THE PATH POLYNOMIAL OF A COMPLETE GRAPH

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Abstract. Let $P_k(x)$ denote the polynomial of the path on $k$ vertices. A complete description of the matrix that is obtained by evaluating $P_k(x)$ at the adjacency matrix of the complete graph, along with computing the effect of evaluating $P_k(x)$ with Laplacian matrices of a path and of a circuit.

Key words. Graph, Adjacency matrix, Laplacian matrix, Characteristic polynomial.

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1. Introduction and preliminaries. For a finite and undirected graph $G$ without loops or multiple edges, with $n$ vertices, let us define the polynomial of $G$, $P_G$, as the characteristic polynomial of its adjacency matrix, $A(G)$, i.e.,

$$P_G(x) = \det(xI_n - A(G)).$$

When the graph is a path with $n$ vertices, we simply call $P_G$ the path polynomial and denote it by $P_n$. Define $A_n$ as the adjacency matrix of a path on $n$ vertices.

For several interesting classes of graphs, $A(G_i)$ is a polynomial in $A(G)$, where $G_i$ is the $i$th distance graph of $G$ ([5]). Actually, for distance-regular graphs, $A(G_i)$ is a polynomial in $A(G)$ of degree $i$, and this property characterizes these kind of graphs ([14]).

In [4], Beezer has asked when a polynomial of an adjacency matrix will be the adjacency matrix of another graph. Beezer gave a solution in the case that the original graph is a path.

**Theorem 1.1 ([4]).** Suppose that $p(x)$ is a polynomial of degree less than $n$. Then $p(A_n)$ is the adjacency matrix of a graph if and only if $p(x) = P_{2i+1}(x)$, for some $i$, with $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

In the same paper, Beezer gave an elegant formula for $P_k(A_n)$ with $k = 1, \ldots, n$, and Bapat and Lal, in [1], completely described the structure of $P_k(A_n)$, for all integers $k$. This result was also reached by Fonseca and Petronilho ([10]) in a non-inductive way.

**Theorem 1.2 ([1],[4],[10]).** For $0 \leq k \leq n - 1$, $n$ being a positive integer,

$$(P_k(A_n))_{ij} = \begin{cases} 1 & \text{if } i + j = k + 2r, \text{ with } 1 \leq r \leq \min \{i, j, n - k\} \\ 0 & \text{otherwise.} \end{cases}$$

In [12], Shi Ronghua obtained some generalizations of the ones achieved by Bapat and Lal. Later, in [10], Fonseca and Petronilho determined the matrix $P_k(C_n)$, where $C_n$ is the adjacency matrix of a circuit on $n$ vertices.

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Consider the permutation \( \sigma = (12 \ldots n) \).

**Theorem 1.3 ([10]).** For any nonnegative integer \( k \),

\[
P_k(C_n) = \sum_{j=0}^{n-1} \sum_{r=0}^{n-1} \delta_{2r,k+2+j-n} P(\sigma^j),
\]

where \( \delta \) is the Kronecker function, \( \sigma \) is the permutation \( (12 \ldots n) \), \( P(\sigma^j) \) is the permutation matrix of \( \sigma^j \) and \( \bar{n} \) runs over the multiples of \( n \).

According to Bapat and Lal (cf. [1]), a graph \( G \) is called *path-positive of order* \( m \) if \( P_k(G) \geq 0 \), for \( k = 1, 2, \ldots, m \), and \( G \) is simply called *path-positive* if it is path-positive of any order. In [3], Bapat and Lal have characterized all graphs that are path-positive. The following corollary is immediate from the theorem above.

**Corollary 1.4.** The circuit \( C_n \) is path-positive.

We define the *complete graph* \( K_n \), to be the graph with \( n \) vertices in which each pair of vertices is adjacent. The adjacency matrix of a complete graph, which we identify also by \( K_n \), is the \( n \times n \) matrix

\[
K_n = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{bmatrix}.
\] (1.1)

In this note, we evaluate \( P_k(K_n) \).

2. **The polynomial** \( P_k \). Let us consider the tridiagonal matrix \( A_k \) whose entries are given by

\[
(A_k)_{ij} = \begin{cases} 
1 & \text{if } |i-j| = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

The expansion of the determinant

\[
\det (xI_k - A_k) = P_k(x)
\]
along the first row or column gives us the recurrence relation

\[
P_k(x) = xP_{k-1}(x) - P_{k-2}(x),
\]

for any positive integer \( k \), with the convention \( P_{-1}(x) = 0 \) and \( P_0(x) = 1 \).

It is well known that

\[
P_k(x) = U_k \left( \frac{x}{2} \right), \quad x \in \mathbb{C}, \quad (k = 0, 1, \ldots),
\]

where \( U_k(x) \) are the Chebyshev polynomials of the second kind.
From (2.2), it is straightforward to prove that

\[(2.3)\quad \frac{P_k(x) - P_k(y)}{x - y} = \sum_{\ell=0}^{k-1} P_\ell(x) P_{k-1-\ell}(y).\]

Then, from (2.1) and (2.3), we may conclude the following lemma.

**Lemma 2.1.** For any positive integer \(k\) and square matrices \(A\) and \(B\),

\[P_k(A) - P_k(B) = \sum_{\ell=0}^{k-1} P_\ell(A) (A - B) P_{k-1-\ell}(B).\]

As in Bapat and Lal [1], note that a connected graph is path-positive if it has a spanning subgraph which is path-positive. Thus we have this immediate corollary from Corollary 1.4.

**Corollary 2.2.** The complete graph \(K_n\) is path-positive.

### 3. Evaluating \(P_k\) of a complete graph.

If a matrix \(A = (a_{ij})\) satisfies the relation

\[a_{ij} = a_{1\sigma^{1-i}(j)}\]

we say that \(A\) is a *circulant matrix*. Therefore, to define a circulant matrix \(A\) is equivalent to presenting an \(n\)-tuple, say \((a_1, \ldots, a_n)\). Then

\[A = \sum_{i=0}^{n-1} a_i P(\sigma^i),\]

and its eigenvalues are given by

\[(3.1)\quad \lambda_h = \sum_{\ell=0}^{n-1} \zeta^{h\ell} a_\ell,\]

where \(\zeta = \exp\left(\frac{2\pi i}{n}\right)\). Given a polynomial \(p(x)\), the image of \(A\) is

\[p(A) = p\left(\sum_{i=0}^{n-1} a_i P(\sigma^i)\right) = n^{-1} \sum_{j=0}^{n-1} \sum_{h=0}^{n-1} \zeta^{-hj} p\left(\sum_{\ell=0}^{n-1} \zeta^{h\ell} a_\ell\right) P(\sigma^j).\]

Then,

\[P_k\left(\sum_{i=0}^{n-1} a_i P(\sigma^i)\right) = n^{-1} \sum_{j=0}^{n-1} \sum_{h=0}^{n-1} \zeta^{-hj} P_k(\lambda_h) P(\sigma^j),\]

where \(\lambda_h\) is defined as in (3.1).
The matrix $K_n$, defined in (1.1), is a circulant matrix and it can be written

$$K_n = \sum_{i=1}^{n-1} P(\sigma^i).$$

By (3.1), $K_n$ has the eigenvalues $\lambda_0 = n - 1$ and $\lambda_\ell = -1$, for $\ell = 1, \ldots, n - 1$. Therefore,

$$P_k(K_n) = P_k \left( \sum_{i=1}^{n-1} P(\sigma^i) \right)$$

$$= n^{-1} \sum_{j=0}^{n-1} \sum_{h=0}^{n-1} \zeta^{-hj} P_k(\lambda_h) P(\sigma^j)$$

$$= n^{-1} \sum_{j=0}^{n-1} \left( P_k(n - 1) + P_k(-1) \sum_{h=1}^{n-1} \zeta^{-hj} \right) P(\sigma^j)$$

$$= P_k(-1) P(\sigma^0) + n^{-1} (P_k(n - 1) - P_k(-1)) \sum_{j=0}^{n-1} P(\sigma^j).$$

Note that $P(\sigma^0)$ is the identity matrix.

We have thus proved the main result of this section:

**Theorem 3.1.** For any nonnegative integer $k$, the diagonal entries of $P_k(K_n)$ are the weighted average $\frac{1}{n} P_k(n - 1) + \frac{n-1}{n} P_k(-1)$ and the off-diagonal entries are $\frac{1}{n} (P_k(n - 1) - P_k(-1))$.

We can easily evaluate the different values of each term of the sum $P_k(K_n)$. According to (2.2),

$$P_k(-1) = \begin{cases} -1 & \text{if } k \equiv 1 \text{ mod } 3 \\ 0 & \text{if } k \equiv 2 \text{ mod } 3 \\ 1 & \text{if } k \equiv 0 \text{ mod } 3 \end{cases}.$$

Another relation already known ([11, p.72]) for $P_k(x)$ is

$$P_k(x) = \sum_{\ell=0}^{[k/2]} (-1)^{\ell} \binom{k-\ell}{\ell} x^{k-2\ell},$$

where $\lfloor z \rfloor$ denotes the greatest integer less or equal to $z$. Therefore we have also

$$P_k(n - 1) - P_k(-1) = \sum_{\ell=0}^{[k/2]} (-1)^{\ell} \binom{k-\ell}{\ell} ((n - 1)^{k-2\ell} - (-1)^{k-2\ell})$$

$$= \sum_{\ell=0}^{[k/2]} \sum_{j=1}^{k-2\ell} (-1)^{k-\ell+j} \frac{(k-\ell)!}{\ell!(k-2\ell-j)!} n^{j-1}.$$
4. Evaluating $P_k$ of some Laplacian matrices. Let $G$ be a graph. Denote $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then

$$L(G) = D(G) - A(G)$$

is the Laplacian matrix of $G$.

In this section, expressions for $P_k(L(A_n))$ and $P_k(L(C_n))$, the path polynomials of the Laplacian matrices of a path and a circuit, respectively, with $n$ vertices, are determined.

Let us consider the following recurrence relation:

$$\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x + 1,$$

$$\tilde{P}_k(x) = (x + 2)\tilde{P}_{k-1}(x) - \tilde{P}_{k-2}(x), \quad \text{for } 2 \leq k \leq n - 1,$$

and

$$\tilde{P}_n(x) = (x + 1)\tilde{P}_{n-1}(x) - \tilde{P}_{n-2}(x).$$

Therefore

$$\tilde{P}_k(x) = U_k \left( \frac{x}{2} + 1 \right) - U_{k-1} \left( \frac{x}{2} + 1 \right), \quad \text{for } 2 \leq k \leq n - 1,$$

and

$$\tilde{P}_n(x) = x U_{n-1} \left( \frac{x}{2} + 1 \right).$$

where $U_k(x)$ are the Chebyshev polynomials of the second kind.

Then the zeroes of $\tilde{P}_n(x)$ are

$$\lambda_j = 2 \cos \frac{j\pi}{n} - 2, \quad j = 0, \ldots, n - 1.$$

The recurrence relation above can be written in the following matricial way:

$$
\begin{bmatrix}
\tilde{P}_0(x) \\
\tilde{P}_1(x) \\
\vdots \\
\tilde{P}_{n-2}(x) \\
\tilde{P}_{n-1}(x)
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 1 & 0 & & & \\
1 & -2 & 1 & & & \\
& & \ddots & \ddots & \ddots & \\
0 & & & 1 & -2 & 1 \\
& & & & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{P}_0(x) \\
\tilde{P}_1(x) \\
\vdots \\
\tilde{P}_{n-2}(x) \\
\tilde{P}_{n-1}(x)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
(4.1)
$$

Thus, for $j = 0, \ldots, n - 1$, the vector

$$
\begin{bmatrix}
\tilde{P}_0(\lambda_j) \\
\tilde{P}_1(\lambda_j) \\
\vdots \\
\tilde{P}_{n-2}(\lambda_j) \\
\tilde{P}_{n-1}(\lambda_j)
\end{bmatrix}
= 
\left( \cos \frac{j\pi}{2n} \right)^{-1}
\begin{bmatrix}
\cos \frac{2\pi}{2n} \\
\cos \frac{3\pi}{2n} \\
\vdots \\
\cos(2n-3)\frac{\pi}{2n} \\
\cos(2n-1)\frac{\pi}{2n}
\end{bmatrix}
$$
is an eigenvector associated to the eigenvalue \( \lambda_j \) of \( -L(A_n) \).

Therefore the matrix \( -L(A_n) \) is diagonalizable and, for \( 0 \leq k \leq n \), the \((i, j)\)th entry of \( P_k(L(A_n)) \) is given by

\[
(P_k(L(A_n)))_{ij} = (-1)^k \sum_{\ell=0}^{n-1} \frac{\tilde{P}_{i-1}(\lambda_\ell) \tilde{P}_{j-1}(\lambda_\ell)}{\sum_{s=1}^{n} \left( \tilde{P}_{s-1}(\lambda_\ell) \right)^2}
\]

which is equal to

\[
\frac{(-1)^k \cos \left( \frac{k\pi}{2n} \right)}{n} + \frac{(-1)^k 2}{n} \sum_{\ell=1}^{n-1} (\cos(2\ell - 1) - 1) \frac{\ell\pi}{2n} \left( \cos(L_n - 1) \right) \cos(2\ell - 1) \frac{\ell\pi}{2n}
\]

If we define

\[
\alpha_p^m = \sum_{\ell=1}^{n-1} \cos \frac{p\pi}{n} \cos \frac{\ell\pi}{n},
\]

then

\[
\alpha_p^m = \frac{1}{2} \left( \alpha_{m-1}^p + \alpha_{m+1}^p \right)
\]

and

\[
(4.2) \quad \alpha_p^m = \frac{1}{2p} \sum_{\ell=0}^{p} \left( \binom{p}{\ell} \right) \alpha_{m+2\ell-p}^0,
\]

with

\[
\alpha_m^0 = n\delta_{m,2n} - \frac{1}{2} (1 + (-1)^m),
\]

where \( \delta \) represents a multiple of \( n \).

Using the trigonometric transformation formula and the Taylor formula

\[
U_k \left( \cos \frac{\ell\pi}{n} - 1 \right) = \sum_{p=0}^{k} \frac{U_k(p)}{p!} (-1)^p \cos \frac{p\ell\pi}{n},
\]

we can state the following proposition.

**Theorem 4.1.** For \( 0 \leq k \leq n \), \( n \) being a positive integer,

\[
(P_k(L(A_n)))_{ij} = \frac{(-1)^k \cos \left( \frac{k\pi}{2n} \right)}{n} + \frac{(-1)^k 2}{n} \sum_{p=0}^{k} \frac{U_k(p)}{p!} \left( \alpha_{i-j}^p + \alpha_{i+j-1}^p \right),
\]

where \( \alpha_m^p \) is defined as in (4.2).
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Note that $U_k^{(p)}(-1)$ can be easily evaluated, since

$$U_k(x) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^\ell \binom{k-\ell}{\ell} (2x)^{k-2\ell},$$

and then

$$U_k^{(p)}(-1) = \sum_{\ell=0}^{\lfloor k/2 \rfloor} (-1)^{k-\ell-p} 2^{k-2\ell} \frac{(k-\ell)!}{\ell!(k-2\ell-p)!}.$$ 

Now, we can find the matrix $P_k(L(C_n))$ using the same techniques of the last section. $L(C_n)$ is the circulant matrix

$$L(C_n) = 2P(\sigma^0) - P(\sigma) - P(\sigma^{n-1}).$$

The eigenvalues of $L(C_n)$ are

$$2 - 2 \cos \frac{2\ell \pi}{n},$$

for $\ell = 0, \ldots, n-1$ and thus

$$P_k(L(C_n)) = P_k\left(2P(\sigma^0) - P(\sigma) - P(\sigma^{n-1})\right)$$

$$= n^{-1} \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} e^{-i2\ell j \pi} U_k\left(1 - \cos \frac{2\ell \pi}{n}\right) P(\sigma^j)$$

$$= (-1)^k \sum_{j=0}^{n-1} \sum_{p=0}^{k} \sum_{\ell=0}^{p} \frac{U_k^{(p)}(-1)}{p!(p-\ell)!2^p} \delta_{j+2\ell-p,n} P(\sigma^j).$$

REFERENCES