_2 \\
& + \varepsilon^{12}\varepsilon^{12} e_1 \otimes e_2 \otimes e_1 \otimes e_2 \\
& + \varepsilon^{12}\varepsilon^{21} e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& + \varepsilon^{21}\varepsilon^{12} e_2 \otimes e_1 \otimes e_1 \otimes e_2 \\
& + \varepsilon^{21}\varepsilon^{21} e_2 \otimes e_1 \otimes e_2 \otimes e_1 \\
& + \varepsilon^{22}\varepsilon^{11} e_2 \otimes e_2 \otimes e_1 \otimes e_1 \\
= {} & e_1 \otimes e_2 \otimes e_1 \otimes e_2 \\
& - e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& - e_2 \otimes e_1 \otimes e_1 \otimes e_2 \\
& + e_2 \otimes e_1 \otimes e_2 \otimes e_1 \\
= {} & (e_1 \wedge e_2) \otimes (e_1 \wedge e_2).
\end{aligned}$$

We will see in section three that this will be the invariant on four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ corresponding to the product of determinants:

$$[\mathbf{v}_1, \mathbf{v}_2][\mathbf{v}_3, \mathbf{v}_4].$$

One more example that we will use later. Consider $\varepsilon^I(2, 4, (23)) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$. All we need to do is to flip, in the above formulas, i_2 with i_3 in the $\varepsilon^{i_1 i_2, i_3, i_4}$. Thus

$$\begin{aligned}
\varepsilon^I(2,4,(23) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4} = {} & \varepsilon^{12}\varepsilon^{12} e_1 \otimes e_1 \otimes e_2 \otimes e_2 \\
& + \varepsilon^{11}\varepsilon^{22} e_1 \otimes e_2 \otimes e_1 \otimes e_2 \\
& + \varepsilon^{12}\varepsilon^{21} e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& + \varepsilon^{21}\varepsilon^{12} e_2 \otimes e_1 \otimes e_1 \otimes e_2
\end{aligned}$$

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$$
\begin{aligned}
&+ \varepsilon^{22}\varepsilon^{11} e_2 \otimes e_1 \otimes e_2 \otimes e_1 \\
&+ \varepsilon^{21}\varepsilon^{21} e_2 \otimes e_2 \otimes e_1 \otimes e_1 \\
= {}& e_1 \otimes e_1 \otimes e_2 \otimes e_2 \\
&- e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
&- e_2 \otimes e_1 \otimes e_1 \otimes e_2 \\
&+ e_2 \otimes e_2 \otimes e_1 \otimes e_1.
\end{aligned}
$$

In section three we will see that this will be the invariant on the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ corresponding to the product of determinants:

$$[\mathbf{v}_1, \mathbf{v}_3][\mathbf{v}_2, \mathbf{v}_4].$$

2. Generators for invariants of $Bil(V, W)$. This section is a quick review of the notation and the results in [5], which we need for the rest of this paper.

Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_k$ be bases for V and W and $e^1, \ldots, e^n$ and $f^1, \ldots, f^k$ be dual bases for V^* and W^*. The goal in [5] is to find the invariant one-dimensional subspaces of $(V \otimes V \otimes W^*)^{\otimes r}$, for each possible r. We will throughout regularly identify $(V \otimes V \otimes W^*)^{\otimes r}$ with $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$.

Let r be a positive integer such that n divides $2r$ and k divides r. For any element σ in the permutation group S_{2r} and any element η in S_r, define

$$v_\sigma = \varepsilon^I(n, 2r, \sigma) e_{i_1} \otimes \ldots \otimes e_{i_{2r}}$$

and

$$w^\eta = \varepsilon_J(k, r, \eta) f^{i_1} \otimes \ldots \otimes f^{i_r}.$$

THEOREM 2.1. *Vector space $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$ has an invariant one-dimensional subspace if and only if n divides $2r$ and k divides r. Every invariant one-dimensional subspace is a linear combination of various $v_\sigma \otimes w^\eta$, where σ and η range through S_{2r} and S_r, respectively.*

For each r, denote the subspace generated by all of the various $v_\sigma \otimes w^\eta$ in $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$ by

$$(V^{\otimes 2r} \otimes (W^*)^{\otimes r})_{\text{inv}}.$$

As shown in [5], for each r the corresponding weights are the same. Hence the sum of any two $v_\sigma \otimes w^\eta$ spans another one-dimensional invariant subspace of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$. Thus, for each r, $(V^{\otimes 2r} \otimes (W^*)^{\otimes r})_{\text{inv}}$ is the invariant subspace of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$ under our group action.

Hence for each r, the theorem is giving us a spanning set for $(V^{\otimes 2r} \otimes (W^*)^{\otimes r})_{\text{inv}}$. Part of the goal of this paper is to produce an algorithm to find the relations among the elements for these spanning sets.

Let us put this into the language of bilinear forms, which will aid us later when we look at specific examples. By making our choice of bases, we can write each

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bilinear map from $V \times V$ to W as a k-tuple of $n \times n$ matrices $(B^1, \ldots, B^k)$, where each $B^\alpha = (B^\alpha_{ij})$. More precisely, if $b \in Bil(V, W)$, then $B^\alpha_{ij} = f^\alpha b(e_i, e_j)$. We can restate the above theorem in terms of the B^α_{ij}.

THEOREM 2.2. *There exists a nonzero homogeneous invariant of degree r on $Bil(V, W)$ only if n divides $2r$ and k divides r. Further, every such homogeneous invariant is a linear combination of various f^σ_η, where*

$$f^\sigma_\eta = \varepsilon^I(n, 2r, \sigma)\varepsilon_J(c, r, \eta)B^{j_1}_{i_1 i_2} \ldots B^{j_r}_{i_{2r-1} i_{2r}}.$$

Again, all of this is in [5].

For an example, let V have dimension two and W have dimension one. Let $r = 1$. Then our vector-valued form B can be represented either as a two form

$$ae^1 \otimes e^1 + be^1 \otimes e^2 + ce^2 \otimes e^1 + de^2 \otimes e^2$$

or as a two by two matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We set

$$v_\sigma = \varepsilon^I(2, 2, \text{identity})e_{i_1} \otimes e_{i_2} = e_1 \otimes e_2 - e_2 \otimes e_1.$$

Since W has dimension one, we must have w^η be the identity. Then $v_\sigma \otimes w^\eta$ acting on B will be

$$b - c$$

and is zero precisely when the matrix B is symmetric.

3. Relations among invariants for the general linear group.

3.1. Nontrivial relations. For $Gl(n, \mathbf{C})$ acting on a vector space V, classically not only are the invariants known, but also so are the relations. Everything in this section is well-known. We will first discuss the second fundamental theorem in the language of brackets, or determinants. This is the invariant language for which the second fundamental theorem is the most clear. We then will state the second fundamental theorem for the two cases that we need in this paper, namely for $V \otimes V$ and W^*.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n column vectors in $\mathbf{C}^n$. The general linear group $Gl(n, \mathbf{C})$ acts on the vectors in $\mathbf{C}^n$ by multiplication on the left. Classically, the *bracket* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, denoted by $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$, is defined to be the determinant of the $n \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Thus by definition

$$[\mathbf{v}_1, \ldots, \mathbf{v}_n] = \det(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

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By basic properties of the determinant, we have that the bracket is an invariant, since

$$\begin{aligned}[A\mathbf{v}_1, \dots, A\mathbf{v}_n] &= \det(A\mathbf{v}_1, \dots, A\mathbf{v}_n)\\ &= \det(A)\det(\mathbf{v}_1, \dots, \mathbf{v}_n)\\ &= \det(A)[\mathbf{v}_1, \dots, \mathbf{v}_n]\end{aligned}$$

The punchline of the first fundamental theorem in this language is that the only invariants are combinations of various brackets and hence of determinants. (See p. 22 in [3] or p. 45 in [13].)

The second fundamental theorem reflects the fact that the determinant of a matrix with two identical rows is zero. Choose $n+1$ column vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$ in $\mathbf{C}^n$. Label the entries of the vector $\mathbf{v}_i = (v_{ij})$, for $1 \le j \le n$. Let $\mathbf{e}_k$ denote the vectors in the standard basis for $\mathbf{C}^n$. Thus all of the entries in $\mathbf{e}_k$ are zero, except in the kth entry, which is one. Then

$$\begin{aligned}[\mathbf{v}_i, \mathbf{e}_2, \dots, \mathbf{e}_n] &= \det(\mathbf{v}_i, \mathbf{e}_2, \dots, \mathbf{e}_n)\\ &= v_{i1}\end{aligned}$$

Now consider the $(n+1)\times(n+1)$ matrix

$$V = \begin{pmatrix} v_{11} & v_{21} & \cdots & v_{(n+1)1}\\ v_{11} & v_{21} & \cdots & v_{(n+1)1}\\ v_{12} & v_{22} & \cdots & v_{(n+1)2}\\ \vdots & \vdots & \vdots & \vdots\\ v_{1n} & v_{2n} & \cdots & v_{(n+1)n} \end{pmatrix}.$$

Since its top two rows are identical, its determinant is zero. Then

$$\begin{aligned}0 &= \det(V)\\ &= \sum_{k=1}^{n+1}(-1)^{k+1}v_{k1}\det(\mathbf{v}_1, \dots, \hat{\mathbf{v}}_k, \dots, \mathbf{v}_{n+1})\\ &= \sum_{k=1}^{n+1}(-1)^{k+1}[\mathbf{v}_1, \dots, \hat{\mathbf{v}}_k, \dots, \mathbf{v}_{n+1}][\mathbf{v}_k, \mathbf{e}_2, \dots, \mathbf{e}_n],\end{aligned}$$

where $\hat{\mathbf{v}}_k$ means delete the $\mathbf{v}_k$ term. This equation is a relation among brackets. The punchline of the second fundamental theorem is that all nontrivial relations are of the form

$$\sum_{k=1}^{n+1}(-1)^{k+1}[\mathbf{v}_1, \dots, \hat{\mathbf{v}}_k, \dots, \mathbf{v}_{n+1}][\mathbf{v}_k, \mathbf{w}_2, \dots, \mathbf{w}_n] = 0,$$

where $\mathbf{w}_2, \dots, \mathbf{w}_n$ are any column vectors.

We will define *nontrivial* in the next subsection. Basically the trivial relations stem from the fact that rearranging the columns of a matrix will only change the

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determinant by at most a sign. Thus rearranging the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and then taking the bracket will give us, up to sign, the same invariant.

Now to quickly put this into the language of tensors, first for the relations for $V \otimes V$. Using the notation of the last section, let n denote the dimension of V and let $2r = nu$. We know that all invariants for a given r are generated by all possible $v_\sigma = \varepsilon^I(n, 2r, \sigma) e_{i_1} \otimes \ldots \otimes e_{i_{2r}}$. All of these invariants have the same weight. Thus, for each r, the sum of any two v_σ spans another invariant one dimensional subspace in $(V \otimes V)^{\otimes r}$. Denote the subspace of $(V \otimes V)^{\otimes r}$ spanned by the various v_σ, with $\sigma \in S_{2r}$, by

$$(V \otimes V)^r_{\text{inv}}.$$

Then the first fundamental theorem in this context can be interpreted as giving a spanning set for $(V \otimes V)^r_{\text{inv}}$, for each r. For each r, denote the vector space with basis indexed by the v_σ by

$$(V \otimes V)^r_0.$$

There is thus an onto linear transformation

$$(V \otimes V)^r_0 \to (V \otimes V)^r_{\text{inv}}.$$

We want to find a spanning set of the kernel of this map. This will be a set of relations among the generators of the invariants.

We first need some notation. Let $\{i_1, \ldots, i_{n+1}\}$ be a subset of $n+1$ distinct elements chosen from $\{1, 2, \ldots, 2r\}$ and let $\sigma \in S_{2r}$. For $1 \le j \le n+1$, define $\sigma_j \in S_{2r}$ by setting $\sigma_j(k) = \sigma(k)$ if k is not in $\{i_1, \ldots, i_{n+1}\}$ and

$$\sigma_j(i_k) = \begin{cases} \sigma(i_k) & \text{if } k < j \\ \sigma(i_{k-1}) & \text{if } j < k \\ \sigma(i_{n+1}) & \text{if } j = k \end{cases}$$

For a fixed σ and subsequence $\{i_1, \ldots, i_{n+1}\}$, let $A(\sigma, \{i_1, \ldots, i_{n+1}\})$ denote the subset of S_{2r} consisting of the σ_j. Then the earlier stated second fundamental theorem can be reformulated in this context as:

THEOREM 3.1. *All nontrivial relations for $(V \otimes V)^{\otimes r}$ are linear combinations of*

$$\sum_{\sigma_j \in A(\sigma, \{i_1, \ldots, i_{n+1}\})} (-1)^{j+1} v_{\sigma_j} = 0,$$

for all possible subsets $\{i_1, \ldots, i_{n+1}\}$ and all possible $\sigma \in S_{2r}$. Again, the term *nontrivial* means the same as before and is hence simply dealing with the fact that if you flip two columns of a matrix, the corresponding determinants changes sign.

Note that this theorem states that the relations are linear for generators v_σ with $\sigma \in S_{2r}$. Thus we are indeed capturing a spanning set for the kernel of the map $(V \otimes V)^r_0 \to (V \otimes V)^r_{\text{inv}}$, for each r. Fixing r, denote the vector space with basis indexed by each of the above relations and by each of the trivial relations by

$$(V \otimes V)^r_1.$$

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Then we have an exact sequence

$$(V \otimes V)_1^r \to (V \otimes V)_0^r \to (V \otimes V)_{\mathrm{inv}}^r,$$

an exact sequence that is a linear algebra description of both the first and second fundamental theorems for this particular group action.

The relations for the invariants of the general linear group $Gl(k, C)$ acting on W^* are similar. Here we have $r = kv$. The invariants are generated by $w^\eta = \varepsilon_J(k, r, \eta) f^{i_1} \otimes \ldots \otimes f^{i_r}$. Choose $k+1$ distinct elements $\{i_1, \ldots, i_{k+1}\}$ from $\{1, 2, \ldots, r\}$ and an $\eta \in S_r$. Let $B(\eta, \{i_1, \ldots, i_{k+1}\})$ denote the set of all $\eta_j \in S_r$, for $1 \leq j \leq k+1$, defined by setting $\eta_j(l) = \eta(l)$ if i is not in $\{i_1, \ldots, i_{k+1}\}$ and

$$\eta_j(i_l) = \begin{cases} \eta(i_l) & \text{if } l < j \\ \eta(i_{l-1}) & \text{if } j < l \\ \eta(i_{n+1}) & \text{if } j = l \end{cases}$$

Then

Theorem 3.2. *All nontrivial relations for $(W^*)^{\otimes r}$ are linear combinations of*

$$\sum_{\eta_j \in B(\eta, \{i_1, \ldots, i_{k+1}\})} (-1)^j w^{\eta_j} = 0,$$

for all possible $\{i_1, \ldots, i_{k+1}\}$ and $\eta \in S_r$. The proofs of theorems 3 and 4 are in [13] on pp. 70-76. Weyl uses the bracket notation, but the equivalence is straightforward. A matrix approach is in section II.3, on page 71, in [1].

Mirroring what we did above, we know that the invariant linear subspaces for $(W^*)^{\otimes r}$ are generated by all possible w^η, for $\eta \in S_r$ and that all of these invariants have the same weight. Thus for each r, the various w^η span an invariant subspace of $(W^*)^{\otimes r}$. Denote this subspace by

$$(W^r)_{\mathrm{inv}}.$$

For each r, let the vector space with basis indexed by the w^η be $(W^*)_0^r$. Let the vector space with basis indexed by the above relations for the various w^η and by the trivial relations be denoted by $(W^*)_1^r$. Then we have the exact sequence

$$(W^*)_1^r \to (W^*)_0^r \to (W^*)_{\mathrm{inv}}^r,$$

Now for an example. We first will write down a relation in the bracket notation, give the translation in terms of tensors and then see that this explicit relation is in the above list. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}$ be any four column vectors in $\mathbf{C}^2$. Then by explicit calculation we have

$$[\mathbf{v}_1, \mathbf{v}_2][\mathbf{v}_3, \mathbf{w}] - [\mathbf{v}_1, \mathbf{v}_3][\mathbf{v}_2, \mathbf{w}] + [\mathbf{v}_2, \mathbf{v}_3][\mathbf{v}_1, \mathbf{w}] = 0.$$

In the dimension two vector space W^*, with basis f^1, f^2, consider the corresponding relation

$$\Sigma = w^{(1)} - w^{(23)} + w^{(132)}$$

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$$
\begin{aligned}
&= (f^1 \otimes f^2 \otimes f^1 \otimes f^2 - f^1 \otimes f^2 \otimes f^2 \otimes f^1 \\
&\quad - f^2 \otimes f^1 \otimes f^1 \otimes f^2 + f^2 \otimes f^1 \otimes f^2 \otimes f^1) \\
&\quad + (f^2 \otimes f^1 \otimes f^1 \otimes f^2 - f^2 \otimes f^2 \otimes f^1 \otimes f^1 \\
&\quad - f^1 \otimes f^1 \otimes f^2 \otimes f^2 + f^1 \otimes f^2 \otimes f^2 \otimes f^1) \\
&\quad + (f^1 \otimes f^1 \otimes f^2 \otimes f^2 - f^2 \otimes f^1 \otimes f^2 \otimes f^1 \\
&\quad - f^1 \otimes f^2 \otimes f^1 \otimes f^2 + f^2 \otimes f^2 \otimes f^1 \otimes f^1) \\
&= 0.
\end{aligned}
$$

Now to show that the relation Σ is in the above list. We have $r = 4$. Let $\eta \in S_4$ be the identity permutation. Let our sequence $\{i_1, i_2, i_3\}$ be simply $\{1, 2, 3\}$. Then η_1 is the permutation (132), η_2 is the permutation (23) and η_3 is the identity permutation. Thus the relation Σ is an example of the relation:

$$w^{\eta_3} - w^{\eta_2} + w^{\eta_1} = 0.$$

3.2. Trivial relations. All of this section is still classical.

It can be directly checked, continuing with our example for the two dimensional vector space W^*, that

$$w^{(1)} + w^{(123)} + w^{(132)} = 0.$$

Here the invariant $w^{(123)}$ is trivially related to the invariant $w^{(23)}$ (more specifically, $w^{(123)} = -w^{(23)}$). This is easiest to see in bracket notation, as this is just reflecting that

$$[\mathbf{v}_3, \mathbf{v}_1][\mathbf{v}_2, \mathbf{w}] = -[\mathbf{v}_1, \mathbf{v}_3][\mathbf{v}_2, \mathbf{w}],$$

which in turn simply reflects that fact that the sign of a determinant changes when we flip two columns.

This is the source of all relations that we want to call *trivial*. Rearranging the columns of a matrix will not change the determinant if the rearrangement is given by an even permutation of the permutation group and will change the determinant by a sign if the rearrangement is given by an odd permutation of the permutation group.

We will give the explicit criterion for trivial relations for the case of a k dimensional vector space W^*. As always, let $r = kv$. Our goal is to determine, given $\sigma, \tau \in S_r$, when

$$w^\sigma = \pm w^\tau.$$

This happens when we have the following v equalities of sets:

$$
\begin{aligned}
\{\sigma^{-1}(1), \ldots, \sigma^{-1}(k)\} &= \{\tau^{-1}(1), \ldots, \tau^{-1}(k)\} \\
\{\sigma^{-1}(k+1), \ldots, \sigma^{-1}(2k)\} &= \{\tau^{-1}(k+1), \ldots, \tau^{-1}(2k)\} \\
&\vdots \\
\{\sigma^{-1}((v-1)k+1), \ldots, \sigma^{-1}(kv)\} &= \{\tau^{-1}((v-1)k+1), \ldots, \tau^{-1}(kv)\}
\end{aligned}
$$

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Each set on the right is thus a permutation of the corresponding set on the left. We will have $w^\sigma = w^\tau$ if there are an even number of odd permutations taking the left hand side of the above set equalities to the right and $w^\sigma = -w^\tau$ if there are an odd number.

Consider our initial example $w^{(123)} = -w^{(23)}$ when W^* is two dimensional. Let $\sigma = (123)$ and $\tau = (23)$. Then

$$\sigma^{-1}(1) = 3, \sigma^{-1}(2) = 1, \sigma^{-1}(3) = 2, \sigma^{-1}(4) = 4$$

and

$$\tau^{-1}(1) = 1, \tau^{-1}(2) = 3, \sigma^{-1}(3) = 2, \sigma^{-1}(4) = 4.$$

Then $\{\sigma^{-1}(1), \sigma^{-1}(2)\}$ is an odd permutation of $\{\tau^{-1}(1), \tau^{-1}(2)\}$, while $\{\sigma^{-1}(3), \sigma^{-1}(4)\}$ is exactly the same as $\{\tau^{-1}(1), \tau^{-1}(2)\}$, reflecting the fact that $w^{(123)} = -w^{(23)}$.

4. A Second Fundamental Theorem for $(V \otimes V \otimes W^*)^{\otimes r}$. We have the two exact sequences

$$(V \otimes V)_1^r \to (V \otimes V)_0^r \to (V \otimes V)_{inv}^r$$

and

$$(W^*)_1^r \to (W^*)_0^r \to ((W^*)^r)_{\text{inv}}.$$

Tensoring either of these exact sequences by a complex vector space will maintain the exactness. The point of [5] is that the natural map

$$(V \otimes V)_0^r \otimes (W^*)_0^r \to (V \otimes V)_{\text{inv}}^r \otimes (W^*)_{\text{inv}}^r$$

is onto. We want to find the kernel of this map.

We have the following commutative double exact sequence:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
(V \otimes V)_1^r \otimes (W^*)_{\text{inv}}^r & \to & (V \otimes V)_0^r \otimes (W^*)_{\text{inv}}^r & \to & (V \otimes V)_{\text{inv}}^r \otimes (W^*)_{\text{inv}}^r & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
(V \otimes V)_1^r \otimes (W^*)_0^r & \to & (V \otimes V)_0^r \otimes (W^*)_0^r & \to & (V \otimes V)_{\text{inv}}^r \otimes (W^*)_0^r & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
(V \otimes V)_1^r \otimes (W^*)_1^r & \to & (V \otimes V)_0^r \otimes (W^*)_1^r & \to & (V \otimes V)_{\text{inv}}^r \otimes (W^*)_1^r & \to & 0
\end{array}$$

with

$$(V \otimes V)_0^r \otimes (W^*)_0^r \to (V \otimes V)_{\text{inv}}^r \otimes (W^*)_{\text{inv}}^r$$

from the above double exact sequence being onto. All of the above maps are linear transformations of vector spaces. A second fundamental theorem for vector-valued forms will be a description of the kernel of this map.

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THEOREM 4.1. *Under the natural maps from the above double exact sequence, the kernel of the map from* $(V \otimes V)_0^r \otimes (W^*)_0^r$ *to* $(V \otimes V)_{inv}^r \otimes (W^*)_{inv}^r$ *is*

$$(V \otimes V)_1^r \otimes (W^*)_0^r \oplus (V \otimes V)_0^r \otimes (W^*)_1^r.$$

The proof is a routine diagram chase.

Thus by standard arguments involving commutative diagrams, the following sequence of vector spaces is exact:

$$\begin{aligned}(V \otimes V)_1^r \otimes (W^*)_0^r \oplus (V \otimes V)_0^r \otimes (W^*)_1^r &\to (V \otimes V)_0^r \otimes (W^*)_0^r \\ &\to (V \otimes V)_{\text{inv}}^r \otimes (W^*)_{\text{inv}}^r \\ &\to 0.\end{aligned}$$

In another language, this theorem can be stated as:

THEOREM 4.2 (A Second Fundamental Theorem). *Among invariants of vector-valued bilinear forms, there exist relations of the following type:*

$$((V \otimes V)_1 \otimes (W^*)_0) \oplus ((V \otimes V)_0 \otimes (W^*)_1).$$

All relations are linear combinations of the above relations.

5. An example of a relation. Let V and W^* both have dimension two. Recall our example of a relation for W:

$$\Sigma = w^1 - w^{(23)} + w^{(132)}.$$

Here $k = 2$ and $r = 4$. Choose $(23)(67) \in S_8$. Then

$$v_{(23)(67)} = \varepsilon^I(2, 8, (23)(67)) e_{i_1} \otimes \ldots \otimes e_{i_{2r}}$$

$$\begin{aligned}
= {} & e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& - e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \\
& - e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& + e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \\
& - e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& + e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \\
& + e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 - e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \\
& - e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1.
\end{aligned}$$

Then we have the relation

$$\begin{aligned}v_{(23)(67)} \otimes \Sigma &= v_{(23)(67)} \otimes w^{(1)} - v_{(23)(67)} \otimes w^{(23)} + v_{(23)(67)} \otimes w^{(132)} \\ &= 0,\end{aligned}$$

which can now be directly checked.

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6. Finding Relations for Bilinear Forms. Most people, though, are not that interested in invariant one-dimensional subspaces of $(V \otimes V \otimes W^*)^{\otimes r}$ for various positive integers r, but are more interested in invariants of bilinear forms. As discussed in section 1.3, this means that we are interested in the algebra of homogeneous polynomials in $\mathbb{C}[V^* \otimes V^* \otimes W]$ that are invariant under the previously defined group action by $Aut(V) \times Aut(W)$. But the homogeneous polynomials of degree r can be identified to elements in the symmetric space $(V^* \otimes V^* \otimes W)^{\odot r}$. We thus want to find the invariant lines in the dual space and hence the invariant one-dimensional subspaces of $(V \otimes V \otimes W^*)^{\odot r}$. So far all we have are the invariant one dimensional subspaces of $(V \otimes V \otimes W^*)^{\otimes r}$.

Denote the vector space spanned by the invariant one-dimensional subspaces in $(V \otimes V \otimes W^*)^{\odot r}$ by

$$(V \otimes V \otimes W^*)^{\odot r}_{\text{inv}}.$$

There is a natural onto map from $(V \otimes V)^r_{\text{inv}} \otimes W^r_{\text{inv}}$ to $(V^* \otimes V^* \otimes W)^{\odot r}_{\text{inv}}$. This is simply the restriction to $(V \otimes V)^r_{\text{inv}} \otimes W^r_{\text{inv}}$ of the symmetrizing map

$$S : (V \otimes V \otimes W^*)^{\otimes r} \to (V \otimes V \otimes W^*)^{\odot r}.$$

We need, though, to check that an element of $(V \otimes V \otimes W^*)^{\otimes r}$ that generates a one-dimensional invariant subspace still generates a one-dimensional invariant subspace after the application of the map S.

The permutation group S_r acts naturally on both $(W^*)^{\otimes r}$ and on $(V \otimes V)^{\otimes r}$. Let $\tau \in S_r$. Then for any $\eta \in S_r$, it can be directly checked that

$$\tau(w^\eta) = w^{\eta \cdot \tau^{-1}},$$

another element in our list of generators.

Similarly, for any $\tau \in S_r$, we will have $\tau(v_\sigma)$ be in our list of generators, for any v_σ. Here the notation is a bit cumbersome. Each $\tau \in S_r$ will induce an element $\hat{\tau} \in S_{2r}$, where if $\tau(i) = j$, then

$$\hat{\tau}(2i-1) = 2j-1$$
$$\hat{\tau}(2i) = 2j.$$

Then it can be checked that

$$\tau(v_\sigma) = v_{\sigma \cdot \hat{\tau}^{-1}}.$$

Thus $\tau(v_\sigma)$ is another invariant.

Then, as is implicit in [5], we have:

Theorem 6.1 (First Fundamental Theorem for Bilinear Forms). *The vector space $(V \otimes V \otimes W^*)^{\odot r}$ has an invariant one-dimensional subspace if and only if n divides $2r$ and k divides r. Every invariant one-dimensional subspace is a linear combination of various $S(v_\sigma \otimes w^\eta)$, where σ and η range through S_{2r} and S_r, respectively.*

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We are interested in the relations among the various $S(v_\sigma \otimes w^\eta)$. Since $(V \otimes V \otimes W^*)^{\odot r}$ is contained in $(V \otimes V \otimes W^*)^{\otimes r}$, any relation in $(V \otimes V \otimes W^*)^{\odot r}$ must be in the relations for $(V \otimes V \otimes W^*)^{\odot r}$.

Hence we just need to map all of our previous relations to $(V \otimes V \otimes W^*)^{\odot r}$ via S. If there is a relation of invariants in $(V \otimes V \otimes W^*)^{\odot r}$, we have already captured it.

Thus we have

Theorem 6.2 (Second Fundamental Theorem for Bilinear Forms). *All nontrivial relations among the nonzero elements $S(v_\sigma \otimes w^\eta)$, where σ and η range through S_{2r} and S_r, respectively, are linear combinations of all*

$$S(v_\sigma \otimes \sum_{\eta_j \in B(\eta,\{i_1,\ldots,i_{k+1}\})} (-1)^j w^{\eta_j})$$

and

$$S(\sum_{\sigma_j \in A(\sigma,\{i_1,\ldots,i_{n+1}\})} (-1)^{j+1} v_{\sigma_j} \otimes w^\eta).$$

We had to use the term "nonzero" in the above theorem. Some of our invariants in $(V \otimes V \otimes W^*)^{\otimes r}$ will be mapped to zero under S. None of the above describes the kernel of S. As we will see in the next section, this does happen. In fact, if we consider the example of symmetrizing map $S : (W^*)^{\otimes r} \to (W^*)^{\odot r}$, then it is not at all obvious that every $v_\sigma \otimes w^\eta$ is not sent to zero, since

$$S(w^\eta) = 0$$

for all $\eta \in S_r$, which can be directly checked. (This just reflects that the geometric fact that there are no invariants for a singe vector in a vector space under the group action of the automorphisms of the vector space, since any vector can be sent to any other vector.) Again, we will see an example in the next section that there are nontrivial relations.

Thus we have an algorithm for finding the invariants and for finding the relations. For each r, we just map all of our generators in $(V \otimes V \otimes W^*)^{\otimes r}$ to $(V \otimes V \otimes W^*)^{\odot r}$ by S, disposing of those that map to zero. All the remaining relations will be already be accounted for by applying S to the previous relations.

7. An Example. We now translate the above relations into the language of invariant polynomials of bilinear forms, for the particular case of an element in $(V \otimes V \otimes W^*)^{\otimes 4}$. Let

$$B = \left(\begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^1 & b_{22}^1 \end{pmatrix}, \begin{pmatrix} b_{11}^2 & b_{12}^2 \\ b_{21}^2 & b_{22}^2 \end{pmatrix} \right)$$

be a bilinear form from $V \times V$ to W. As a tensor in $V^* \otimes V^* \otimes W$, this bilinear form becomes the tensor:

$$\begin{aligned} b &= b_{ij}^k e^i \otimes e^j \otimes f_k \\ &= b_{11}^1 e^1 \otimes e^1 \otimes f_1 + b_{12}^1 e^1 \otimes e^2 \otimes f_1 + b_{21}^1 e^2 \otimes e^1 \otimes f_1 + b_{22}^1 e^2 \otimes e^2 \otimes f_1 \\ &\quad + b_{11}^2 e^1 \otimes e^1 \otimes f_2 + b_{12}^2 e^1 \otimes e^2 \otimes f_2 + b_{21}^2 e^2 \otimes e^1 \otimes f_2 + b_{22}^1 e^2 \otimes e^2 \otimes f_2. \end{aligned}$$

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To see what the invariants $v_{(23)(67)} \otimes w^{(1)}$, $v_{(23)(67)} \otimes w^{(23)}$ and $v_{(23)(67)} \otimes w^{(132)}$ are in terms of the variables b^k_{ij}, we have each act on the tensor $b \otimes b \otimes b \otimes b$. Then, after a painful calculation, we get that

$$\begin{aligned}
v_{(23)(67)} \otimes w^{(1)}(b \otimes b \otimes b \otimes b) &= 0 \\
v_{(23)(67)} \otimes w^{(23)}(b \otimes b \otimes b \otimes b) &= 4b^1_{11}b^1_{22}b^2_{11}b^2_{22} + 4b^1_{12}b^1_{22}b^2_{11}b^2_{21} \\
&\quad -8b^1_{12}b^1_{21}b^2_{11}b^2_{22} - 8b^1_{11}b^1_{22}b^2_{12}b^2_{21} \\
&\quad +4b^1_{11}b^1_{21}b^2_{12}b^2_{22} + 4b^1_{21}b^1_{22}b^2_{11}b^2_{12} \\
&\quad -2b^1_{22}b^1_{22}b^2_{11}b^2_{11} - 2b^1_{11}b^1_{11}b^2_{22}b^2_{22} \\
&\quad +4b^1_{12}b^1_{21}b^2_{12}b^2_{21} - 2b^1_{21}b^1_{21}b^2_{12}b^2_{12} \\
&\quad -2b^1_{12}b^1_{12}b^2_{21}b^2_{21} + 4b^1_{11}b^1_{12}b^2_{21}b^2_{22} \\
v_{(23)(67)} \otimes w^{(132)}(b \otimes b \otimes b \otimes b) &= 4b^1_{11}b^1_{22}b^2_{11}b^2_{22} + 4b^1_{12}b^1_{22}b^2_{11}b^2_{21} \\
&\quad -8b^1_{12}b^1_{21}b^2_{11}b^2_{22} - 8b^1_{11}b^1_{22}b^2_{12}b^2_{21} \\
&\quad +4b^1_{11}b^1_{21}b^2_{12}b^2_{22} + 4b^1_{21}b^1_{22}b^2_{11}b^2_{12} \\
&\quad -2b^1_{22}b^1_{22}b^2_{11}b^2_{11} - 2b^1_{11}b^1_{11}b^2_{22}b^2_{22} \\
&\quad +4b^1_{12}b^1_{21}b^2_{12}b^2_{21} - 2b^1_{21}b^1_{21}b^2_{12}b^2_{12} \\
&\quad -2b^1_{12}b^1_{12}b^2_{21}b^2_{21} + 4b^1_{11}b^1_{12}b^2_{21}b^2_{22}
\end{aligned}$$

Since $v_{(23)(67)} \otimes w^{(23)}(b \otimes b \otimes b \otimes b) = v_{(23)(67)} \otimes w^{(132)}(b \otimes b \otimes b \otimes b)$ and since $v_{(23)(67)} \otimes w^{(1)}(b \otimes b \otimes b \otimes b) = 0$, we see that our relations from the tensor language do translate to relations on the invariant polynomials in the b^k_{ij} terms.

Also, note that this means that

$$S(v_{(23)(67)} \otimes w^{(1)}) = 0$$

and hence $v_{(23)(67)} \otimes w^{(1)}$ is in the kernel of the symmetrizing map $S : (V \otimes V \otimes W^*)^{\otimes r} \to (V \otimes V \otimes W^*)^{\odot r}$.

Of course, it is difficult to see what these invariants and relations actually measure. There must be geometry behind these formulas, though it is almost always hidden. There are times that we can understand some of the information contained in the formulas. As an example, we will now see why we chose $r = 4$ for our example. We will see that for when the rank of V and W are both two, this is the first time we would expect any interesting invariants for bilinear forms. Consider the relation

$$v_{(1)} - v_{(23)} + v_{(132)} = 0$$

for $V^* \otimes V^*$. (This is the V-analogue of our earlier $w^1 - w^{(23)} + w^{(132)} = 0$.) Next we construct an invariant of $(W)_0$ by letting $k = 2$, $r = 2$, and $\eta = \mathrm{id}$. This gives us the invariant $w^\eta = f^1 \otimes f^2 - f^2 \otimes f^1$. Then a relation is

$$(v_{(1)} - v_{(23)} + v_{(132)}) \otimes w^\eta = 0.$$

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But this relation is not at all interesting when made into a statement about polynomials on the entries of a bilinear form, since each invariant becomes the zero polynomial:

$$(v_{(1)} \otimes w^{\eta})(b \otimes b) = 0$$
$$(v_{(23)} \otimes w^{\eta})(b \otimes b) = 0$$
$$(v_{(132)} \otimes w^{\eta})(b \otimes b) = 0,$$

following from a direct calculation.

When $n = k = 2$, the first time that interesting invariants can occur is indeed when $r = 4$ (which is why our example has $r = 4$), as we will now see. Start with our bilinear form

$$B = (B_1, B_2) = \left(\begin{pmatrix} b^1_{11} & b^1_{12} \\ b^1_{21} & b^1_{22} \end{pmatrix}, \begin{pmatrix} b^2_{11} & b^2_{12} \\ b^2_{21} & b^2_{22} \end{pmatrix} \right)$$

and consider the polynomial

$$\begin{aligned} P(x, y) &= \det(xB_1 + yB_2) \\ &= \det \begin{pmatrix} xb^1_{11} + yb^2_{11} & xb^1_{12} + yb^2_{12} \\ xb^1_{21} + yb^2_{21} & xb^1_{22} + yb^2_{22} \end{pmatrix} \\ &= (b^1_{11}b^1_{22} - b^1_{12}b^1_{21})x^2 + (b^1_{11}b^2_{22} + b^1_{22}b^2_{11} - b^1_{12}b^2_{21} - b^1_{21}b^2_{12})xy \\ &\quad +(b^2_{11}b^2_{22} - b^2_{12}b^2_{21})y^2, \end{aligned}$$

a polynomial that Mizner [8] used in the study of codimension two CR structures and which was mentioned earlier, independently, by Griffiths and Harris in the study of codimension two subvarieties of complex projective space [6]. Note that $P(x, y)$ is homogeneous of degree two in the variables x and y. Let $A \in GL(n, \mathbf{C})$ act on our bilinear form B. Thus we have B becoming $(A^T B_1 A, A^T B_2 A)$. Then the polynomial $P(x, y)$ transforms as follows:

$$\det(xA^T B_1 A + yA^T B_2 A) = |\det(A)|^2 \det(xB_1 + yB_2) = |\det(A)|^2 P(x, y).$$

By looking at this polynomial, we have effectively eliminated the influence of the $GL(n, \mathbf{C})$ action. In other words, one method for generating invariants of the bilinear form B under the action of $GL(n, \mathbf{C}) \times GL(k, \mathbf{C})$ is to find invariants of the polynomial $P(x, y)$ under the action of $GL(k, \mathbf{C})$. The action of $GL(k, \mathbf{C})$ is just the standard change of basis on the variables x and y. For degree two homogeneous polynomials in two variables, it is well know that the only invariant is the discriminant. (See Chapter One of [11]; recall, for the polynomial $Ax^2 + Bxy + Cy^2$, that the discriminant is $B^2 - 4AC$.) Thus for our bilinear form B, the invariant corresponding to the discriminant of the polynomial $\det(xB_1 + yB_2)$ will be

$$(b^1_{11}b^2_{22} + b^1_{22}b^2_{11} - b^1_{12}b^2_{21} - b^1_{21}b^2_{12})^2 - 4(b^1_{11}b^1_{22} - b^1_{12}b^1_{21})(b^2_{11}b^2_{22} - b^2_{12}b^2_{21}),$$

which can be checked is

$$(-1/2)v_{(23)(67)} \otimes w^{(23)}.$$

Again, we have added this last part only to emphasis that there is geometry and meaning (though largely unexplored) behind the mechanical, almost crude, invariants that this paper generates.

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8. Conclusions. There are many questions remaining. One difficulty is in determining what a particular invariant or relation means. This paper and [5] just give lists, with no clue as to which have any type of important meaning, save for the invariants associated to the Mizner polynomial. (Of course, this is one of the difficulties in almost all of classical invariant theory). Even harder is to determine when two vector-valued forms are equivalent. It is highly unlikely that the algebraic techniques in this paper will answer this question.

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REFERENCES

[1] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris. *Geometry of Algebraic Curves*, Vol. I. Grundlehren der mathematischen Wissenschaften 267, Springer-Verlag, New York, 1985.
[2] William Fulton and Joe Harris. *Representation Theory.* Springer-Verlag, New York, 1991.
[3] J.A. Dieudonne and J.B. Carrell. Invariant theory, old and new. *Adv. Math.*, 4:1–89, 1970.
[4] E. Formanek. The invariants of $n \times n$ matrices. In *Invariant Theory* (S.S. Koh, Ed.), LNM 1268, Springer-Verlag, New York, 1987.
[5] Thomas Garrity and Robert Mizner. Invariants of vector-valued bilinear and sesquilinear forms. *Linear Algebra Appl.*, 218:225–237, 1995.
[6] P. Griffiths and J. Harris. Algebraic geometry and local differential geometry. *Ann. Scient. Ec. Norm. Sup.*, 12:355–432, 1979.
[7] Zachary J. Grossman. Relations and syzygies in classical invariant theory for vector-valued bilinear forms. Williams College thesis, 1999.
[8] Robert I. Mizner. CR structures of codimension 2. *J. Differential Geom.*, 30(1):167–190, 1989.
[9] V.L. Popov. *Groups, Generators, Syzygies, and Orbits in Invariant Theory.* Translations of Mathematical Monographs, vol. 100, AMS, Providence, 1992.
[10] C. Procesi. The invariant theory of $n \times n$ matrices. *Adv. Math.*, 19:306–381, 1976.
[11] Peter J. Olver. *Classical Invariant Theory.* London Mathematical Society Student Texts 44, Cambridge University Press, 1999.
[12] Bernd Sturmfels. *Algorithms in Invariant Theory.* Springer-Verlag, New York, 1993.
[13] Hermann Weyl. *The Classical Groups.* Princeton University Press, Princeton, NJ, 1939.
[14] Takeo Yokonuma. *Tensor Spaces and Exterior Algebra.* Translations of Mathematical Monographs, vol. 108, AMS, Providence, 1992.