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STRONGLY REGULAR GRAPHS: VALUES OF λ AND μ FOR WHICH THERE ARE ONLY FINITELY MANY FEASIBLE $(v, k, \lambda, \mu)^*$

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Dedicated to the memory of Prof. Dom de Caen

Abstract. Given λ and μ, it is shown that, unless one of the relations $(\lambda - \mu)^2 = 4\mu,\ \lambda - \mu = -2,\ (\lambda - \mu)^2 + 2(\lambda - \mu) = 4\mu$ is satisfied, there are only finitely many parameter sets (v, k, λ, μ) for which a strongly regular graph with these parameters could exist. This result is used to prove as special cases a number of results which have appeared in previous literature.

Key words. Strongly Regular Graph, Adjacency Matrix, Feasible Parameter Set.

AMS subject classifications. 05C50, 05E30.

1. Properties of Strongly Regular Graphs. In this section we present some well known results needed in Section 2. The results can be found in [11, ch.21] or [8, ch.10]. A *strongly regular graph with parameters* (v, k, λ, μ), denoted $SRG(v, k, \lambda, \mu)$, is a k-regular graph on v vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and for every pair of non-adjacent vertices there are μ vertices adjacent to both. We assume throughout that a strongly regular graph G is connected and that G is not a complete graph. Consequently, k is an eigenvalue of the adjacency matrix of G with multiplicity 1 and

$$v - 1 > k \geq \mu > 0 \quad \text{and} \quad k - 1 > \lambda \geq 0. \tag{1.1}$$

Counting the number of edges in G connecting the vertices adjacent to a vertex x and the vertices not adjacent to x in two ways we obtain

$$k(k - \lambda - 1) = (v - k - 1)\mu. \tag{1.2}$$

Thus if we are given three of the parameters v, k, λ, μ, the fourth is uniquely determined. If A is the adjacency matrix of an $SRG(v, k, \lambda, \mu)$, we have the equation

$$A^2 = kI + \lambda A + \mu(J - A - I). \tag{1.3}$$

Since eigenvectors with eigenvalue $\theta \neq k$ are orthogonal to the all-ones vector, then by (1.3) the remaining eigenvalues must satisfy the equation

$$\theta^2 - (\lambda - \mu)\theta - (k - \mu) = 0. \tag{1.4}$$

Thus the eigenvalues of G are

$$k, \quad \text{and} \quad \theta_1, \theta_2 = \frac{(\lambda - \mu) \pm \sqrt{\Delta}}{2}, \tag{1.5}$$

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where

$$\Delta = (\lambda - \mu)^2 + 4(k - \mu) > 0. \tag{1.6}$$

Because the sum of the eigenvalues equals trace$(A) = 0$ it follows easily (see equations (2.4) and (2.5)) that the corresponding multiplicities are

$$1, \quad \text{and} \quad m_1, m_2 = \frac{1}{2}\left(v - 1 \pm \frac{(v-1)(\mu - \lambda) - 2k}{\sqrt{\Delta}}\right). \tag{1.7}$$

Since the values m_1 and m_2 are multiplicities of eigenvalues, they must be integers. The complement of an $SRG(v, k, \lambda, \mu)$ is an $SRG(v, \overline{k}, \overline{\lambda}, \overline{\mu})$, where

$$\overline{k} = v - k - 1,\ \overline{\lambda} = v - 2k + \mu - 2, \quad \text{and} \quad \overline{\mu} = v - 2k + \lambda.$$

From these parameters we see that $\overline{\lambda} = v - 2k + \mu - 2 \geq 0$ is a necessary condition for an $SRG(v, k, \lambda, \mu)$ to exist.

Definition 1.1. Parameter sets of nonnegative integers (v, k, λ, μ) where (1.1) and (1.2) hold, m_1 and m_2 are positive integers, and $\overline{\lambda} \geq 0$, are called *feasible parameter sets.*

The term *feasible* is used here only to mean that certain preliminary requirements are met; a feasible parameter set need not be the parameter set of an *srg*. There are other conditions that the parameters of an *srg* must satisfy (e.g. the *Krein Conditions* [11, p.237]). However, it turns out to be more convenient to work with the restricted list of conditions in the definition and apply additional conditions to our results in Table 2.1 for specific cases.

Given λ and μ, we show in Section 2 that, unless λ and μ satisfy one of three relations, there are only finitely many feasible parameter sets (v, k, λ, μ). We give formulas for the parameters, eigenvalues, and eigenvalue multiplicities for each of the three cases for which there may be infinitely many feasible parameter sets. In Section 3 we present some consequences of the results in Section 2, some of which have appeared in the literature.

Strongly regular graphs with $m_1 = m_2$ are called *conference graphs.* We have the following two well-known results on conference graphs [11, p.235].

Lemma 1.2. *Let G be a strongly regular graph. Then G is a conference graph if and only if G is an $SRG(v, \frac{1}{2}(v-1), \frac{1}{4}(v-5), \frac{1}{4}(v-1))$.*

Lemma 1.3. *Let G be a strongly regular graph with parameters (v, k, λ, μ) and eigenvalues k, θ_1 and θ_2. If G is not a conference graph, then θ_1 and θ_2 are integers and Δ is a perfect square.*

2. Main Result. The proof technique used here is a natural extension of that used by Hoffman and Singleton [9] in their well-known result on diameter two Moore graphs (discussed in Section 3), and later by Biggs [2, p.102] in his result on strongly regular graphs with no triangles (Lemma 3.2) and by Berlekamp, van Lint, and Seidel [1, sec.2] in their result on strongly regular graphs with $\lambda - \mu = -1$ (Lemma 3.5). Suppose that λ and μ are given. We wish to determine the parameters v and k such that the parameter set (v, k, λ, μ) is feasible. By Lemma 1.2 there is at most

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one feasible parameter set with $m_1 = m_2$, so we assume throughout that $m_1 \neq m_2$. Then, by Lemma 1.3, Δ is a perfect square. Thus there is a positive integer s such that

$$\Delta = (\lambda-\mu)^2 + 4(k-\mu) = s^2 \tag{2.1}$$

and so

$$k = \mu - \frac{(\lambda-\mu)^2 + s^2}{4}. \tag{2.2}$$

Also, from identity (1.2),

$$v - 1 = k + \frac{k(k-1-\lambda)}{\mu}. \tag{2.3}$$

Also

$$m_1 + m_2 = v - 1, \tag{2.4}$$

$$\theta_1 m_1 + \theta_2 m_2 = -k. \tag{2.5}$$

Substituting the values for θ_1 and θ_2 from (1.5) into equation (2.5) we get

$$2k + (\lambda-\mu)(m_1+m_2) + s(m_1-m_2) = 0.$$

Substituting into this equation for $v - 1 = m_1 + m_2$ as given in (2.3), then for k as given in (2.2), and finally multiplying through by 16 we get

$$\begin{aligned}&\big(8\mu + (\lambda-\mu)(4\mu - (\lambda-\mu)^2 - 4 - 4\lambda + 4\mu + s^2)\big)\\ &\big(4\mu - (\lambda-\mu)^2 + s^2\big) = 16s\mu(m_2 - m_1).\end{aligned} \tag{2.6}$$

Finally, taking both sides of (2.6) modulo s we get

$$\big(4\mu - (\lambda-\mu)^2\big)\big(8\mu + (\lambda-\mu)(8\mu - 4\lambda - 4 - (\lambda-\mu)^2)\big) \equiv 0 \pmod{s}.$$

Factoring this we get

$$c(\lambda,\mu) = \big(4\mu - (\lambda-\mu)^2\big)(\lambda-\mu+2)\big((\lambda-\mu)^2 + 2\lambda - 6\mu\big) \equiv 0 \pmod{s}. \tag{2.7}$$

Thus, given λ, μ, if $c(\lambda,\mu)$ is nonzero, then there are only finitely many possibilities for s and consequently only finitely many possibilities for v, k. However, if $c(\lambda,\mu) = 0$, then the congruence (2.7) provides no restrictions on s. We now have the following theorem.

THEOREM 2.1. *Let λ and μ be fixed. Then there are only finitely many feasible parameters sets (v, k, λ, μ), unless λ and μ satisfy one of the following three relations:*

$$\text{(a) } (\lambda-\mu)^2 = 4\mu;\ \text{(b) } \lambda-\mu = -2;\ \text{(c) } (\lambda-\mu)^2 + 2(\lambda-\mu) = 4\mu. \tag{2.8}$$

Consequently, unless λ and μ satisfy one of (2.8) there are only finitely many graphs with parameters v, k, λ, and μ. For λ and μ fixed, each divisor of $c(\lambda,\mu)$ realizes a

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possible value of s and thus a possible value of k using (2.1). The parameter v is then determined by (2.3). The multiplicities m_1 and m_2 are given by (1.7).

If one of the relations in (2.8) holds then there may be infinitely many v and k such that (v, k, λ, μ) is feasible. We shall see examples of such parameter sets in the next section. The parameter sets (v, k, λ, μ) for which λ and μ satisfy one of the three equations in (2.8) are summarized in the following table.

Case	v	k	λ	μ	m_1
(a)	$\frac{(r^2+r-t)(r^2-r-t)}{t^2}$	r^2	t^2+2t	t^2	$\frac{r(r^2+r-t)(r-t-1)}{2t^2}$
(b)	$\frac{(r^2+\mu)^2-r^2}{\mu}$	$r^2+\mu-1$	$\mu-2$	μ	$\frac{(r^2+r+\mu)(r^2+\mu-1)}{2\mu}$
(c)	$\frac{r^2(r^2-1)}{t(t+1)}$	r^2+t	t^2+3t	t^2+t	$\frac{(r^2+t)(r-t-1)(r+1)}{2t(t+1)}$

Table 1

In each of the three cases, $\lambda - \mu$ is even, so we may take $\lambda - \mu = 2t$ (in case (b), t=-1). Consequently s must be even, so we make the substitution $s = 2r$, $r > 0$. The eigenvalues are $\theta_1, \theta_2 = t \pm r$. Each choice of r, t in cases (a),(c) or of r, μ in case (b) with $\lambda \geq 0,\ \mu > 0$ such that $k > \lambda + 1$ will yield a feasible parameter set if $\overline{\lambda} \geq 0$ and $m_1, m_2 = v - 1 - m_1$ are positive integers. Note that t may be negative. Equation (1.2) was used in deriving the parameters, and so will be satisfied. The inequalities $v - 1 > k \geq \mu$ will then hold as well.

From the formulas for the parameters in each of these cases, it is evident that it is not enough for s to be a divisor of $c(\lambda, \mu)$ for the parameter set to be feasible. In general for each of these three cases there is not much that can be said about finding feasible parameter sets without specifying a value for t in cases (a), (c) or a value for μ in case (b). However, if there is one value of r for a given t or μ that gives a feasible parameter set, then any positive integer greater than r and equivalent to r modulo the denominators of $v,\ m_1$ and m_2 will also yield positive integer values for each of these parameters. Furthermore $\overline{\lambda} \geq 0$ for all r greater than some constant, since the coefficient of r^4, the term of highest degree in $\overline{\lambda}$, is positive in each case. Thus, if there is one value for r that gives a feasible parameter set, then there are infinitely many values of r for a given t or μ.

Strongly regular graphs with $\lambda - \mu = -2$ correspond to symmetric 2-$(v, k+1, \mu)$ designs with $\mu \geq 2$ that have a polarity with all points absolute (see [7, pp.13, 42-43] for definitions). There are infinitely many symmetric 2-$(v, k+1, 1)$ designs (the projective planes), but only finitely many symmetric 2-$(v, k+1, \mu)$ designs are known for each $\mu \geq 2$. Thus it is currently unknown whether there are infinitely many $SRG(v, k, \mu - 2, \mu)$ for a given $\mu \geq 2$.

For each of the relations in (2.8) there are a number of graphs known with λ and μ satisfying that relation. For example, the *triangular graph* $T(n)$ is the graph whose vertices are the 2-subsets of an n-set where two vertices are adjacent if they intersect. The graph $T(n)$ is an $SRG(\binom{n}{2}, 2(n-2), n-2, 4)$. The graph $T(10)$ is an $SRG(45, 16, 8, 4)$, satisfying the first relation. The complement of the graph $T(6)$ is an $SRG(15, 6, 1, 3)$, satisfying the second relation. Finally, the Higman-Sims graph [11, p.327] is an $SRG(100, 22, 0, 6)$ satisfying the third relation. There are, however,

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no pairs λ and μ known for which there are infinitely many strongly regular graphs with parameters (v, k, λ, μ). In fact, we shall see in the next section that there are pairs λ and μ for which there are infinitely many feasible parameter sets but for which there are no constructions known.

3. Applications. In this section we present some consequences of the results of the previous section that have appeared in previous literature. The first application is the one that motivated the result in Theorem 2.1.

EXAMPLE 3.1. If n is the number of vertices of a k-regular graph of diameter 2, then

$$n \leq 1 + k + k(k-1) = k^2 + 1.$$

Graphs in which equality holds are called *Moore graphs of diameter 2*, and are easily seen to be strongly regular with parameters

$$(k^2 + 1, k, 0, 1).$$

Here $c(\lambda, \mu) = -15$, so $s \in \{1, 3, 5, 15\}$, by (2.7). Also $s^2 = \Delta = 4k - 3$ and we obtain the well-known constraint that $k = \frac{s^2+3}{4} \in \{1, 3, 7, 57\}$, unless $m_1 = m_2$. If $m_1 = m_2$, then $k^2 + 1 = 2k + 1$, so $k = 2$.

It is easy to see that a Moore graph with $k = 1$ cannot exist. Unique graphs are known for $k \in \{2, 3, 7\}$. The five cycle C_5 is an $SRG(5, 2, 0, 1)$. The Petersen graph is the complement of $T(5)$ and is an $SRG(10, 3, 0, 1)$. The Hoffman-Singleton graph [8, sec.5.9] is an $SRG(50, 7, 0, 1)$. The case $k = 57$ is undecided.

In Example 3.1, we saw that the Moore graphs of diameter 2 are the strongly regular graphs with $\lambda = 0$ and $\mu = 1$. Two natural classes of strongly regular graphs have been suggested as generalizations of Moore graphs: those with $\lambda = 0$ (Cameron and van Lint [7, ch.8]), and those with $\mu = 1$ (Bose and Dowling [3]).

In [2, p.102] N. Biggs proves the following result on strongly regular graphs with $\lambda = 0$.

LEMMA 3.2.

1. *If $\mu \notin \{2, 4, 6\}$ then the parameter set $(v, k, 0, \mu)$ is feasible if and only if k is one of a finite list of values for each given μ.*
2. *The parameter set $(v, k, 0, 2)$ is feasible if and only if $k = r^2 + 1$ where $r \not\equiv 0 \pmod 4$, $r \geq 2$.*
3. *The parameter set $(v, k, 0, 4)$ is feasible if and only if $k = r^2$ where r is any positive integer.*
4. *The parameter set $(v, k, 0, 6)$ is feasible if and only if $k = r^2 - 3$ where $r \not\equiv 0 \pmod 4$, $r \geq 3$.*

Proof. Substituting $\lambda = 0$ into equation (2.7), we get

$$c(0, \mu) = \mu^2(\mu - 2)(\mu - 4)(\mu - 6).$$

If $\mu \notin \{2, 4, 6\}$, then $c(0, \mu) \neq 0$, so the number of possible values of $s = \sqrt{\Delta}$, a divisor of $c(0, \mu)$, is finite.

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If $\mu \in \{2, 4, 6\}$, then, by inspection from Table 2.1 we have parameter sets with $\lambda = 0$ when $\lambda = \mu - 2$ ($\mu = 2$), $(\lambda - \mu)^2 = 4\mu$ ($\mu = 4,\ t = -2$), and $(\lambda - \mu)^2 + 2(\lambda - \mu) = 4\mu$ ($\mu = 6,\ t = -3$). The values of k follow from the formulas for the parameters. The conditions on r follow from the eigenvalue multiplicity formulas. For example, when $\mu = 6$, then $t = -3$ and the formulas for parameter sets with $(\lambda-\mu)^2+2(\lambda-\mu) = 4\mu$ give $k = r^2-2$ and $m_1 = (r^2-3)(r+2)(r+1) \equiv 0 \pmod{12}$. Substituting $r = 0, 1, 2, \ldots, 11$ in the last condition yields the equivalent condition $r \not\equiv 0 \pmod 4$. The condition $r \geq 3$ ensures that $m_1, m_2 > 0$. □

We can construct a 2-design, possibly with repeated blocks, from a strongly regular graph G with $\lambda = 0$. The points of the design are the vertices adjacent to a given vertex, and the blocks are the vertices not adjacent to that vertex. A point is incident to a block if the corresponding vertices are adjacent in G. The resulting design is a 2-$(k, \mu, \mu - 1)$ design. This observation leads to further constraints on the feasible parameters with $\lambda = 0$ [7, ch.8]. The trivial examples of strongly regular graphs with $\lambda = 0$ are the complete bipartite graphs $K_{m,m},\ m \geq 2$ with parameters $(2m, m, 0, m)$. Other than the known constructions of diameter 2 Moore graphs, there are only four known nontrivial examples of strongly regular graphs with $\lambda = 0$ [7, ch.8]. Each of these strongly regular graphs is determined uniquely by its parameters. One example is the Higman-Sims graph mentioned in Section 2. The other examples are the Clebsch graph, the Gewirtz graph and the second subconstituent of the Higman-Sims graph which have parameters $(16, 5, 0, 2)$, $(56, 10, 0, 2)$, and $(77, 16, 0, 4)$ respectively [11, ch.21].

We can also prove the following result about strongly regular graphs with $\mu = 1$ suggested by Bose and Dowling as a generalization of Moore graphs. Strongly regular graphs with $\mu = 1$ are also discussed by Kantor [10].

Lemma 3.3.

1. *If $\lambda \neq 3$ then the parameter set $(v, k, \lambda, 1)$ is feasible if and only if k is one of a finite list of values for each given λ.*
2. *The parameter set $(v, k, 3, 1)$ is feasible if and only if $k = r^2$ where r is even, $r \geq 4$.*

Proof. Substituting $\mu = 1$ into equation (2.7) we get

$$c(\lambda, 1) = (\lambda - 3)(\lambda + 1)^2(\lambda^2 - 5).$$

If $\lambda \neq 3$ then $c(\lambda, 1) \neq 0$ so the number of possible divisors of $s = \sqrt{\Delta}$ is finite. By inspection from Table 2.1, if $\lambda = 3$ then we have case (a) with $t = 1$ since $\mu = 1$. The value of k follows from the formulas for the parameters. The conditions on r follow from the eigenvalue multiplicity formulas. The remaining feasibility conditions are then satisfied. □

If $\mu = 1$, the induced subgraph on the neighbours of a vertex x must be a disjoint union of $(\lambda+1)$-cliques, so $(\lambda+1)|k$, and counting the number of $(\lambda+2)$-cliques in G, we get that $(\lambda+1)(\lambda+2)|vk$ [6]. Currently the only known constructions for strongly regular graphs with parameters $(v, k, \lambda, 1)$ have $\lambda = 0$ (the Moore graphs of diameter 2). Kantor [10] shows that the case $(v, k, 1, 1)$ is impossible. The smallest unsettled case is (400,21,2,1) [5, sec.1.17].

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By examining each of the relations given in (2.8), we saw in Section 2 that the difference between λ and μ is always even. It is also non-zero in each of the three cases. Thus we have the following simple observation.

LEMMA 3.4. *Let λ and μ be fixed. If $\lambda - \mu$ is zero or odd, then there are only finitely many feasible parameter sets (v, k, λ, μ).*

The following result, due to Berlekamp, van Lint, and Seidel, [1], is a special case of Lemma 3.4.

LEMMA 3.5. *For each λ and μ with $\lambda - \mu = -1$ there are only finitely many feasible parameter sets (v, k, λ, μ).*

Strongly regular graphs with $\lambda - \mu = -1$ include the conference graphs and the Moore graphs. They are equivalent to examples of the following generalization of block designs due to Bridges and Ryser [4]. A *binary (v, k, μ)-system on r and s* is a pair X, Y of $(0, 1)$-matrices of order v satisfying the matrix equation

$$XY = YX = (k - \mu)I + \mu J,\ k \neq \mu,$$

where X and Y have constant row sums r and s respectively. Strongly regular graphs with $\lambda = \mu - 1$ and adjacency matrix A, are equivalent to binary (v, k, μ)-systems on k and $k + 1$ with $X = A$ and $Y = A + I$.

The conditions in (2.8) partition the parameter sets considered in Theorem 2.1 into three families. We observed earlier that the case $\lambda - \mu = -2$ corresponds to 2-$(v, k+1, \mu)$ designs having a polarity with all points absolute. There are no pairs λ and μ with $\lambda - \mu = -2$ for which infinitely many strongly regular graphs are known. However, there are infinite classes of graphs with $\lambda - \mu = -2$ if we allow λ and μ to vary. For example, the complements of the symplectic graphs [8, p.242] have parameters $(2^{2r} - 1,\ 2^{2r-1} - 2,\ 2^{2r-2} - 3,\ 2^{2r-2} - 1)$. It would be interesting to know if either of the other two conditions are associated with, or at least contain, an infinite family of combinatorial structures.

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