Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 468-472, October 2019.

A NOTE ON VARIANTS OF ZERO FORCING*

LON MITCHELL†

Abstract. A small improvement is made to the zero-forcing variants defined by Butler, Grout, and Hall (2015) for matrices with a given number of negative eigenvalues, resulting in a better value for the Barioli-Fallat tree and one negative eigenvalue.

Key words. Zero forcing, Barioli-Fallat tree.

AMS subject classifications. 05C50.

1. Introduction. Each n-by-n Hermitian matrix $M = (m_{ij})$ has an associated simple graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{ij \mid i < j,\, m_{ij} \neq 0\}$. Conversely, starting with a simple graph G, we can consider the set $S(G)$ of all Hermitian matrices whose graph is G. The maximum nullity among the matrices in $S(G)$ is an invariant of G, denoted $M(G)$.

Zero forcing is a combinatorial game played on the graph G that gives an upper bound, the zero-forcing number $Z(G)$, for $M(G)$ [1]. A modified game gives an upper bound, $Z_+(G)$, for the maximum nullity of positive semidefinite matrices in $S(G)$ [2]. Butler, Grout, and Hall [4] defined variants of these games and related numbers $Z_q(G)$ that interpolate between $Z_+(G) = Z_0(G)$ and $Z(G) = Z_n(G)$. They noted that for most trees T on ten or fewer vertices, Z_q usually gives a tight bound for the maximum nullity of matrices in $S(T)$ having exactly q negative eigenvalues, $M_q(G)$. One notable exception is the Barioli-Fallat tree BF [3], shown in Figure 1, which has $M_1(BF) = 2$ but $Z_1(BF) = 3$.

In this note, we will introduce an improvement to Butler, Grout, and Hall's original definition for Z_q, resulting in new values $\dot{Z}_q(G)$ that are tighter bounds in many cases, including providing a tight bound for M_1 of the Barioli-Fallat tree.

2. Definitions. The *change rule*, given a graph with vertices that are either "filled" or "unfilled", is to fill a vertex v that is the unique unfilled neighbor of a filled vertex u (v is said to be *forced* by u). This rule is derived from the following idea: If we consider a vector in the null space of a matrix $M \in S(G)$ that has zero entries corresponding to the filled vertices and v and u are as above, then matrix multiplication implies the entry corresponding to v must also be zero.

Definition 2.1. (Z_q-Forcing Game) All the vertices of the graph G are initially unfilled. There are two players, known as Player (who has tokens) and Opponent. Player will repeatedly apply one of the following three options until all vertices are filled:

1. For one token, Player can fill any vertex.
2. At no cost, Player can apply the change rule on the entire graph G.
3. Let the vertices currently filled be denoted by B, and let $W_1, \ldots, W_k$ be the vertex sets of the connected components of $G - B$. Player selects at least $q + 1$ of the W_i and announces the selection

*Received by the editors on June 6, 2018. Accepted for publication on August 5, 2019. Handling Editor: Leslie Hogben.
†University of South Florida St. Petersburg, USA (lonmitchell@mail.usf.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 468-472, October 2019.

to Opponent. Opponent then selects a nonempty subset of these components, say $\{W_{i_1}, \dots, W_{i_l}\}$, and announces it to Player. At no cost, Player can apply the change rule on $G[B \cup W_{i_1} \cup \dots \cup W_{i_l}]$.

We can try to understand this game as follows: Consider a matrix $M \in S(G)$ expressed as

$$M = \begin{bmatrix} M_1 & 0 & \cdots & 0 & B_1^T \\ 0 & M_2 & \cdots & 0 & B_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_k & B_k^T \\ B_1 & B_2 & \cdots & B_k & C \end{bmatrix}, \tag{2.1}$$

where each M_i corresponds to the vertices of W_i and C corresponds to the filled vertices B at a given stage in the zero-forcing process. Let X be the subspace of null vectors of M whose support is contained in $G - B$. Let

$$x = \begin{bmatrix} x_1 & x_2 & \cdots & x_k & 0 \end{bmatrix}^*$$

be a column vector in X whose support is the union of all supports occurring in X (where each x_i corresponds to the vertices of W_i). If we are able to conclude that $\sum_{i \in S} B_i x_i = 0$ for some subset S of $\{1, \dots, k\}$, then we can apply the change rule on the subgraph induced by B and the W_i with $i \in S$, since showing that an entry of x must be zero is equivalent to showing that entry is zero in every vector in X.

Assuming a fixed number of negative eigenvalues allows this sort of argument, as follows: An *isotropic* subspace for M is a subspace of vectors y such that $y^* M y = 0$.

THEOREM 2.2. ([4]) *Let A be a Hermitian matrix and R an isotropic subspace for A of dimension more than $\min\{p, q\}$, where p and q are the number (counting multiplicities) of positive and negative eigenvalues, respectively. Then R contains a nonzero vector in the null space of A.*

Suppose then we know that M has exactly q negative eigenvalues and $p \geq q$. Take any $q+1$ of the W_i, say $W_1, \dots, W_{q+1}$. Since $Mx = 0$, for each j we have $M_j x_j = 0$. If $q = 0$, then the row span of B_j is in the row span of M_j for each j (often called the row and column inclusion property of positive semidefinite matrices), meaning $B_j x_j = 0$ for each j. In particular, if M is positive semidefinite, then we can apply the change rule to B and any component. If $q > 0$ and $B_j x_j \neq 0$ for each j with $1 \leq j \leq q+1$, if we extend each x_j to an n-by-1 column vector y_j whose entries corresponding to W_j taken from x_j and which has zeros elsewhere, we find the y_j are linearly independent and span an isotropic subspace for M. By Theorem 2.2, there is some nontrivial linear combination of the y_j that is a null vector: $M \sum_{i=1}^{q+1} b_i y_i = \sum_{i=1}^{q+1} b_i B_i y_i = 0$. Since an entry of y_j corresponding to W_j is zero if and only if the corresponding entry of x_j is zero, we can apply the change rule to the subgraph of G induced by B and those W_j whose corresponding coefficient b_j is nonzero in the above sum.

The design of the game to include Opponent reflects that, in general, q-zero forcing can not be applied to all matrices at once (as it can with $q = 0$ or $q = n$), but must be considered on a matrix-by-matrix basis. Each matrix $M \in S(G)$ with exactly q negative eigenvalues gives a strategy for Opponent: If Player selects $W_1, \dots, W_{q+1}$ and $B_j x_j = 0$ for some j, then Opponent returns W_j; otherwise, Opponent returns those W_i such that the corresponding b_i is nonzero. Butler, Grout, and Hall showed that this strategy forces Player to spend at least as many tokens as the nullity of M [4]. In particular, M cannot have nullity more than $Z_q(G)$, so that $Z_q(G) \geq M_q(G)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 468-472, October 2019.

3. Improvement. In this section, we will combine two key elements of the game to suggest an improvement. We will let $\dot{Z}_q$ denote the resulting parameter. First, note that the change rule can be applied to B and any number of components W_i (as determined by Opponent) and the implementation of the change rule may involve as few as one filled vertex (once we have the null vector, applying the change rule amounts to studying the result of multiplying the null vector by a single row of M). Thus, the only way the q value appears in the game is in deciding how many components W_i we must have in order to convince Opponent to look for a vector in the null space that allows application of the change rule. Second, the strategies used by Opponent to show that $Z_q(G) \geq M_q(G)$ come from actual matrices $M \in S(G)$. Thus, in altering the game to restrict the play of Opponent, as long as we still allow Opponent to act based on any $M \in S(G)$, we can still conclude $\dot{Z}_q(G) \geq M_q(G)$.

Definition 3.1. ($\dot{Z}_q$-Forcing Game) All the vertices of the graph G are initially unfilled. There are two players, known as Player (who has tokens) and Opponent. Before the game begins, for each induced subgraph H of G, Opponent must specify a value $e(H)$ subject to the following conditions:

- $0 \leq e(H) \leq q = e(G)$;
- if $H_1, \ldots, H_k$ are the connected components of $H - B$ for some subgraph H and set of vertices B, then $\sum_{i=1}^{k} e(H_i) \leq e(H)$.

Next, Player will repeatedly apply one of the following three options until all vertices are filled:

1. For one token, Player can fill any vertex.
2. At no cost, Player can apply the change rule on the entire graph G.
3. Let the vertices currently filled be denoted by B, and let $W_1, \ldots, W_k$ be the vertex sets of the connected components of $G - B$. Player selects m of the W_i and a subset B' of B such that, if H is the subgraph induced by B' and the selected W_i, then $m > e(H)$. Player then announces the selection to Opponent. Opponent selects a nonempty subset of the components, say $\{W_{i_1}, \ldots, W_{i_l}\}$, and announces it to Player. At no cost, Player can apply the change rule on $G[B' \cup W_{i_1} \cup \cdots \cup W_{i_l}]$.

To see how this new definition takes advantage of the key elements discussed above, consider a matrix M as in Equation (2.1). First, if M has exactly q negative eigenvalues, using Cauchy eigenvalue interlacing, we know that the M_j combined have at most q negative eigenvalues. Initially, that does not provide any benefit. However, once some vertices of, for example, W_1 are filled, we can see if we can apply the change rule using just vertices of W_1. If M_1 has fewer than q negative eigenvalues, then we need not provide as many connected components of W_1 to Opponent to convince Opponent that a null vector for those components exists. This comes from the fact that if a vector x is an isotropic vector for a matrix A whose support is some subset of vertices S, then x is also an isotropic vector for any principal submatrix of A corresponding to a superset of S. Thus, we may be able to tailor our choices of components and filled vertices based on the e distribution specified by Opponent.

Second, Opponent can still get a strategy for the game from each matrix $M \in S(G)$: Choose $e(H)$ to be the number of negative eigenvalues of the principal submatrix of M corresponding to H; play exactly as before when Player specifies certain components (but now Player is a little craftier in knowing when a null vector will exist). As in the proof by Butler, Grout, and Hall [4], Opponent's play using the matrix strategy can still protect a null vector supported on the unfilled vertices that has largest support, forcing Player to pay a token for each dimension of the null space (and perhaps more).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 468-472, October 2019.

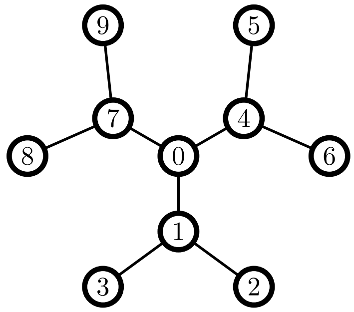

FIGURE 1. *The Barioli-Fallat tree.*

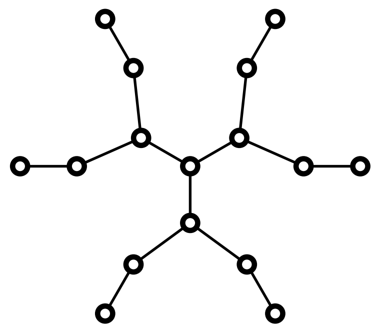

FIGURE 2. *The extended Barioli-Fallat tree.*

EXAMPLE 3.2. Consider the Barioli-Fallat tree BF as in Figure 1. Removing the center vertex leaves three isomorphic subtrees. Regardless of how Opponent assigns e values, two of the three can be considered to have no negative eigenvalues. Without loss of generality, let $e(G[4,5,6]) = 1$. We have $\dot{Z}_1(BF) > 1$ since starting with one filled vertex must leave 5 or 6 (or both) unfilled regardless of how the game is played. On the other hand, if we start with, for example, 3 and 5 filled, we can force 1 and 4 using the global rule. Next, since $e(G[1,2]) = 0$, we can use positive semidefinite zero forcing to fill 2. Then, the global rule can be applied in turn to 0, 6, and 7. Finally, since $e(G[7,8,9]) = 0$, we can again use positive semidefinite zero forcing to fill 8 and 9 in turn. Thus, $\dot{Z}_1(BF) = 2$.

EXAMPLE 3.3. The star $K_{1,m}$ provides an example where there is no difference between $\dot{Z}_q$ and Z_q, since Opponent can define e to equal q for any connected subgraph.

REMARK 3.4. Butler, Grout, and Hall [4] also discussed the extended Barioli-Fallat tree, EBF, as shown in Figure 2. Adding information about the diagonal entries (looped and unlooped vertices) gives an improvement on the traditional zero forcing number, $\hat{Z}(G)$. They defined $\hat{Z}_q(G)$ in a corresponding fashion and found EBF to be problematic for $\hat{Z}_1$ just as BF is for Z_1. The same technique of Example 3.2 works to show $\dot{Z}_1(EBF) = 2$. It would be interesting to know if $\dot{Z}_1$ correctly gives the corresponding maximum nullity for all trees.

REMARK 3.5. Butler, Grout, and Hall [4] showed that, for the tree T in Figure 3, $Z_0(T) = 1$, $Z_1(T) = 2$, and $Z_2(T) = 3$, while Z_2 of the disjoint union $T \cup T$ is 5. For $Z_q(G \cup H)$, they proposed instead considering $Z_s(G)$ and $Z_t(H)$ where $s + t = q$. The new game incorporates this idea, so that, in general,

$$\dot{Z}_q(G \cup H) = \max_{s+t=q} (\dot{Z}_s(G) + \dot{Z}_t(H)).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 468-472, October 2019.

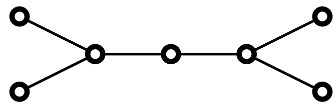

FIGURE 3. *The tree T in Remark 3.5.*

4. Further remarks. Butler, Grout, and Hall [4] contend that "on all but one tree with 10 or fewer vertices, $Z_q(G)$ is a tight bound for the maximum nullity of a matrix associated with the tree that has q negative eigenvalues". Of course, this is not true for large values of q relative to n. For example, for a path P_n on n vertices, $Z_n(P_n) = Z(P_n) = 1$ but the nullity of an n-by-n matrix with exactly n negative eigenvalues is zero. However, their remark does seem to be true if we consider matrices with *at most* q negative eigenvalues, and in fact they point out their main result is still true in this case (using the same proof): The nullity of $A \in S(G)$ when A has at most q negative eigenvalues is at most $Z_q(G)$.

For the purposes of applying the change rule in the new game, principal submatrices that have no negative eigenvalues are helpful. But row inclusion also holds for negative semidefinite matrices, so it should be just as helpful to find submatrices that have no positive eigenvalues. The current definition, which relies just on negative eigenvalues could also be improved to consider the minimum of p and q as in Theorem 2.2. Then the $\dot{Z}_m$-forcing game could be applied where $m = \min\{p, q\}$. As above, high values of m relative to n could be ignored. For example, $m \geq n/2$ would force nullity zero! We could incorporate this, for example, by defining $\bar{Z}_m(G) = \min\{\dot{Z}_m(G), n - 2m\}$ and $\bar{Z}(G) = \max_m \bar{Z}_m(G)$. From the definitions and because $Z_q(G) \leq Z(G)$ for each q, $\bar{Z}(G) \leq Z(G)$. It would be interesting to know if graphs exist where $\bar{Z}(G)$ is strictly less than $Z(G)$.

REFERENCES

[1] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.

[2] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433:401–411, 2010.

[3] F. Barioli and S.M. Fallat. On two conjectures regarding an inverse eigenvalue problem for acyclic symmetric matrices. *Electronic Journal of Linear Algebra*, 11:41–50, 2004.

[4] S. Butler, J. Grout, and H.T. Hall. Using variants of zero forcing to bound the inertia set of a graph. *Electronic Journal of Linear Algebra*, 30:1–18, 2015.