Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 102-105, May 2003

ELA

RECOGNITION OF HIDDEN POSITIVE ROW DIAGONALLY DOMINANT MATRICES*

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Abstract. A hidden positive row diagonally dominant (hprdd) matrix is a square matrix A for which there exist square matrices C and B so that $AC = B$ and each diagonal entry of B and C is greater than the sum of the absolute values of the off-diagonal entries in its row. A linear program with $5n^2 - 4n$ variables and $2n^2$ constraints is defined that takes as input an $n \times n$ matrix A and produces C and B satisfying the above conditions if and only if they exist. A 4×4 symmetric positive definite matrix that is not an hprdd matrix is presented.

Key words. Factorization of matrices, Linear inequalities, P-matrices.

AMS subject classifications. 15A23, 15A39, 15A48.

1. Introduction and Terminology. An $n \times n$ matrix $A = (a_{ij})$ with real entries is called *row diagonally dominant* if, for $i = 1, \ldots, n$, we have

$$|a_{ii}| > \sum_{i \neq j} |a_{ij}|.$$

The matrix A is called a *P-matrix* if the principal minors of A are all positive. The problem of determining if A is not a P-matrix is known to be NP-complete [2]. Two of the most important subclasses of the class of P-matrices are the class of positive definite matrices and the class of M-matrices. M-matrices are P-matrices that have non-positive off-diagonal elements. Membership in each of these subclasses can be checked in polynomial time. The class of M-matrices has been generalized to the class of *hidden M* matrices. A P-matrix A is hidden M if there are matrices B and C such that $AC = B$, where C and B both have non-positive off-diagonal entries and C is an M-matrix. Membership in the hidden M class can also be tested in polynomial time (see [4]). It has recently been (see [7]) proved that A is a P-matrix whenever one can find row diagonally dominant matrices B and C, with positive diagonal entries, so that $AC = B$. We will call matrices satisfying this sufficient condition for P-matrices *hidden positive row diagonally dominant*, or *hprdd*.

Tsatsomeros [7] proved that every hidden-M matrix is *hprdd*. No example of a P-matrix that is not *hprdd* was known until now, and there was some speculation (see [7]) that there might not be such a matrix. Part of the difficulty in producing such a matrix was that no efficient algorithm had been published for checking the *hprdd* property. We present a linear program with $5n^2-4n$ variables and $2n^2$ constraints that produces the required B and C if they exist. If the input matrix is not *hprdd*, the solution to the dual linear program provides a proof that the matrix is not *hprdd*. Experimentation

*Received by the editors on 16 January 2003. Accepted for publication on 27 April 2003. Handling Editor: Bryan Shader.
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Volume 10, pp. 102-105, May 2003

with randomly generated positive definite matrices yielded the following example of a positive definite matrix that is not *hprdd*:

$$\begin{bmatrix} 4 & 4 & 7 & -9 \\ 4 & 16 & -7 & -2 \\ 7 & -7 & 30 & -24 \\ -9 & -2 & -24 & 25 \end{bmatrix}.$$

2. The Linear Program. The set of vectors K in $\mathbb{R}^n$ satisfying an inequality $x_i > \sum_{i \neq j} |x_j|$ is the interior of a cone that has $2(n-1)$ extreme rays, each of the form $e_i \pm e_j$ for standard basis vectors $e_i \neq e_j$. The set of pairs (C, B) of $n \times n$ matrices can be identified with $\mathbb{R}^{2n^2}$. Define $U = \{(C, B) \in \mathbb{R}^{2n^2}$: B and C are both positive row diagonally dominant$\}$. Then U is the Cartesian product of $2n$ copies of K. It is the interior of a convex cone that has the $4n(n-1)$ extreme rays of the form $(E_{ii} \pm E_{ij}, 0)$, $i \neq j$ or $(0, E_{ii} \pm E_{ij})$, $i \neq j$ where E_{ij} is the $n \times n$ matrix that has 1 in entry (i, j) and 0 everywhere else.

Let A be an $n \times n$ matrix. Define $V = \{(C, B) \in \mathbb{R}^{2n^2} : AC = B\}$. Note that V is a subspace of $\mathbb{R}^{2n^2}$ that is spanned by the n^2 vectors of the form (E_{ij}, AE_{ij}). A matrix A is hprdd if and only if $V \cap U$ is not empty.

In order to define a linear program to check for the hprdd property of a matrix A, let H be a $2n^2 \times n^2$ matrix that has as its columns the vectors (E_{ij}, AE_{ij}) (suitably identified with column vectors in $\mathbb{R}^{2n^2}$), and let G be a $2n^2 \times 4n(n-1)$ matrix that has as its columns the vectors $(E_{ii} \pm E_{ij}, 0), i \neq j$ and $(0, E_{ii} \pm E_{ij}), i \neq j$. Then $U \cap V$ is nonempty if and only if there exist vectors x and y, with $y > 0$, so that $Hx = Gy$. The components of y are positive because V is the interior of the cone generated by the columns of G. The problem of finding x and y, if they exist, is a linear program with $2n^2$ rows and $5n^2 - 4n$ columns.

Most linear program solvers require the inequality $y > 0$ to be of the form $\geq$. We will therefore require that the components of y each be at least 1. This does not affect the feasibility of the problem, because V is closed under multiplication by positive scalars. We also look for a solution that minimizes the sum of the components of y.

3. Infeasibility Proof. From Theorem 11.2 of Rockafellar [5] we conclude that if the subspace V and the non-empty open convex set U are disjoint, there must be a hyperplane L containing V so that one of the open half-spaces associated with L contains U. Clearly, U is in an open half-space associated with L if and only if its closure is in the corresponding closed half-space and an element of the closure of U is in the open half-space. When V is the span of the columns of a matrix H, as above, and U is the interior of the cone generated by the columns of a matrix G, the existence of such an L follows from the theorem of Stiemke [6].

The dot product of two elements (R, S) and (C, B) in $\mathbb{R}^{2n^2}$ is $(R, S) \cdot (C, B) = \#(R \circ C + S \circ B)$ where $\circ$ is the entrywise product and $\#$ is the sum of the entries function. Let L be the set of (C, B) in $\mathbb{R}^{2n^2}$ for which $(R, S) \cdot (C, B) = 0$. Then $V \subseteq L$ if and only if $(R, S) \cdot (E_{ij}, AE_{ij}) = 0$ for all pairs (i, j). Note that AE_{ij} is the matrix with column i of A in its j^{th} column and zeros everywhere else. It follows

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that $\#(R \circ E_{ij}) = r_{ij}$ and that $\#(S \circ AE_{ij})$ is the usual dot product of column j of S and column i of A. The inclusion $V \subseteq L$ is therefore equivalent to the equation $R + A^T S = 0$.

The requirement that the closure of U be in one of the closed half-spaces determined by L is equivalent to the set of $4n(n-1)$ inequalities $(R,S)\cdot(E_{ii} \pm E_{ij}, 0) \geq 0$, $i \neq j$ and $(R,S)\cdot(0, E_{ii} \pm E_{ij}) \geq 0$, $i \neq j$. The inequalities $(R,S)\cdot(E_{ii} \pm E_{ij}, 0) \geq 0$ for a given pair (i,j) with $i \neq j$ are clearly equivalent to the inequality $r_{ii} \geq |r_{ij}|$, and the inequalities $(R,S)\cdot(0, E_{ii} \pm E_{ij}) \geq 0$ for a given pair (i,j) with $i \neq j$ are clearly equivalent to the inequality $s_{ii} \geq |s_{ij}|$.

In order for the open set U, which is in a closed halfspace determined by L, to be in the corresponding open halfspace, we need one of the extreme rays of the closure of U to miss L. Thus, for some $i \neq j$, we need one of the following: $r_{ii} > r_{ij}, r_{ii} > -r_{ij}, s_{ii} > s_{ij}$, or $s_{ii} > -s_{ij}$. For a pair (R,S) that satisfies the inequalities $r_{ii} \geq |r_{ij}|$ and $s_{ii} \geq |s_{ij}|$ whenever $i \neq j$, the satisfaction of at least one such strict inequality is equivalent to $(R,S) \neq (0,0)$.

It is not possible for $U \cap V$ to be nonempty if U is in one of the open half-spaces defined by L (which contains V). We therefore have the following theorem.

Theorem 3.1. *Let A be an $n \times n$ matrix. Exactly one of the following holds:*
1. *There exist row diagonally dominant matrices B and C with positive diagonal entries and with $AC = B$, or*
2. *There exist square matrices R and S, not both 0, satisfying $R + A^T S = 0$ and, for all $1 \leq i,j \leq n$, $r_{ii} \geq |r_{ij}|$ and $s_{ii} \geq |s_{ij}|$.*

4. A non-hprdd matrix that is positive definite. We implemented the linear program of the previous section using the free software "MPL 4.2 for Windows." The 6×6 symmetric positive definite matrix of [1] that is not hidden M encouraged us to test symmetric positive definite matrices. (See [3] for a 3×3 positive definite matrix that is not hidden M.) It is also easy to generate random symmetric positive definite matrices by randomly generating their Cholesky factors. The smallest counterexample we found is the matrix

$$\begin{bmatrix} 4 & 4 & 7 & -9 \\ 4 & 16 & -7 & -2 \\ 7 & -7 & 30 & -24 \\ -9 & -2 & -24 & 25 \end{bmatrix},$$

for which the proof that it is not hprdd is provided by the matrices

$$R = \begin{bmatrix} 11 & 8 & 2 & -11 \\ 23 & 24 & -24 & -7 \\ 10 & -16 & 23 & -22 \\ -22 & -8 & -23 & 23 \end{bmatrix} \text{ and } S = \begin{bmatrix} 24 & -24 & -9 & 23 \\ -8 & 8 & 8 & -5 \\ -3 & 8 & 8 & -1 \\ 6 & 0 & 6 & 6 \end{bmatrix}.$$

We should point out that this approach does not resolve the question of whether or not there exist 3×3 P-matrices satisfying the second alternative of the Theorem. To produce a P matrix satisfying the second alternative where its transpose does not is an interesting challenge.

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Acknowledgment. The author would like to thank Teresa Chu for showing him this problem, for suggesting the name "hidden prdd," and for pointing out the example in [3]. He would also like to thank M. Tsatsomeros and C. R. Johnson for their comments on a first draft of the paper. The anonymous referee also provided helpful comments.

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