Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 155-162, June 2003

ELA

THE PATH POLYNOMIAL OF A COMPLETE GRAPH*

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Abstract. Let $P_k(x)$ denote the polynomial of the path on k vertices. A complete description of the matrix that is the obtained by evaluating $P_k(x)$ at the adjacency matrix of the complete graph, along with computing the effect of evaluating $P_k(x)$ with Laplacian matrices of a path and of a circuit.

Key words. Graph, Adjacency matrix, Laplacian matrix, Characteristic polynomial.

AMS subject classifications. 05C38, 05C50.

1. Introduction and preliminaries. For a finite and undirected graph G without loops or multiple edges, with n vertices, let us define the *polynomial of* G, P_G, as the characteristic polynomial of its adjacency matrix, $A(G)$, *i.e.*,

$$P_G(x) = \det\left(xI_n - A(G)\right).$$

When the graph is a path with n vertices, we simply call P_G the *path polynomial* and denote it by P_n. Define A_n as the adjacency matrix of a path on n vertices.

For several interesting classes of graphs, $A(G_i)$ is a polynomial in $A(G)$, where G_i is the ith distance graph of G ([5]). Actually, for distance-regular graphs, $A(G_i)$ is a polynomial in $A(G)$ of degree i, and this property characterizes these kind of graphs ([14]).

In [4], Beezer has asked when a polynomial of an adjacency matrix will be the adjacency matrix of another graph. Beezer gave a solution in the case that the original graph is a path.

Theorem 1.1 ([4]). *Suppose that $p(x)$ is a polynomial of degree less than n. Then $p(A_n)$ is the adjacency matrix of graph if and only if $p(x) = P_{2i+1}(x)$, for some i, with $0 \le i \le \lfloor \frac{n}{2} \rfloor - 1$.*

In the same paper, Beezer gave an elegant formula for $P_k\left(A_n\right)$ with $k = 1, \ldots, n$, and Bapat and Lal, in [1], completely described the structure of $P_k\left(A_n\right)$, for all integers k. This result was also reached by Fonseca and Petronilho ([10]) in a non-inductive way.

Theorem 1.2 ([1],[4],[10]). *For $0 \le k \le n-1$, n being a positive integer,*

$$\left(P_k\left(A_n\right)\right)_{ij} = \begin{cases} 1 & \text{if } i+j = k+2r \text{ , with } 1 \le r \le \min\{i,j,n-k\} \\ 0 & \text{otherwise.} \end{cases}$$

In [12], Shi Ronghua obtained some generalizations of the ones achieved by Bapat and Lal. Later, in [10], Fonseca and Petronilho determined the matrix $P_k\left(C_n\right)$, where C_n is the adjacency matrix of a circuit on n vertices.

*Received by the editors on 15 March 2003. Accepted for publication on 02 May 2003. Handling Editor: Ravindar B. Bapat.

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Consider the permutation $\sigma = (12\ldots n)$.

THEOREM 1.3 ([10]). *For any nonnegative integer k,*

$$P_k\left(C_n\right) = \sum_{j=0}^{n-1}\sum_{r=0}^{n-1} \delta_{2r,k+2+j-\dot{n}} P\left(\sigma^j\right),$$

where δ is the Kronecker function, σ is the permutation $(12\ldots n)$, $P\left(\sigma^j\right)$ is the permutation matrix of σ^j and $\dot{n}$ runs over the multiples of n.

According to Bapat and Lal (cf. [1]), a graph G is called *path-positive of order* m if $P_k(G) \geq 0$, for $k = 1, 2, \ldots, m$, and G is simply called *path-positive* if it is path-positive of any order. In [3], Bapat and Lal have characterized all graphs that are path-positive. The following corollary is immediate from the theorem above.

COROLLARY 1.4. *The circuit C_n is path-positive.*

We define the *complete graph* K_n, to be the graph with n vertices in which each pair of vertices is adjacent. The adjacency matrix of a complete graph, which we identify also by K_n, is the $n \times n$ matrix

$$K_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.1}$$

In this note, we evaluate $P_k(K_n)$.

2. The polynomial P_k. Let us consider the tridiagonal matrix A_k whose entries are given by

$$\left(A_k\right)_{ij} = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The expansion of the determinant

$$\det\left(xI_k - A_k\right) = P_k(x)$$

along the first row or column gives us the recurrence relation

$$P_k(x) = xP_{k-1}(x) - P_{k-2}(x), \tag{2.1}$$

for any positive integer k, with the convention $P_{-1}(x) = 0$ and $P_0(x) = 1$.

It is well known that

$$P_k(x) = U_k\left(\frac{x}{2}\right), \quad x \in \mathbb{C}, \quad (k = 0, 1, \ldots), \tag{2.2}$$

where $U_k(x)$ are the Chebyshev polynomials of the second kind.

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From (2.2), it is straightforward to prove that

$$\frac{P_k(x)-P_k(y)}{x-y}=\sum_{\ell=0}^{k-1}P_\ell(x)\,P_{k-1-\ell}(y). \tag{2.3}$$

Then, from (2.1) and (2.3), we may conclude the following lemma.

Lemma 2.1. *For any positive integer k and square matrices A and B,*

$$P_k(A)-P_k(B)=\sum_{\ell=0}^{k-1}P_\ell(A)\,(A-B)\,P_{k-1-\ell}(B).$$

As in Bapat and Lal [1], note that a connected graph is path-positive if it has a spanning subgraph which is path-positive. Thus we have this immediate corollary from Corollary 1.4 .

Corollary 2.2. *The complete graph K_n is path-positive.*

3. Evaluating P_k of a complete graph. If a matrix $A=(a_{ij})$ satisfies the relation

$$a_{ij}=a_{1\sigma^{1-i}(j)}$$

we say that A is a *circulant matrix*. Therefore, to define a circulant matrix A is equivalent to presenting an $n-$tuple, say $(a_1,\ldots,a_n)$. Then

$$A=\sum_{i=0}^{n-1}a_iP\left(\sigma^i\right),$$

and its eigenvalues are given by

$$\lambda_h=\sum_{\ell=0}^{n-1}\zeta^{h\ell}a_\ell, \tag{3.1}$$

where $\zeta=\exp\left(i\frac{2\pi}{n}\right)$. Given a polynomial $p(x)$, the image of A is

$$p(A)=p\left(\sum_{i=0}^{n-1}a_iP\left(\sigma^i\right)\right)=n^{-1}\sum_{j=0}^{n-1}\sum_{h=0}^{n-1}\zeta^{-hj}p\left(\sum_{\ell=0}^{n-1}\zeta^{h\ell}a_\ell\right)P\left(\sigma^j\right).$$

Then,

$$P_k\left(\sum_{i=0}^{n-1}a_iP\left(\sigma^i\right)\right)=n^{-1}\sum_{j=0}^{n-1}\sum_{h=0}^{n-1}\zeta^{-hj}P_k\left(\lambda_h\right)P\left(\sigma^j\right),$$

where λ_h is defined as in (3.1).

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The matrix K_n, defined in (1.1), is a circulant matrix and it can be written

$$K_n = \sum_{i=1}^{n-1} P\left(\sigma^i\right).$$

By (3.1), K_n has the eigenvalues $\lambda_0 = n-1$ and $\lambda_\ell = -1$, for $\ell = 1, \ldots, n-1$. Therefore,

$$\begin{aligned} P_k\left(K_n\right) &= P_k\left(\sum_{i=1}^{n-1} P\left(\sigma^i\right)\right) \\ &= n^{-1} \sum_{j=0}^{n-1} \sum_{h=0}^{n-1} \zeta^{-hj} P_k\left(\lambda_h\right) P\left(\sigma^j\right) \\ &= n^{-1} \sum_{j=0}^{n-1} \left(P_k\left(n-1\right) + P_k\left(-1\right) \sum_{h=1}^{n-1} \zeta^{-hj}\right) P\left(\sigma^j\right) \\ &= P_k\left(-1\right) P\left(\sigma^0\right) + n^{-1}\left(P_k\left(n-1\right) - P_k\left(-1\right)\right) \sum_{j=0}^{n-1} P\left(\sigma^j\right). \end{aligned}$$

Note that $P\left(\sigma^0\right)$ is the identity matrix.

We have thus proved the main result of this section:

THEOREM 3.1. *For any nonnegative integer k, the diagonal entries of $P_k\left(K_n\right)$ are the weighted average $\frac{1}{n}P_k(n-1) + \frac{n-1}{n}P_k(-1)$ and the off-diagonal entries are $\frac{1}{n}P_k(n-1) - \frac{1}{n}P_k(-1)$.*

We can easily evaluate the different values of each term of the sum $P_k\left(K_n\right)$. According to (2.2),

$$P_k\left(-1\right) = \left\{ \begin{array}{rl} -1 & \text{if } k \equiv 1 \bmod 3 \\ 0 & \text{if } k \equiv 2 \bmod 3 \\ 1 & \text{if } k \equiv 0 \bmod 3 \end{array} \right. .$$

Another relation already known ([11, p.72]) for $P_k(x)$ is

$$P_k(x) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \left(\begin{array}{c} k-\ell \\ \ell \end{array} \right) x^{k-2\ell},$$

where $\lfloor z \rfloor$ denotes the greatest integer less or equal to z. Therefore we have also

$$\begin{aligned} P_k\left(n-1\right) - P_k\left(-1\right) &= \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \left(\begin{array}{c} k-\ell \\ \ell \end{array} \right) \left((n-1)^{k-2\ell} - (-1)^{k-2\ell}\right) \\ &= n \sum_{\ell=0}^{\lfloor k/2 \rfloor} \sum_{j=1}^{k-2\ell} (-1)^{k-j+\ell} \frac{(k-\ell)!}{\ell! j! (k-2\ell-j)!} n^{j-1}. \end{aligned}$$

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4. Evaluating P_k of some Laplacian matrices. Let G be a graph. Denote $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then

$$L(G) = D(G) - A(G)$$

is the *Laplacian matrix of* G.

In this section, expressions for $P_k(L(A_n))$ and $P_k(L(C_n))$, the path polynomials of the Laplacian matrices of a path and a circuit, respectively, with n vertices, are determined.

Let us consider the following recurrence relation:

$$\widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x + 1,$$

$$\widetilde{P}_k(x) = (x+2)\widetilde{P}_{k-1}(x) - \widetilde{P}_{k-2}(x), \quad \text{for } 2 \leq k \leq n-1,$$

and

$$\widetilde{P}_n(x) = (x+1)\widetilde{P}_{n-1}(x) - \widetilde{P}_{n-2}(x).$$

Therefore

$$\widetilde{P}_k(x) = U_k\left(\frac{x}{2}+1\right) - U_{k-1}\left(\frac{x}{2}+1\right), \quad \text{for } 2 \leq k \leq n-1,$$

and

$$\widetilde{P}_n(x) = xU_{n-1}\left(\frac{x}{2}+1\right).$$

where $U_k\left(x\right)$ are the Chebyshev polynomials of the second kind.

Then the zeroes of $\widetilde{P}_n(x)$ are

$$\lambda_j = 2\cos\frac{j\pi}{n} - 2, \qquad j = 0, \ldots, n-1.$$

The recurrence relation above can be written in the following matricial way:

$$x\begin{bmatrix} \widetilde{P}_0(x) \\ \widetilde{P}_1(x) \\ \vdots \\ \widetilde{P}_{n-2}(x) \\ \widetilde{P}_{n-1}(x) \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -1 \end{bmatrix} \begin{bmatrix} \widetilde{P}_0(x) \\ \widetilde{P}_1(x) \\ \vdots \\ \widetilde{P}_{n-2}(x) \\ \widetilde{P}_{n-1}(x) \end{bmatrix} + \widetilde{P}_n(x)\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Thus, for $j = 0, \ldots, n-1$, the vector

$$\begin{bmatrix} \widetilde{P}_0(\lambda_j) \\ \widetilde{P}_1(\lambda_j) \\ \vdots \\ \widetilde{P}_{n-2}(\lambda_j) \\ \widetilde{P}_{n-1}(\lambda_j) \end{bmatrix} = \left(\cos\frac{j\pi}{2n}\right)^{-1} \begin{bmatrix} \cos\frac{j\pi}{2n} \\ \cos 3\frac{j\pi}{2n} \\ \vdots \\ \cos(2n-3)\frac{j\pi}{2n} \\ \cos(2n-1)\frac{j\pi}{2n} \end{bmatrix} \tag{4.1}$$

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is an eigenvector associated to the eigenvalue λ_j of $-L(A_n)$.

Therefore the matrix $-L(A_n)$ is diagonalizable and, for $0 \leq k \leq n$, the (i,j)th entry of $P_k\left(L(A_n)\right)$ is given by

$$\left(P_k\left(L(A_n)\right)\right)_{ij} = (-1)^k \sum_{\ell=0}^{n-1} \frac{\widetilde{P}_{i-1}\left(\lambda_\ell\right) P_k\left(\lambda_\ell\right) \widetilde{P}_{j-1}\left(\lambda_\ell\right)}{\sum_{s=1}^{n}\left(\widetilde{P}_{s-1}\left(\lambda_\ell\right)\right)^2}$$

which is equal to

$$\frac{(-1)^k \cos\left(\frac{k\pi}{2}\right)}{n} + \frac{(-1)^k 2}{n} \sum_{\ell=1}^{n-1} \cos(2i-1)\frac{\ell\pi}{2n} U_k\left(\cos\frac{\ell\pi}{n} - 1\right)\cos(2j-1)\frac{\ell\pi}{2n}.$$

If we define

$$\alpha_m^p = \sum_{\ell=1}^{n-1} \cos m\frac{\ell\pi}{n} \cos^p \frac{\ell\pi}{n},$$

then

$$\alpha_m^p = \frac{1}{2}\left(\alpha_{m-1}^{p-1} + \alpha_{m+1}^{p-1}\right)$$

and

$$\alpha_m^p = \frac{1}{2^p} \sum_{\ell=0}^{p} \binom{p}{\ell} \alpha_{m+2\ell-p}^0, \tag{4.2}$$

with

$$\alpha_m^0 = n\delta_{m,2\dot{n}} - \frac{1}{2}(1+(-1)^m),$$

where $\dot{n}$ represents a multiple of n.

Using the trigonometric transformation formula and the Taylor formula

$$U_k\left(\cos\frac{\ell\pi}{n} - 1\right) = \sum_{p=0}^{k} \frac{U_k^{(p)}(-1)}{p!} \cos^p \frac{\ell\pi}{n},$$

we can state the following proposition.

THEOREM 4.1. *For $0 \leq k \leq n$, n being a positive integer,*

$$\left(P_k L(A_n)\right)_{ij} = \frac{(-1)^k \cos\left(\frac{k\pi}{2}\right)}{n} + \frac{(-1)^k 2}{n} \sum_{p=0}^{k} \frac{U_k^{(p)}(-1)}{p!} \left(\alpha_{i-j}^p + \alpha_{i+j-1}^p\right),$$

where α_m^p is defined as in (4.2).

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Note that $U_k^{(p)}(-1)$ can be easily evaluated, since

$$U_k(x) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \binom{k-\ell}{\ell} (2x)^{k-2\ell},$$

and then

$$U_k^{(p)}(-1) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^{k-\ell-p} 2^{k-2\ell} \frac{(k-\ell)!}{\ell!(k-2\ell-p)!}.$$

Now, we can find the matrix $P_k(L(C_n))$ using the same techniques of the last section. $L(C_n)$ is the circulant matrix

$$\begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & \mathbf{0} & \\ & \ddots & \ddots & \ddots & \\ & \mathbf{0} & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{bmatrix}.$$

Hence

$$L(C_n) = 2P\left(\sigma^0\right) - P(\sigma) - P\left(\sigma^{n-1}\right).$$

The eigenvalues of $L(C_n)$ are

$$2 - 2\cos\frac{2\ell\pi}{n},$$

for $\ell = 0, \ldots, n-1$ and thus

$$\begin{aligned} P_k L(C_n) &= P_k\left(2P\left(\sigma^0\right) - P(\sigma) - P\left(\sigma^{n-1}\right)\right) \\ &= n^{-1} \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} e^{-i\frac{2\ell j\pi}{n}} U_k\left(1 - \cos\frac{2\ell\pi}{n}\right) P\left(\sigma^j\right) \\ &= (-1)^k \sum_{j=0}^{n-1} \sum_{p=0}^{k} \sum_{\ell=0}^{p} \frac{U_k^{(p)}(-1)}{\ell!(p-\ell)!2^p}\, \delta_{j+2\ell-p,\dot{n}} P\left(\sigma^j\right). \end{aligned}$$

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