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THE A_α-SPECTRUM OF GRAPH PRODUCT*

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Abstract. Let $A(G)$ and $D(G)$ denote the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. Define

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

for any real $\alpha \in [0,1]$. The collection of eigenvalues of $A_\alpha(G)$ together with multiplicities is called the A_α-*spectrum* of G. Let $G\square H$, $G[H]$, $G \times H$ and $G \oplus H$ be the Cartesian product, lexicographic product, directed product and strong product of graphs G and H, respectively. In this paper, a complete characterization of the A_α-spectrum of $G\square H$ for arbitrary graphs G and H, and $G[H]$ for arbitrary graph G and regular graph H is given. Furthermore, A_α-spectrum of the generalized lexicographic product $G[H_1, H_2, \ldots, H_n]$ for n-vertex graph G and regular graphs H_i's is considered. At last, the spectral radii of $A_\alpha(G \times H)$ and $A_\alpha(G \oplus H)$ for arbitrary graph G and regular graph H are given.

Key words. A_α-spectrum, Cartesian product, Lexicographic product, Generalized lexicographic product.

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1. Introduction. In this paper, we are concerned with simple finite undirected graphs. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees of G and $A(G)$ be the adjacency matrix of G. The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. In [9], Nikiforov proposes to study the convex combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ defined by

$$A_\alpha(G) := \alpha D(G) + (1-\alpha)A(G), \quad 0 \le \alpha \le 1.$$

Note that $A_0(G) = A(G)$ and $A_{1/2}(G) = 1/2Q(G)$ and $A_1(G) = D(G)$, $A_\alpha(G)$ runs from $A(G)$ to $D(G)$ with essentially $Q(G)$ in the middle of the way, and it was claimed in [9, 10] that the matrices $A_\alpha(G)$ can underpin a unified theory of $A(G)$ and $Q(G)$. In [10], several results about the $A_\alpha(G)$-matrices of trees are given. In [9] and [12], the authors search for the positive semidefiniteness of $A_\alpha(G)$. For more properties of $A_\alpha(G)$, we refer the readers to [2, 6, 7, 8, 9, 10, 11, 12].

Let M be an $n \times n$ real symmetric matrix. Denote the eigenvalues of M by $\lambda_1(M) \ge \lambda_2(M) \ge \cdots \ge \lambda_n(M)$. The collection of eigenvalues of M together with multiplicities is called the spectrum of M, denoted by $Spec(M)$. In particular, $\lambda_1(M)$ is called the spectral radius of M and $\lambda_n(M)$ is called the least eigenvalues of M.

In this paper, the identity matrix of appropriate order is denoted by I, I_m and $J_{m\times n}$ denote the identity matrix of order m and the all one $m \times n$ matrix, respectively. Furthermore, we write j_m for the column m-vector of ones and 0 for the all zeros matrix of the appropriate notations. We use $[n]$ to denote the set of $\{1, 2, \ldots, n\}$.

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Let $G\Box H$, $G[H]$, $G \times H$ and $G \oplus H$ be the Cartesian product, lexicographic product, directed product and strong product of graphs G and H, respectively. This paper is organized as follows. In the next section, we recall some basic definitions of those graph products. In Section 3, we give a complete characterization of the A_α-spectrum of $G\Box H$ for arbitrary graph G and arbitrary graph H. In Sections 4, we give the characterization of A_α-spectrum of $G[H]$ for arbitrary graph G and regular graph H. In Section 5, we consider A_α-spectrum of the generalized lexicographic product $G[H_1, H_2, \ldots, H_n]$ for n-vertex graph G and regular graphs H_i's. In the last section, we give the spectral radii of $A_\alpha(G \times H)$ and $A_\alpha(G \oplus H)$ for arbitrary graph G and regular graph H.

2. Preliminaries. In this section, we will given some basic definitions.

The Cartesian product, direct product, the strong product and the lexicographic product are defined as follows, also see [1, 3, 4, 5, 13, 14].

The *Cartesian product* $G\Box H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

The *direct product* $G \times H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

The *strong product* $G \oplus H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G\Box H) \cup E(G \times H)$.

The *lexicographic product* $G[H]$ (also called the *composition*) of graphs G and H, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$, in which two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.

The lexicographic product was generalized in [14] as follows: Consider a graph G whose vertex set is $\{v_1, v_2, \ldots, v_n\}$ and graphs H_i, $i = 1, 2, \ldots, n$, with vertex sets $V(H_i)s$ two by two disjoints. The *generalized composition* $G[H_1, H_2, \ldots, H_n]$ is the graph such that

$$V(G[H_1, H_2, \ldots, H_n]) = \bigcup_{i=1}^{n} V(H_i)$$

and

$$E(G[H_1, H_2, \ldots, H_n]) = \bigcup_{i=1}^{n} E(H_i) \cup \bigcup_{v_i v_j \in E(G)} E(H_i \vee H_j),$$

where $G_i \vee G_j$ denotes the join of the graphs G_i and G_j. It is obvious that $G[H, \ldots, H]$ is exactly the graph $G[H]$.

3. The spectrum of $A_\alpha(G\Box H)$. In this section, we will characterize the spectrum of $A_\alpha(G\Box H)$ for arbitrary graphs G of order n and H of order m. Let $A \otimes B$ denote the Kronecker product [12] of two matrix $A = (a_{ij})$ and $B = (b_{i,j})$, i.e., $A \otimes B = (a_{ij}B)$. Some basic properties of the Kronecker product are $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. Moreover, if both A and B are invertible matrices, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$; if both A and B are orthogonal matrices, then $A \otimes B$ is also an orthogonal matrix.

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It is well known that

$$A(G\Box H) = A(G)\otimes I_m + I_n \otimes A(H)$$

and

$$d_{G\Box H}((v_i,u_j)) = d_G(v_i) + d_H(u_j),$$

i.e.,

$$D(G\Box H) = D(G)\otimes I_m + I_n \otimes D(H).$$

Thus, we have that

$$A_\alpha(G\Box H) = A_\alpha(G)\otimes I_m + I_n \otimes A_\alpha(H). \tag{3.1}$$

THEOREM 3.1. *Let G and H be any graph with order n and m. If $Spec(A_\alpha(G)) = \{\lambda_1(A_\alpha(G)),\ldots,\lambda_n(A_\alpha(G))\}$ and $Spec(A_\alpha(H)) = \{\lambda_1(A_\alpha(H)),\ldots,\lambda_m(A_\alpha(H))\}$, then*

$$Spec(A_\alpha(G\Box H)) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\{\lambda_i(A_\alpha(G)) + \lambda_j(A_\alpha(H))\}.$$

Proof. Let

$$X = [X_1\ X_2\ \cdots\ X_n]$$

be an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalue $\lambda_1(A_\alpha(G))$, $\lambda_2(A_\alpha(G)),\ldots,\lambda_n(A_\alpha(G))$. Let

$$Y = [Y_1\ Y_2\ \cdots\ Y_m]$$

be an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalue $\lambda_1(A_\alpha(H))$, $\lambda_2(A_\alpha(H)),\ldots,\lambda_m(A_\alpha(H))$. Then

$$X^T A_\alpha(G) X = \begin{pmatrix} \lambda_1(A_\alpha(G)) & & & \\ & \lambda_2(A_\alpha(G)) & & \\ & & \ddots & \\ & & & \lambda_n(A_\alpha(G)) \end{pmatrix} \tag{3.2}$$

and

$$Y^T A_\alpha(H) Y = \begin{pmatrix} \lambda_1(A_\alpha(H)) & & & \\ & \lambda_2(A_\alpha(H)) & & \\ & & \ddots & \\ & & & \lambda_m(A_\alpha(H)) \end{pmatrix}. \tag{3.3}$$

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Note that $X \otimes Y$ is an orthogonal matrix, and

$$\begin{aligned}(X\otimes Y)^T A_\alpha(G\Box H)(X\otimes Y) &= (X\otimes Y)^T A_\alpha(G)\otimes I_m + I_n\otimes A_\alpha(H)(X\otimes Y)\\ &= (X^T A_\alpha(G)X)\otimes(Y^TY) + (X^TX)\otimes(Y^TA_\alpha(H)Y)\\ &= \begin{pmatrix}\lambda_1(A_\alpha(G)) & & & \\ & \lambda_2(A_\alpha(G)) & & \\ & & \ddots & \\ & & & \lambda_n(A_\alpha(G))\end{pmatrix}\otimes I_m\\ &\quad + I_n\otimes\begin{pmatrix}\lambda_1(A_\alpha(H)) & & & \\ & \lambda_2(A_\alpha(H)) & & \\ & & \ddots & \\ & & & \lambda_m(A_\alpha(H))\end{pmatrix}.\end{aligned}$$

Thus, we have our conclusion. □

4. The spectrum of $A_\alpha(G[H])$. In this section, we will characterize the spectrum of $A_\alpha(G[H])$ for arbitrary graph G and regular graph H.

Recall that $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. Then, we can see that for p-regular graph G, $A_\alpha(G) = p\alpha I + (1-\alpha)A(G)$. Hence, the following lemma is obvious:

Lemma 4.1. *Let H be a p-regular graph with $V(H) = \{u_1, u_2, \dots, u_m\}$. If $p \geq \lambda_2(H) \geq \cdots \geq \lambda_m(H)$ are the spectrum of $A(H)$, then*

$$Spec(A_\alpha(H)) = \{p, \alpha p + (1-\alpha)\lambda_2(H), \dots, \alpha p + (1-\alpha)\lambda_m(H)\}.$$

Furthermore, if $Y = [j_m\ Y_2\ \cdots\ Y_m]$ is an orthogonal matrix whose columns $j_m, Y_2, \dots, Y_m$ are eigenvectors corresponding to the eigenvalues $p, \lambda_2(H), \dots, \lambda_m(H)$, respectively, then Y is also an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalues $p, \alpha p + (1-\alpha)\lambda_2(H), \dots, \alpha p + (1-\alpha)\lambda_m(H)$ of $A_\alpha(H)$, respectively.

Theorem 4.2. *Let G be a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, H be a p-regular graph with $V(H) = \{u_1, u_2, \dots, u_m\}$, respectively. If $p \geq \lambda_2(H) \geq \cdots \geq \lambda_m(H)$ are the spectrum of $A(H)$, then*

$$Spec(A_\alpha(G[H])) = \bigcup \big\{\alpha p + (1-\alpha)\lambda_j(H) + \alpha m d_G(v_i)\big\} \cup Spec(C),$$

where $C = pI_n + A_\alpha(G)$.

Proof. Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G and $d_G(v_i)$ be the degree of v_i of G for $i = 1, 2, \dots, n$. It is obvious that

$$A(G[H]) = \begin{pmatrix} A(H) & a_{12}J_{m\times m} & \cdots & a_{1n}J_{m\times m}\\ a_{21}J_{m\times m} & A(H) & \cdots & a_{2n}J_{m\times m}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}J_{m\times m} & a_{n2}J_{m\times m} & \cdots & A(H)\end{pmatrix} = I_n\otimes A(H) + A(G)\otimes J_{m\times m}$$

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and

$$D(G[H]) = \begin{pmatrix} (p+d_G(v_1)m)I_m & 0 & \cdots & 0 \\ 0 & (p+d_G(v_2)m)I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p+d_G(v_n)m)I_m \end{pmatrix}$$
$$= I_n \otimes D(H) + mD(G) \otimes I_m.$$

Then we have that

$$\begin{aligned} A_\alpha(G[H]) &= \alpha\big(I_n \otimes D(H) + mD(G) \otimes I_m\big) + (1-\alpha)\big(I_n \otimes A(H) + A(G) \otimes J_{m\times m}\big) \\ &= I_n \otimes A_\alpha(H) + \alpha m D(G) \otimes I_m + (1-\alpha)A(G) \otimes J_{m\times m}. \end{aligned}$$

For $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, m$, we first prove that $\alpha p + (1-\alpha)\lambda_j(H) + \alpha m d_G(v_i)$ is an eigenvalue of $A_\alpha(G[H])$.

Let $Y = [j_m \ Y_2 \ \cdots \ Y_m]$ be an orthogonal matrix whose columns $j_m, Y_2, \cdots, Y_m$ are eigenvectors corresponding to the eigenvalues $p, \lambda_2(H), \ldots, \lambda_m(H)$, respectively. By Lemma 4.1, for $j = 2, 3, \ldots, m$, $A_\alpha(H)Y_j = \big(\alpha p + (1-\alpha)\lambda_j(H)\big)Y_j$ and $j_m^T Y_j = 0$. Let $e_i = (\underbrace{0, 0, \ldots, 1}_{i}, \ldots, 0)^T$ for $i = 1, 2, \ldots, n$. We have that

$$\begin{aligned} A_\alpha(G[H])(e_i \otimes Y_j) &= \big(I_n \otimes A_\alpha(H) + \alpha m D(G) \otimes I_m + (1-\alpha)A(G) \otimes J_{m\times m}\big)(e_i \otimes Y_j) \\ &= e_i \otimes A_\alpha(H)Y_j + \alpha m D(G)e_i \otimes Y_j + (1-\alpha)A(G)e_i \otimes (J_{m\times m}Y_j) \\ &= \big(\alpha p + (1-\alpha)\lambda_j(H)\big)(e_i \otimes Y_j) + \alpha m d_G(v_i)(e_i \otimes Y_j) + 0 \\ &= \big(\alpha p + (1-\alpha)\lambda_j(H) + \alpha m d_G(v_i)\big)(e_i \otimes Y_j). \end{aligned}$$

Hence, $e_i \otimes Y_j$ is an eigenvector of $A_\alpha(G[H])$ corresponding to $\alpha p + (1-\alpha)\lambda_j(H) + \alpha m d_G(v_i)$.

For $i = 1, 2, \ldots, n$, let X_i be the eigenvector of $A_\alpha(G)$ corresponding to $\lambda_i(A_\alpha(G))$. Then

$$\begin{aligned} A_\alpha(G[H])(X_i \otimes j_m) &= \big(I_n \otimes A_\alpha(H) + \alpha m D(G) \otimes I_m + (1-\alpha)A(G) \otimes J_{m\times m}\big)(X_i \otimes j_m) \\ &= X_i \otimes \big(A_\alpha(H)j_m\big) + \alpha m D(G)X_i \otimes j_m + (1-\alpha)A(G)X_i \otimes \big(J_{m\times m}j_m\big) \\ &= p(X_i \otimes j_m) + \alpha m D(G)X_i \otimes j_m + m(1-\alpha)A(G)X_i \otimes j_m \\ &= p(X_i \otimes j_m) + mA_\alpha(G)X_i \otimes j_m \\ &= \big(p + m\lambda_i(A_\alpha(G))\big)(X_i \otimes j_m). \end{aligned}$$

Hence, $X_i \otimes j_m$ is an eigenvector of $A_\alpha(G[H])$ corresponding to $p + m\lambda_i(A_\alpha(G)$.

Note that $(e_{i_1} \otimes Y_{j_1})^T(e_{i_2} \otimes Y_{j_2}) = 0$ if $(i_1, j_1) \neq (i_2, j_2)$, and $(e_{i_1} \otimes Y_j)^T(e_{i_2} \otimes j_m) = 0$ for any $i_1, i_2 \in [n]$ and $j \in [m] \setminus \{1\}$, i.e., all these eigenvectors are orthogonal, hence we have our conclusion. ∎

5. The spectrum of $A_\alpha(G[H_1, H_2, \ldots, H_n])$.

Theorem 5.1. *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and for $i = 1, 2, \ldots, n$, let H_i be a p_i-regular graph with order m_i, respectively. Let $A(G) = (a_{ij})$ be the adjacency matrix of G and for $i = 1, 2, \ldots, n$, $s_i = \sum_{j \in N_G(v_i)} m_j$. If $p_i \geq \lambda_2(H_i) \geq \cdots \geq \lambda_{m_i}(H_i)$ are the spectrum of $A(H_i)$, then*

$$Spec(A_\alpha(G[H_1, H_2, \ldots, H_n])) = \bigcup_{i=1}^{n} \bigcup_{j=2}^{m_i} \big\{\alpha(p_i + s_i) + (1-\alpha)\lambda_j(H_i)\big\} \cup Spec(C),$$

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where

$$C=\begin{pmatrix} p_1+\alpha s_1 & (1-\alpha)a_{12}\sqrt{m_1m_2} & \cdots & (1-\alpha)a_{1n}\sqrt{m_1m_n}\\ (1-\alpha)a_{21}\sqrt{m_2m_1} & p_2+\alpha s_2 & \cdots & (1-\alpha)a_{2n}\sqrt{m_2m_n}\\ \vdots & \vdots & \ddots & \vdots\\ (1-\alpha)a_{n1}\sqrt{m_nm_1} & (1-\alpha)a_{n2}\sqrt{m_nm_2} & \cdots & p_n+\alpha s_n \end{pmatrix}.$$

Proof. It is obvious that

$$A(G[H_1,H_2,\ldots,H_n])=\begin{pmatrix} A(H_1) & a_{12}J_{m_1\times m_2} & \cdots & a_{1n}J_{m_1\times m_n}\\ a_{21}J_{m_2\times m_1} & A(H_2) & \cdots & a_{2n}J_{m_2\times m_n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}J_{m_n\times m_1} & a_{n2}J_{m_n\times m_2} & \cdots & A(H_n) \end{pmatrix}$$

and

$$D(G[H_1,H_2,\ldots,H_n])=\begin{pmatrix} (p_1+s_1)I_{m_1} & 0 & \cdots & 0\\ 0 & (p_2+s_2)I_{m_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & (p_n+s_n)I_{m_n} \end{pmatrix}.$$

Then we have that

$$\begin{aligned} A_\alpha(G[H_1,H_2,\ldots,H_n]) &= \alpha D(G[H_1,H_2,\ldots,H_n])+(1-\alpha)A(G[H_1,H_2,\ldots,H_n])\\ &= \begin{pmatrix} A_\alpha(H_1)+\alpha s_1I_{m_1} & (1-\alpha)a_{12}J_{m_1\times m_2} & \cdots & (1-\alpha)a_{1n}J_{m_1\times m_n}\\ (1-\alpha)a_{21}J_{m_2\times m_1} & A_\alpha(H_2)+\alpha s_2I_{m_2} & \cdots & (1-\alpha)a_{2n}J_{m_2\times m_n}\\ \vdots & \vdots & \ddots & \vdots\\ (1-\alpha)a_{n1}J_{m_n\times m_1} & (1-\alpha)a_{n2}J_{m_n\times m_2} & \cdots & A_\alpha(H_n)+\alpha s_nI_{m_n} \end{pmatrix}. \end{aligned}$$

For $i=1,2,\ldots,n$ and $j=2,3,\ldots,m$, we first prove that $\alpha(p_i+s_i)+(1-\alpha)\lambda_j(H)$ is an eigenvalue of $A_\alpha(G[H_1,H_2,\ldots,H_n])$.

Let $Y_i=[j_{m_i}\ Y_{i2}\ \cdots\ Y_{im_i}]$ be an orthogonal matrix whose columns $j_{m_i},Y_{i2},\ldots,Y_{im_i}$ are eigenvectors corresponding to the eigenvalues $p_i,\lambda_2(H_i),\ldots,\lambda_{m_i}(H_i)$, respectively. By Lemma 4.1, for $j=2,3,\ldots,m_i$, $A_\alpha(H_i)Y_{ij}=\big(\alpha p_i+(1-\alpha)\lambda_j(H_i)\big)Y_{ij}$ and $j_{m_i}^TY_{ij}=0$. Let $Y'_{ij}=(\mathbf{0}_{1\times m_1},\mathbf{0}_{1\times m_2},\ldots,Y_{ij}^T,\ldots,\mathbf{0}_{1\times m_n})^T$. Note that

$$J_{m_j\times m_i}Y_{ij}=\mathbf{0}_{m_j\times 1}$$

and

$$(A_\alpha(H_i)+\alpha s_iI_{m_i})Y_{ij}=\big(\alpha(p_i+s_i)+(1-\alpha)\lambda_j(H_i)\big)Y_{ij}.$$

So, we have that

$$A_\alpha(G[H_1,\ldots,H_n])Y'_{ij}=\big(\alpha(p_i+s_i)+(1-\alpha)\lambda_j(H_i)\big)Y'_{ij}.$$

Hence, Y'_{ij} is an eigenvector of $A_\alpha(G[H_1,\ldots,H_n])$ corresponding to $\alpha(p_i+s_i)+(1-\alpha)\lambda_j(H_i)$.

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Let $X = [X_1 \ \cdots \ X_n]$ be an orthogonal matrix whose column $X_i = (x_{i1}, \ldots, x_{in})^T$ is an eigenvector corresponding to the eigenvalue $\lambda_i(C)$. Then $CX_i = \lambda_i(C)X_i$ and $X_i^T X_j = 0$ for $i \neq j$. Let

$$\begin{aligned} X_i' &= (\underbrace{\frac{x_{i1}}{\sqrt{m_1}}, \ldots, \frac{x_{i1}}{\sqrt{m_1}}}_{m_1}, \underbrace{\frac{x_{i2}}{\sqrt{m_2}}, \ldots, \frac{x_{i2}}{\sqrt{m_2}}}_{m_2} \cdots, \underbrace{\frac{x_{in}}{\sqrt{m_n}}, \ldots, \frac{x_{in}}{\sqrt{m_n}}}_{m_n})^T \\ &= (\frac{x_{i1}}{\sqrt{m_1}} j_{m_1}^T, \frac{x_{i2}}{\sqrt{m_2}} j_{m_2}^T, \ldots, \frac{x_{in}}{\sqrt{m_n}} j_{m_n}^T)^T. \end{aligned}$$

As

$$(A_\alpha(H_r) + \alpha s_r I_{m_r}) \frac{x_{ir}}{\sqrt{m_r}} j_{m_r} = (p_r + \alpha s_r) \frac{x_{ir}}{\sqrt{m_r}} j_{m_r},$$

and for $t \in [n] \setminus \{r\}$,

$$J_{m_r \times m_t} \frac{x_{it}}{\sqrt{m_t}} j_{m_t} = \sqrt{m_r m_t} \frac{x_{it}}{\sqrt{m_r}} j_{m_r}.$$

We have that

$$\begin{aligned} A_\alpha(G[H_1, \ldots, H_n])X_i' &= \begin{pmatrix} \frac{1}{\sqrt{m_1}} \big((p_1 + \alpha s_1)x_{i1} + \sum_{t \in [n]\setminus\{1\}} (1-\alpha) a_{1t} \sqrt{m_1 m_t} x_{it}\big) j_{m_1} \\ \frac{1}{\sqrt{m_2}} \big((p_2 + \alpha s_2)x_{i2} + \sum_{t \in [n]\setminus\{2\}} (1-\alpha) a_{2t} \sqrt{m_2 m_t} x_{it}\big) j_{m_2} \\ \vdots \\ \frac{1}{\sqrt{m_n}} \big((p_n + \alpha s_n)x_{in} + \sum_{t \in [n]\setminus\{n\}} (1-\alpha) a_{nt} \sqrt{m_n m_t} x_{it}\big) j_{m_n} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_i \frac{x_{i1}}{\sqrt{m_1}} j_{m_1} \\ \lambda_i \frac{x_{i2}}{\sqrt{m_2}} j_{m_2} \\ \vdots \\ \lambda_i \frac{x_{in}}{\sqrt{m_n}} j_{m_n} \end{pmatrix} = \lambda_i(C) X_i'. \end{aligned}$$

Hence, X_i' is an eigenvector of $A_\alpha(G[H_1, H_2, \ldots, H_n])$ corresponding to $\lambda_i(C)$.

Note that $(Y'_{i_1 j_1})^T Y'_{i_2 j_2} = 0$ for $(i_1, j_1) \neq (i_2, j_2)$ and $(Y'_{i_1 j_1})^T X_i' = 0$ for any $i, i_1 \in [n]$ and $j_1 \in [m_{i_1}] \setminus \{1\}$, i.e., all these eigenvectors are orthogonal, thus we have our conclusion. ∎

6. The spectral radii of $A_\alpha(G \times H)$ and $A_\alpha(G \oplus H)$. In this section, we will characterize the spectra radii of $A_\alpha(G \times H)$ and $A_\alpha(G \oplus H)$ for arbitrary graph G and regular graph H. It is obvious that

$$A(G \times H) = A(G) \otimes A(H)$$

and

$$A(G \oplus H) = A(G \Box H) + A(G \times H) = A(G) \otimes I_m + I_n \otimes A(H) + A(G) \otimes A(H).$$

As

$$d_{G \times H}((v_i, u_j)) = d_G(v_i) \times d_H(u_j)$$

and

$$d_{G \oplus H}((v_i, u_j)) = d_G(v_i) + d_H(u_j) + d_G(v_i) \times d_H(u_j),$$

we can see that

$$D(G \otimes H) = D(G) \otimes D(H)$$

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and

$$D(G \oplus H) = D(G) \otimes I_m + I_n \otimes D(H) + D(G) \otimes D(H).$$

Thus, we have that

$$A_\alpha(G \times H) = \alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes A(H) \tag{6.4}$$

and

$$A_\alpha(G \oplus H) = A_\alpha(G) \otimes I_m + I_n \otimes A_\alpha(H) + \alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes A(H). \tag{6.5}$$

Recall that for p regular graph of order m, j_m is an eigenvector of G corresponding to the spectral radius p.

Theorem 6.1. *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, H be a p-regular graph with $V(H) = \{u_1, u_2, \ldots, u_m\}$, respectively. Let $\lambda_1(A_\alpha(G))$ be the spectral radius of $A_\alpha(G)$. Then,*

$$\lambda_1(A_\alpha(G \times H)) = p\lambda_1(A_\alpha(G)), \lambda_1(A_\alpha(G \oplus H)) = p\lambda_1(A_\alpha(G)) + \lambda_1(A_\alpha(G)) + p.$$

Proof. Let $X_1 = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $A_\alpha(G)$, i.e., $x_i > 0$, $X_1^T X_1 = 1$ and $A_\alpha(G)X_1 = \lambda_1(A_\alpha(G))X_1$. By (6.4), we have that

$$\begin{aligned}
A_\alpha(G \times H)(X_1 \otimes j_m) &= (\alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes A(H))(X_1 \otimes j_m) \\
&= (\alpha D(G)X_1) \otimes (D(H)j_m) + ((1-\alpha)A(G)X_1) \otimes (A(H)j_m) \\
&= (\alpha p D(G)X_1) \otimes (j_m) + (p(1-\alpha)A(G)X_1) \otimes (j_m) \\
&= (pA_\alpha(G)X_1) \otimes j_m \\
&= p\lambda_1(A_\alpha(G)X_1))X_1 \otimes j_m.
\end{aligned}$$

Thus, we have that $p\lambda_1(A_\alpha(G))$ is an eigenvalue of $A_\alpha(G \times H)$. Note that every entries of $X_1 \otimes j_m$ are positive, and hence, by the Perron-Frobenius theorem, we know that

$$\lambda_1(A_\alpha(G \times H)) = p\lambda_1(A_\alpha(G)).$$

Similarly, we can see that $X_1 \otimes j_m$ is also an eigenvector of $A_\alpha(G \oplus H)$ corresponding to $p\lambda_1(A_\alpha(G)) + \lambda_1(A_\alpha(G)) + p$, and thus,

$$\lambda_1(A_\alpha(G \oplus H)) = p\lambda_1(A_\alpha(G)) + \lambda_1(A_\alpha(G)) + p$$

as desired. ∎

Note that the graphs $G \times H \cong H \times G$ and $G \oplus H \cong H \oplus G$, the following corollary is obvious by Theorem 6.1.

Corollary 6.2. *Let H be a connected graph with $V(H) = \{u_1, u_2, \ldots, u_m\}$,$G$ be a p- regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, respectively. Let $\lambda_1(A_\alpha(H))$ be the spectral radius of $A_\alpha(H)$. Then,*

$$\lambda_1(A_\alpha(G \times H)) = p\lambda_1(A_\alpha(H)), \lambda_1(A_\alpha(G \oplus H)) = p\lambda_1(A_\alpha(H)) + \lambda_1(A_\alpha(H)) + p.$$

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