Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

THE MERRIS INDEX OF A GRAPH*

FELIX GOLDBERG† AND GREGORY SHAPIRO†

Abstract. In this paper the sharpness of an upper bound, due to Merris, on the independence number of a graph is investigated. Graphs that attain this bound are called *Merris graphs*. Some families of Merris graphs are found, including Kneser graphs $K(v, 2)$ and non-singular regular bipartite graphs. For example, the Petersen graph and the Clebsch graph turn out to be Merris graphs. Some sufficient conditions for non-Merrisness are studied in the paper. In particular it is shown that the only Merris graphs among the joins are the stars. It is also proved that every graph is isomorphic to an induced subgraph of a Merris graph and conjectured that almost all graphs are *not* Merris graphs.

Key words. Laplacian matrix, Laplacian eigenvalues, Merris index, Merris graph, Independence number.

AMS subject classifications. 05C50, 05C69, 15A42.

1. Introduction. Let G be a finite simple graph. The *Laplacian matrix* $L(G)$ (which we often write as simply L) is defined as the difference $D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of the vertex degrees of G. It is not hard to show that L is a singular M-matrix and that it is positive semidefinite. The eigenvalues of L will be denoted by $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. It is easy to show that $\lambda_2 = 0$ if and only if G is not connected.

Suppose that G is a d-regular graph. Then $L(G) = dI - A(G)$ and we have the following well-known fact:

PROPOSITION 1.1. *If G is a d-regular graph, then λ is an eigenvalue of $L(G)$ if and only if $d - \lambda$ is an eigenvalue of $A(G)$* This simple observation will be useful in the proofs of Theorems 3.5, 4.2 and 5.2.

We now fix some notation and terminology (for undefined graph-theoretic and matrix-theoretic terms we refer the reader to [3] and [10], respectively).

A set S of vertices in a graph G is called *independent* if no two vertices in S are connected by an edge. The maximum cardinality of such a set is called the *independence number* of G and denoted $\alpha(G)$.

We denote by $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ) the maximum and minimum degrees of a vertex in G, respectively. The degree of a vertex v is denoted by $d(v)$. For any vertex v and a subset $T \subseteq V$ we denote by $e(v, T)$ the set of edges between v and a vertex in T.

The vertex connectivity of G is denoted by $\nu(G)$. The matching number and the vertex-covering number of G are denoted by $\mu(G)$ and $\tau(G)$, respectively.

The disjoint union of two graphs G_1,G_2 will be denoted by $G_1 \cup G_2$ and their join by $G_1 \vee G_2$. Their Cartesian product will be denoted by $G_1 \square G_2$ (following [11]).

*Received by the editors on 31 July 2003. Accepted for publication on 13 August 2003. Handling Editor: Daniel Hershkowitz.

†Department of Mathematics, Technion-IIT, Haifa 32000, Israel (felixg@tx.technion.ac.il, sgorg10@t2.technion.ac.il). The first author is partially supported by a scholarship from the Chais Family Foundation Program for Exceptionally Gifted Students at Technion-IIT.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

For any subset $I \subseteq \mathbb{R}$ we let $m_G(I)$ stand for the number of the eigenvalues of $L(G)$ that fall inside I (counting multiplicities).

A graph G is called *Laplacian integral* if all eigenvalues of $L(G)$ are integers. A k-regular graph is called *singular* or *non-singular* according to whether k is or is not an eigenvalue of $L(G)$.

In [14] Merris has obtained the following result:

THEOREM 1.2. *Let G be a graph on n vertices. Then,*

$$m_G([\delta, n]) \geq \alpha(G), \tag{1.1}$$

$$m_G([0, \Delta]) \geq \alpha(G). \tag{1.2}$$

We remark that for regular graphs Theorem 1.2 is equivalent to a result of Cvetković about the eigenvalues of the adjacency matrix [4].

In this paper we shall consider the sharpness of inequality (1.1). We shall denote the quantity $m_G([\delta, n])$ by $M(G)$ and call it the *Merris index* of G. Graphs satisfying $M(G) = \alpha(G)$ will be called *Merris graphs*.

To get started, we obtain the following result:

PROPOSITION 1.3. *Let G be a disconnected Merris graph. Then every connected component of G is a Merris graph.*

Proof. Let G_1 be one of the components of G and let G_2 be the subgraph of G induced by the vertices that do not belong to G_1. The spectrum of G is the union of the spectra of G_1 and G_2 and also: $\delta(G) = min\{\delta(G_1), \delta(G_2)\}$. From the foregoing two observations we conclude that: $M(G) \geq M(G_1) + M(G_2) \geq \alpha(G_1) + \alpha(G_2)$.

On the other hand, $\alpha(G) = \alpha(G_1) + \alpha(G_2)$. Recalling that $M(G) = \alpha(G)$ we see that $M(G_1) = \alpha(G_1)$ and $M(G_2) = \alpha(G_2)$. The result now follows by induction on the number of connected components. $\square$

However, the converse of Proposition 1.3 is not true, since the addition of an isolated vertex to a Merris graph yields a graph that is not Merris.

One of our results (Theorem 7.5) has an immediate analogue for (1.2) but in general it appears that the behaviour of $m_G([\delta, n])$ and $m_G([0, \Delta])$ may be quite different and thus $m_G([0, \Delta])$ is likely to require a separate study.

The organization of the paper is as follows: in Section 2 we collect in one place for the readers' convenience some of the known results that we use; in Section 3 we consider some simple families of graphs and see which of them are Merris graphs. In Section 4 we show that all non-singular regular bipartite graphs are Merris and conclude that incidence graphs of symmetric designs are Merris.

In Section 5 we show that the Kneser graphs $K(v, 2)$ are Merris graphs. This implies that the Petersen graph is a Merris graph. In Section 6 we list some Merris graphs which are at the moment considered "sporadic".

Subsequently, in Section 7 we find some sufficient conditions for strict inequality to hold in (1.1) and obtain a structure theorem for Merris graphs (although an explicit classification is not achieved). Then in Section 8 we show that the only joins of graphs that are Merris graphs are the stars. Finally, in Section 9 we prove that every graph

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

is isomorphic to an induced subgraph of a Merris graph. On the other hand, we conjecture that almost all graphs are not Merris graphs.

2. Some known results used in this paper. First we recall the following result of Fiedler [6]:

THEOREM 2.1. *For any graph $G \neq K_n$ holds $\lambda_2(G) \leq \nu(G)$.*

Since always $\nu(G) \leq \delta(G)$ we have that:

COROLLARY 2.2. *For any graph $G \neq K_n$ holds $\lambda_2(G) \leq \delta(G)$.*

We use the following form of the interlacing inequality for Laplacian matrices [9]:

THEOREM 2.3. *Let $\tilde{G}$ be a graph on n vertices. Suppose G is a spanning subgraph of $\tilde{G}$ obtained by removing an edge. Then the $(n-1)$ largest eigenvalues of $L(G)$ interlace the eigenvalues of $L(\tilde{G})$.*

We also make use of the following computation of the Laplacian eigenvalues of a join of graphs; see [15]:

THEOREM 2.4. *Let $G = G_1 \vee G_2$ be a graph on n vertices, with G_1 and G_2 having n_1 and n_2 vertices, respectively. Suppose that the eigenvalues of $L(G_1)$ are $0, \mu_1, \dots, \mu_{n_1}$ and that the eigenvalues of $L(G_2)$ are $0, \lambda_1, \dots, \lambda_{n_2}$. Then the eigenvalues of $L(G)$ are $0, \mu_1 + n_2, \dots, \mu_{n_1} + n_2, \lambda_1 + n_1, \dots, \lambda_{n_2} + n_1, n$.*

The next fact is well-known. We prove it for the sake of completeness.

PROPOSITION 2.5. *For any graph G holds $2\mu(G) \geq \tau(G)$.*

Proof. Let M be a maximum matching in G. We claim that $A = \bigcup_{e \in M} e$ is a vertex-covering of G. Indeed, if there were some edge f not covered by A, then $M \cup f$ would have been a matching, contradicting the maximality of M. $\square$

The following classic theorem is due to Gallai [7]:

THEOREM 2.6. *For any graph G on n vertices holds $\alpha(G) + \tau(G) = n$.*

We will need a recent result of Ming and Wang [16, Theorem 4]:

THEOREM 2.7. *Let G be a connected graph on n vertices so that $n > 2\mu(G)$. Then $m_G((2, n]) \geq \mu(G)$.*

As an immediate consequence of a well-known majorization theorem of Schur [18] (cf. [10, p.193]) we have that:

LEMMA 2.8. *Let G be a graph on n vertices. Then $\lambda_n(G) \geq \Delta(G)$.*

We also use the following result of Teranishi [19]:

THEOREM 2.9. *Let G be a connected Laplacian integral graph on p vertices, for some prime p. Then G is the join of two Laplacian integral graphs.*

3. Some simple families of graphs. We begin by noting that stars yield equality in both bounds of Theorem 1.2:

PROPOSITION 3.1. *Let $G = K_{1,n-1}$. Then equality holds in* (1.1) *and* (1.2).

Proof. The Laplacian eigenvalues of G in this case are: $0, n, 1, \dots, 1$. On the other hand: $\delta(G) = 1, \Delta(G) = n-1, \alpha(G) = n-1$. $\square$

However, it turns out that equality doesn't have to hold simultaneously in (1.1) and (1.2). This can be illustrated by the following proposition:

PROPOSITION 3.2. *Let $G = K_n$. Then equality holds in* (1.2) *but not in* (1.1).

Now we consider paths, bearing in mind that $\delta(P_n) = 1$ and $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. First we state a result that has been obtained by some explicit computations (in MATLAB):

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

PROPOSITION 3.3. (1) *For* $1 \leq n \leq 5$ *or* $n = 7$, $M(P_n) = \alpha(P_n)$.
(2) *For* $n = 6$ *or* $8 \leq n \leq 15$, $M(P_n) > \alpha(P_n)$.

Now we are in a position to handle longer paths.

THEOREM 3.4. *For* $n \geq 16$, $M(P_n) > \alpha(P_n)$.

Proof. We use induction on n. Note that the graph $G = P_{\lfloor \frac{n}{2} \rfloor} \cup P_{\lceil \frac{n}{2} \rceil}$ is a spanning subgraph of P_n, obtained by deleting one edge. Also, we make the simple observations that:

$$\alpha(G) = \alpha(P_{\lfloor \frac{n}{2} \rfloor}) + \alpha(P_{\lceil \frac{n}{2} \rceil}), \tag{3.1}$$

$$\alpha(G) + 1 \geq \alpha(P_n), \tag{3.2}$$

$$m_G([1,n]) = m_{P_{\lfloor \frac{n}{2} \rfloor}}([1,n]) + m_{P_{\lceil \frac{n}{2} \rceil}}([1,n]). \tag{3.3}$$

Now either by the induction assumption or by Proposition 3.3(2) (depending on the value of n) we have that:

$$m_{P_{\lfloor \frac{n}{2} \rfloor}}([1,n]) \geq \alpha(P_{\lfloor \frac{n}{2} \rfloor}) + 1, \tag{3.4}$$

$$m_{P_{\lceil \frac{n}{2} \rceil}}([1,n]) \geq \alpha(P_{\lceil \frac{n}{2} \rceil}) + 1. \tag{3.5}$$

We apply the interlacing inequality (in the form stated in Theorem 2.3) to obtain:

$$m_{P_n}([1,n]) \geq m_G([1,n]). \tag{3.6}$$

Finally we collate the equations and inequalities established so far ((3.1)-(3.6)):

$$m_{P_n}([1,n]) \geq m_G([1,n]) = m_{P_{\lfloor \frac{n}{2} \rfloor}}([1,n]) + m_{P_{\lceil \frac{n}{2} \rceil}}([1,n])$$

$$\geq \alpha(P_{\lfloor \frac{n}{2} \rfloor}) + 1 + \alpha(P_{\lceil \frac{n}{2} \rceil}) + 1 = \alpha(G) + 2 \geq \alpha(P_n) + 1. \quad \square$$

We can also determine which cycles are Merris graphs:

THEOREM 3.5. $M(C_n) = \alpha(C_n)$ *if and only if* n *is congruent to* 1 *or* 2 *modulo* 4.

Proof. The independence numbers of C_n is $\lfloor \frac{n}{2} \rfloor$. Since cycles are 2-regular, their Laplacian eigenvalues may be recovered from those of the adjacency matrix that are given in [1, p. 17], using Proposition 1.1. The verification of the assertion is now straightforward. $\square$

A similar result holds for hypercubes (we omit the proof, which consists of little more than an appeal to the formulas for the Laplacian eigenvalues of Q_m that can be found in, say, [17]):

THEOREM 3.6. $M(Q_m) = \alpha(Q_m)$ *if and only if* m *is odd.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

4. Non-singular regular bipartite graphs. Let us introduce a large class of Merris graphs:

THEOREM 4.1. *Let G be a non-singular k-regular bipartite graph on n vertices. Then G is a Merris graph.*

Proof. Since G is regular and bipartite, both partitions of G must be of equal cardinality. Therefore, $\alpha(G) = \frac{n}{2}$. By Theorem 1.2 we have that both $m_G([\delta, n])$ and $m_G([0, \Delta])$ are greater than or equal to $\frac{n}{2}$.

On the other hand, since k is not a Laplacian eigenvalue of G, $m_G([\delta, n]) + m_G([0, \Delta]) = n$. We conclude, then, that $m_G([\delta, n]) = m_G([0, \Delta]) = \frac{n}{2}$ $\square$

A complete description of the class of non-singular regular bipartite graphs is unknown. However, we can point out an important subclass thereof:

THEOREM 4.2. *Let G be the incidence graph of a symmetric $2-(v, k, \lambda)$ design. Then G is a Merris graph.*

Proof. The k-regularity and bipartiteness of G follow directly from its being the incidence graph of a symmetric design. It remains to point out that Koolen and Moulton have observed in [12] that the adjacency eigenvalues of such a graph are: $k, -k, \sqrt{k-\lambda}, -\sqrt{k-\lambda}$ (with appropriate multiplicities). Therefore, by Proposition 1.1, k is not a Laplacian eigenvalue of G. Thus, G is non-singular and we are done by Theorem 4.1. $\square$

5. The Kneser graphs $K(v, 2)$. The results of Sections 3 and 4 might have suggested that Merris graphs necessarily have large independence numbers. However, in this section we are going to meet an infinite family of graphs whose independence number is $O(\sqrt{n})$, where n is, as usually, the number of vertices.

We recall that the *Kneser graph* $K(v, r)$ has $\binom{v}{r}$ vertices corresponding to the r-subsets of some set of cardinality v and two vertices are connected by an edge if and only if they represent disjoint sets. The independence number of a Kneser graph is given by the following well-known theorem of Erdős, Ko and Rado [5]:

THEOREM 5.1. *If $v > 2r$, then $\alpha(K(v, r)) = \binom{v-1}{r-1}$.*

For the proof of the following result we shall use the concept of *strongly regular graph.* A k-regular graph on n vertices is said to be strongly regular with parameters (n, k, λ, μ) if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. (This definition is due to Bose [2].) Strongly regular graphs have been studied quite extensively (for instance, see [8, Chapter 10]).

Now we can state and prove the following result.

THEOREM 5.2. *For every $v \geq 5$, $K(v, 2)$ is a Merris graph.*

Proof. The graph $K(v, 2)$ is a strongly regular graph with parameters $(\binom{v}{2}, \binom{v-2}{2}, \binom{v-4}{2}, \binom{v-3}{2})$. Therefore, we can compute the eigenvalues of its *adjacency* matrix by the standard formulas (see [8, Section 10.2]) for strongly regular graphs: $\binom{v-2}{2}$ with multiplicity 1, 1 with multiplicity $\frac{v(v-3)}{2}$ and $3-v$ with multiplicity $v-1$.

Since $K(v, 2)$ is regular we infer from Proposition 1.1 that its Merris index is equal to the number of non-positive eigenvalues of the adjacency matrix. In the previous paragraph, we have seen that this number is $v-1$. Finally, by Theorem 5.1 we have that $M(K(v, 2)) = \alpha(K(v, 2)) = v-1$. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

Since the Petersen graph is isomorphic to $K(5,2)$ we have that:

COROLLARY 5.3. *The Petersen graph is a Merris graph.*

6. Some sporadic Merris graphs. In this section we list some interesting Merris graphs that we are unable to classify at the moment:

(1) The Clebsch graph. This is the unique strongly regular graph with parameters $(16,5,0,2)$; see [8, Theorem 10.6.4]. Its Merris index is 5.

(2) The prisms $C_7\square P_2$ and $C_8\square P_2$ with Merris indices 6 and 8, respectively.

(3) The graph obtained from the 5-cycle by adding a new vertex that is adjacent to two consecutive vertices on the cycle. It has Merris index 3.

Some other strongly regular graphs and prisms are Merris but we do not have a way of determining Merrisness from a parameter set for them as yet.

7. Some sufficient conditions for non-Merrisness. We begin with the following result:

THEOREM 7.1. *Let G be a connected Laplacian integral graph on $n \geq 3$ vertices with $\delta(G) = 1$, but not a star. Then, $M(G) > \alpha(G)$.*

Proof. The result if trivial if $G = K_n$. Therefore, we may assume that $G \neq K_n$. By Corollary 2.2 and the assumptions on G we have $\lambda_2(G) = 1$ and thus $m_G([\delta, n]) = n-1$. On the other hand, since G is not a star we have $\alpha(G) \leq n-2$. □

It turns out that in general the hypothesis $\delta(G) = 1$ in the theorem may neither be omitted nor be weakened to $\nu(G) = 1$. However, as we shall see in Section 8, if n is prime, then the hypothesis $\delta(G) = 1$ is unnecessary.

THEOREM 7.2. *Let G be a connected graph on n vertices satisfying the following conditions:*
(i) $\delta(G) \leq 2$,
(ii) $n > 3\alpha(G)$,
(iii) $n > 2\mu(G)$.
Then, $M(G) > \alpha(G)$.

Proof. Using assumption (i) we have:

$$m_G([\delta, n]) \geq m_G([2, n]) \geq m_G((2, n]).$$

Now, by applying Theorem 2.7, Proposition 2.5 and Theorem 2.6 (in this order) we have:

$$m_G((2, n]) \geq \mu(G) \geq \frac{\tau(G)}{2} = \frac{n - \alpha(G)}{2}.$$

Therefore $m_G([\delta, n]) \geq \frac{n-\alpha(G)}{2}$ and it remains to apply assumption (ii) to finish the proof. □

We can weaken assumption (iii) in an obvious way to obtain a simpler result:

COROLLARY 7.3. *Let G be a connected graph on n vertices satisfying the following conditions:*
(i) $\delta(G) \leq 2$,
(ii) $n > 3\alpha(G)$,
(iii) *n is odd.*
Then, $M(G) > \alpha(G)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

A family of non-Merris graphs satisfying the assumptions of Corollary 7.3 can be obtained in the following way: take an r-clique for some even $r \geq 4$ and connect it to a triangle by a single edge.

We can now obtain a structure theorem for Merris graphs (although it does not give us an explicit classification):

THEOREM 7.4. *Let G be a Merris graph. Then for every connected component C of G at least one of the following conditions must hold:*
(a) $\delta(C) \geq 3$,
(b) *C has at most $3\alpha(C)$ vertices,*
(c) *C has a perfect matching.*

Proof. Apply Proposition 1.3 and Theorem 7.2. □

The proof of the next result is essentially a variation on the proof of Theorem 1.2 by Merris.

THEOREM 7.5. *Let G be a graph on n vertices. Suppose that there exist a maximum independent set $S \subseteq V$ and a vertex $v \in V \backslash S$ that satisfy the following conditions:*
(i) $d(v) - |e(v, S)| \geq \delta(G)$,
(ii) *For any $s \in S$, if s is adjacent to v then $d(s) > \delta(G)$.*
Then, $M(G) > \alpha(G)$.

Proof. We denote by B the principal submatrix of $L(G)$ that is indexed by the rows and columns corresponding to $S \cup \{v\}$. Now we consider the Geršgorin disks of B. From assumptions (i) and (ii) we see that the leftmost (necessarily real) points of all the Geršgorin disks of B are not less in magnitude than $\delta(G)$. So, by Geršgorin's theorem (see [10, Theorem 6.1.1]) the eigenvalues of B are all greater than or equal to $\delta(G)$. Therefore, by the Cauchy interlacing theorem (see [10, Theorem 4.3.15]) we see that $L(G)$ has at least $\alpha(G) + 1$ eigenvalues greater than or equal to $\delta(G)$. This proves our claim. □

It is not hard to see that an analogue of Theorem 7.5 holds for (2). The proof is the same, *mutatis mutandis.*

THEOREM 7.6. *Let G be a graph on n vertices. Suppose that there exist a maximum independent set $S \subseteq V$ and a vertex $v \in V \backslash S$ that satisfy the following conditions:*
(i) $d(v) + |e(v, S)| \leq \Delta(G)$,
(ii) *For any $s \in S$, if s is adjacent to v then $d(s) < \Delta(G)$.*
Then, $m_G([0, \Delta]) > \alpha(G)$.

A family of non-Merris graphs satisfying the assumptions of Theorem 7.5 can be obtained in the following way: take an s-cycle for some even $s \geq 4$ and join three consecutive vertices to some new vertex v.

8. Joins of graphs. We recall that a graph G is said to be the *join* of two graphs G_1 and G_2 if $V(G)$ is the disjoint union of $V(G_1)$ and $V(G_2)$ and all possible edges between $V(G_1)$ and $V(G_2)$ are present in G. In this section we shall show that a join is a Merris graph if and only if it is a star (which is the join $K_{1,n-1} = K_1 \vee (n-1)K_1$).

Let us assume that $G = G_1 \vee G_2$ and that G_1 and G_2 have n_1 and n_2 vertices,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

respectively. We make the following straightforward observations:

$$\alpha(G) = max\{\alpha(G_1), \alpha(G_2)\}, \tag{8.1}$$

$$\delta(G) = min\{\delta(G_1) + n_2, \delta(G_2) + n_1\}. \tag{8.2}$$

THEOREM 8.1. *Let $G = G_1 \vee G_2$ be a graph on $n \geq 3$ vertices. Then $M(G) = \alpha(G)$ if and only if $G = K_{1,n-1}$.*

Proof. We assume, without loss of generality, that $\alpha(G_1) \geq \alpha(G_2)$ and therefore $\alpha(G) = \alpha(G_1)$. We will have to examine now a number of cases.

First we deal with the case when $\delta(G_1) \geq 1$. Suppose that the non-zero eigenvalues of G_1 are: $\lambda_k, \ldots, \lambda_{n_1}$. Then it follows from Theorem 2.4 that among the eigenvalues of G we will find the following numbers: $\lambda_k + n_2, \ldots, \lambda_{n_1} + n_2, n$. So we can write:

$$m_G([\delta(G_1) + n_2, n]) \geq m_{G_1}([\delta(G_1), n_1]) + 1. \tag{8.3}$$

(We use the assumption that $\delta(G_1) \geq 1$ in stating (8.3)).

Therefore, by (8.2), (8.3), Theorem 1.2 and our assumption that $\alpha(G) = \alpha(G_1)$, respectively:

$$m_G([\delta(G), n]) \geq m_G([\delta(G_1) + n_2, n]) \geq m_{G_1}([\delta(G_1), n_1]) + 1$$

$$\geq \alpha(G_1) + 1 = \alpha(G) + 1.$$

So far we have shown that $M(G) > \alpha(G)$, provided $\delta(G_1) \geq 1$.

Now we assume $\delta(G_1) = 0$. If $\alpha(G_1) < n_1$ then the same arguments as in the previous case show that $M(G) > \alpha(G)$. Therefore we can assume that $\alpha(G_1) = n_1$.

Now if $n_2 \geq 2$ we refine the foregoing argument in which we have counted $\alpha(G_1) = \alpha(G)$ eigenvalues of G that are not less than $\delta(G)$. We note that if μ is the largest eigenvalue of G_2, then by Theorem 2.4, $\mu + n_1$ is an eigenvalue of G. But using Lemma 2.8 we have: $\mu + n_1 \geq \Delta(G_2) + n_1 \geq \delta(G_2) + n_1 \geq \delta(G)$. Therefore, we can increase our count by one and obtain once again that $M(G) > \alpha(G)$.

The only possible case remaining now is that when $\alpha(G_1) = n_1$ and $n_2 = 1$. In other words, when $G = K_{1,n-1}$. By Proposition 3.1, in this case indeed $M(G) = \alpha(G)$. This completes the proof of the theorem. □

COROLLARY 8.2. *Let G be a graph on $n \geq 2$ vertices with $\Delta(G) = n - 1$, but not a star. Then, $M(G) > \alpha(G)$.*

Proof. Since $\Delta(G) = n - 1$ we can write $G = H \vee K_1$ for some graph H on $n - 1$ vertices and now the previous theorem applies. □

The wheel graphs satisfy the assumptions of Corollary 8.2 and therefore are not Merris graphs.

COROLLARY 8.3. *Let G be a connected Laplacian integral graph on p vertices, for some prime p, but not a star. Then, $M(G) > \alpha(G)$.*

Proof. Apply Theorems 2.9 and 8.1. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

9. Embedding and asymptotics. Given a graph G it is natural to ask whether it is isomorphic to an induced subgraph of some Merris graph H (in which case we say that G can be embedded in H). The answer to this question turns out to be always positive. Moreover, we are going to give a simple explicit construction of H.

Namely, H will be the *corona* of G, which is the graph resulting from the addition of a new pendant vertex at every original vertex of G. We remark parenthetically that coronas have been previously studied in relation to domination in graphs; see [3, p. 305].

THEOREM 9.1. *Let G be some graph on n vertices and let H be its corona. Then H is a Merris graph.*

Proof. First, we observe that H has $2n$ vertices and that $\alpha(H) = n$. Let us write down the Laplacian matrix of H:

$$L(H) = \left[\begin{array}{c|c} L(G)+I & -I \\ \hline -I & I \end{array}\right]$$

Suppose that v is an eigenvector of $L(G)$ that corresponds to the eigenvalue λ. We shall be looking for eigenvectors of $L(H)$ that have the following form:

$$w = \left[\begin{array}{c} v \\ \hline \gamma v \end{array}\right]$$

Let us compute:

$$L(H)w = \left[\begin{array}{c} (\lambda+1-\gamma)v \\ \hline (\gamma-1)v \end{array}\right] = (\lambda+1-\gamma)\left[\begin{array}{c} v \\ \hline \frac{\gamma-1}{\lambda+1-\gamma}v \end{array}\right]$$

We now see that for w to be an eigenvector of $L(H)$ it is necessary and sufficient that

$$\gamma = \frac{\gamma-1}{\lambda+1-\gamma}$$

holds. This is a quadratic equation in γ that has two real solutions:

$$\gamma_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2+4}}{2}$$

Therefore,

$$w_i = \left[\begin{array}{c} v \\ \hline \gamma_i v \end{array}\right], i \in \{1,2\}$$

are eigenvectors of $L(H)$ corresponding to the eigenvalues $\lambda + 1 - \gamma_i$. We point out that in this way we have obtained all the eigenvectors and eigenvalues of $L(H)$. It remains to find $M(H)$.

Obviously, $\delta(H) = 1$. On the other hand, for every eigenvalue λ of $L(G)$ we see that:

$$\lambda + 1 - \gamma_1 = \lambda + 1 - \frac{\lambda + \sqrt{\lambda^2+4}}{2} < 1.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

Therefore, $M(H) \leq n$. But we recall that $\alpha(H) = n$ and deduce that $M(H) = \alpha(H) = n$, thereby completing the proof. ☐

Given some property P of graphs, the following assertion holds (its proof has been suggested to us by Brendan McKay [13]):

PROPOSITION 9.2. *If almost all graphs have property P, then every graph is isomorphic to an induced subgraph of a graph that has property P.*

Proof. If G is a fixed graph, then by a well-known result (see [3, p. 379]), almost every graph H has an induced subgraph isomorphic to G. By the assumption that almost all graphs have property P we can choose H to have property P. ☐

Had we known that almost all graphs are Merris graphs, we could have immediately deduced from Proposition 9.2 that every graph can be embedded in a Merris graph. However, we believe that almost all graphs are *not* Merris graphs. We conclude the paper with the formal statement of this conjecture:

CONJECTURE. Let X_n be the number of graphs on n vertices and let M_n be the number of Merris graphs on n vertices. Then $lim_{n\to\infty} \frac{M_n}{X_n} = 0$.

10. Acknowledgments. The authors would like to express their gratitude to Professor Abraham Berman for his inspiring advice and encouragement, to Professor Brendan McKay for suggesting the proof of Proposition 9.2, to Professor Russell Merris for many useful comments on the final draft of this paper and to Yulia Bugaev for an interesting discussion.

REFERENCES

[1] Norman Biggs. *Algebraic Graph Theory.* Cambridge Mathematical Library. Cambridge University Press, 2nd edition, 1993.
[2] R.C. Bose. Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.*, 13:389–419, 1963.
[3] Gary Chartrand and Linda Lesniak. *Graphs and Digraphs.* Chapman and Hall, 3rd edition, 1996.
[4] Dragan M. Cvetković. Graphs and their spectra. *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, 354-356:1–50, 1971.
[5] Paul Erdős, C. Ko, and Richard Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford*, 12:313–318, 1961.
[6] Miroslav Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23 (98):298–305, 1973.
[7] Tibor Gallai. Über extreme Punkt- und Kantenmengen. *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.*, 2:133–138, 1959.
[8] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer, 2001.
[9] Robert Grone, Russell Merris, and V.S. Sunder. The Laplacian spectrum of a graph. *SIAM J. Matrix Anal. Appl.*, 11:218–238, 1990.
[10] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.
[11] Wilfried Imrich and Sandi Klavžar. *Product Graphs: Structure and Recognition.* Wiley-Interscience, 2000.
[12] Jack H. Koolen and Vincent Moulton. Maximal energy bipartite graphs. *Graphs Combin.*, 19:131–135, 2003.
[13] Brendan McKay. Personal communication, March 2003.
[14] Russell Merris. Laplacian matrices of graphs: a survey. *Liniear Algebra Appl.*, 197/8:143–176, 1994.
[15] Russell Merris. Laplacian graph eigenvectors. *Linear Algebra Appl.*, 278:221–236, 1998.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 212-222, August 2003

ELA

[16] Guo Ji Ming and Tan Shang Wang. A relation between the matching number and Laplacian spectrum of a graph. *Linear Algebra Appl.*, 325:71–74, 2001.

[17] Bojan Mohar. Some applications of Laplace eigenvalues of graphs. In G. Hahn and G. Sabidussi, editors, *Graph symmetry: algebraic methods and applications*, volume 497 of *NATO ASI Ser. C*, pages 225–275. Kluwer Academic Publishers, 1997.

[18] Isai Schur. Über eine Klasse von Mittelbildungen mit Anwendungen die Determinanten. *Sitzungsber. Berlin Math. Gesellshaft*, 22:9–20, 1923.

[19] Yasuo Teranishi. The Hoffman number of a graph. *Discrete Math.*, 260:255–265, 2003.