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ON A REFINED OPERATOR VERSION OF YOUNG'S INEQUALITY AND ITS REVERSE*

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Abstract. In this note, some refinements of Young's inequality and its reverse for positive numbers are proved, and using these inequalities, some operator versions and Hilbert-Schmidt norm versions for matrices of these inequalities are obtained.

Key words. Young inequality, Positive operators, Weighted means, Hilbert-Schmidt norm.

AMS subject classifications. 47A63, 47A64, 15A42.

1. Introduction. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In the case when dim $\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices. For $A = (a_{ij}) \in \mathbb{M}_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}},$$

where $s_j(A)$ $(1 \leq j \leq n)$ are the singular values of A. It is known that $\|.\|_2$ is a unitarily invariant norm.

For $A, B \in \mathbb{M}_n$, denote by $A \circ B$ the Schur (Hadamard) product of A and B, that is, the entrywise product.

For positive real numbers a and b, the classical Young inequality says that if $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \leqslant (1-\nu)a + \nu b,$$

with equality if and only if $a = b$. When $\nu = \frac{1}{2}$, the Young inequality reduces to the arithmetic-geometric mean inequality

$$\sqrt{ab} \leqslant \frac{a+b}{2}. \tag{1.1}$$

Throughout, we denote $a^{1-\nu}b^{\nu}$ and $(1-\nu)a + \nu b$, respectively by $a\sharp_\nu b$ and $a\nabla_\nu b$. The Heinz mean is defined as

$$H_\nu(a, b) = \frac{a^{1-\nu}b^{\nu} + a^{\nu}b^{1-\nu}}{2}$$

for $a, b > 0$ and $\nu \in [0, 1]$. It is easy to see that

$$\sqrt{ab} \leqslant H_\nu(a, b) \leqslant \frac{a+b}{2}.$$

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In [9] and [10], F. Kittaneh and Y. Manasrah improved the Young inequality and its reverse as follows:

$$a^{1-\nu}b^{\nu}+r(\sqrt{a}-\sqrt{b})^2\leqslant(1-\nu)a+\nu b\leqslant a^{1-\nu}b^{\nu}+s(\sqrt{a}-\sqrt{b})^2, \tag{1.2}$$

where $r=\min\{\nu,1-\nu\}$ and $s=\max\{\nu,1-\nu\}$.

The authors of [7] and [8] obtained another refinement of the Young inequality as follows:

$$r^2(a-b)^2\leqslant((1-\nu)a+\nu b)^2-(a^{1-\nu}b^{\nu})^2\leqslant s^2(a-b)^2, \tag{1.3}$$

where $r=\min\{\nu,1-\nu\}$ and $s=\max\{\nu,1-\nu\}$.

Recently, J. Zhao and J. Wu [13] obtained the following refinement of inequality (1.2):

$$\begin{aligned}r((ab)^{\frac{1}{4}}-\sqrt{a})^2+\nu(\sqrt{a}-\sqrt{b})^2&\leqslant(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leqslant(1-\nu)(\sqrt{a}-\sqrt{b})^2-r((ab)^{\frac{1}{4}}-\sqrt{b})^2,\end{aligned}$$

where $0\leqslant\nu\leqslant\frac{1}{2}$ and $r=\min\{2\nu,1-2\nu\}$, and

$$\begin{aligned}r((ab)^{\frac{1}{4}}-\sqrt{b})^2+(1-\nu)(\sqrt{a}-\sqrt{b})^2&\leqslant(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\leqslant\nu(\sqrt{a}-\sqrt{b})^2-r((ab)^{\frac{1}{4}}-\sqrt{a})^2,\end{aligned}$$

where $\frac{1}{2}\leqslant\nu\leqslant1$ and $r=\min\{2(1-\nu),1-2(1-\nu)\}$. Also, they obtained the following refinement of inequalities (1.3):

$$\begin{aligned}r(\sqrt{ab}-a)^2+\nu^2(a-b)^2&\leqslant((1-\nu)a+\nu b)^2-(a^{1-\nu}b^{\nu})^2\\&\leqslant(1-\nu)^2(a-b)^2-r(\sqrt{ab}-b)^2,\end{aligned} \tag{1.4}$$

where $0\leqslant\nu\leqslant\frac{1}{2}$ and $r=\min\{2\nu,1-2\nu\}$, and

$$\begin{aligned}r(\sqrt{ab}-b)^2+(1-\nu)^2(a-b)^2&\leqslant((1-\nu)a+\nu b)^2-(a^{1-\nu}b^{\nu})^2\\&\leqslant\nu^2(a-b)^2-r(\sqrt{ab}-a)^2,\end{aligned} \tag{1.5}$$

where $\frac{1}{2}\leqslant\nu\leqslant1$ and $r=\min\{2(1-\nu),1-2(1-\nu)\}$.

Let $A,B\in\mathbb{B}(\mathcal{H})$ be two operators and $\nu\in[0,1]$. The weighted arithmetic mean of A and B, denoted by $A\nabla_{\nu}B$, is defined by:

$$A\nabla_{\nu}B=(1-\nu)A+\nu B.$$

If A and B are positive semidefinite and A is invertible, $\nu-$geometric mean and $\nu-$Heinz mean of A and B are defined respectively, as

$$A\sharp_{\nu}B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\nu}A^{\frac{1}{2}}$$

and

$$H_{\nu}(A,B)=\frac{A\sharp_{\nu}B+A\sharp_{1-\nu}B}{2}.$$

When $\nu=\frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively. It is well known that if A and B are positive invertible operators, then

$$A\nabla_{\nu}B\geqslant A\sharp_{\nu}B,$$

for $0<\nu<1$; see [4, 6] for more information.

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Based on the refined Young inequality (1.4) and its reverse (1.5), J. Zhao and J. Wu [13] proved that if $A, B, X \in \mathbb{M}_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} \nu^2\|AX - XB\|_2^2 + r\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\|_2^2 &\\ &\leqslant \|(1-\nu)AX + \nu XB\|_2^2 - \|A^{1-\nu}XB^{\nu}\|_2^2 \\ &\leqslant (1-\nu)^2\|AX - XB\|_2^2 - r\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\|_2^2, \end{aligned} \tag{1.6}$$

where $0 \leqslant \nu \leqslant \frac{1}{2}$ and $r = \min\{2\nu, 1-2\nu\}$, and

$$\begin{aligned} (1-\nu)^2\|AX - XB\|_2^2 + r\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB\|_2^2 &\\ &\leqslant \|(1-\nu)AX + \nu XB\|_2^2 - \|A^{1-\nu}XB^{\nu}\|_2^2 \\ &\leqslant \nu^2\|AX - XB\|_2^2 - r\|A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX\|_2^2, \end{aligned} \tag{1.7}$$

where $\frac{1}{2} \leqslant \nu \leqslant 1$ and $r = \min\{2(1-\nu), 1-2(1-\nu)\}$. Their results were generalized by Liao and Wu [11], using Kantorovich constant. Similar results can be found in [1, 3].

In addition, in [2], the authors investigated these inequalities, for the cases $\nu \leq 0$ and $\nu \geq 1$. In these cases, they proved the reverse of some of these inequalities. Furthermore, in [12], the numerical version of some of these relations are discussed.

The main aim of this paper is to state a generalization of these inequalities. First we present some generalizations of numerical inequalities. Based on them we prove some refined operator versions of Young's inequality and its reverse. Also some inequalities for the Hilbert-Schmidt norm of matrices are obtained.

In this paper, for $0 < \nu < 1$, the notation $m_k = \lfloor 2^k\nu \rfloor$ is for the largest integer not greater than $2^k\nu$, $r_0 = \min\{\nu, 1-\nu\}$ and $r_k = \min\{2r_{k-1}, 1-2r_{k-1}\}$, for $k \geq 1$. Note that r_k's as functions in ν are piecewise linear in such a way that they vanish at the points $0, \frac{1}{2^k}, \frac{2}{2^k}, \dots, 1$ and take the value $\frac{1}{2}$ at the points $\frac{1}{2^{k+1}}, \frac{3}{2^{k+1}}, \dots, \frac{2^{k+1}-1}{2^{k+1}}$.

2. Numerical results.

We start with some numerical results.

THEOREM 2.1. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla_\nu b \geqslant a\sharp_\nu b + \sum_{k=0}^{\infty} r_k\big[\big(a^{1-\frac{m_k}{2^k}}b^{\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{1-\frac{m_k+1}{2^k}}b^{\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2. \tag{2.8}$$

In addition, if $\nu = \frac{t}{2^n}$ for some $t, n \in \mathbb{N}$, then

$$a\nabla_\nu b = a\sharp_\nu b + \sum_{k=0}^{n-1} r_k\big[\big(a^{1-\frac{m_k}{2^k}}b^{\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{1-\frac{m_k+1}{2^k}}b^{\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2. \tag{2.9}$$

Proof. It is enough to prove that for each $n \in \mathbb{N} \cup \{0\}$,

$$a\nabla_\nu b \geqslant a\sharp_\nu b + \sum_{k=0}^{n} r_k\big[\big(a^{1-\frac{m_k}{2^k}}b^{\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{1-\frac{m_k+1}{2^k}}b^{\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2. \tag{2.10}$$

We prove it by induction. For $n = 0$, we get to the well-known inequality (1.2). Let inequality (2.10) hold for n.

First, let $0 < \nu < \frac{1}{2}$. In this case, we have

$$\begin{aligned} a\nabla_\nu b - r_0(\sqrt{a} - \sqrt{b})^2 &= a\nabla_\nu b - \nu(\sqrt{a} - \sqrt{b})^2 \\ &= 2\nu\sqrt{ab} + (1-2\nu)a \\ &= a\nabla_{2\nu}\sqrt{ab}. \end{aligned}$$

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Applying inequality (2.10) for two positive numbers a and $\sqrt{ab}$ and $2\nu \in (0,1)$, we have

$$\begin{aligned}
a\nabla_\nu b - r_0(\sqrt{a}-\sqrt{b})^2 &= a\nabla_{2\nu}\sqrt{ab}\\
&\geq a\sharp_{2\nu}\sqrt{ab} + \sum_{k=0}^{n} r_{k+1}\big[\big(a^{1-\frac{m_{k+1}}{2^k}}(\sqrt{ab})^{\frac{m_{k+1}}{2^k}}\big)^{\frac{1}{2}}\\
&\quad - \big(a^{1-\frac{m_{k+1}+1}{2^k}}(\sqrt{ab})^{\frac{m_{k+1}+1}{2^k}}\big)^{\frac{1}{2}}\big]^2\\
&= a\sharp_\nu b + \sum_{k=1}^{n+1} r_k\big[\big(a^{1-\frac{m_k}{2^k}}b^{\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{1-\frac{m_k+1}{2^k}}b^{\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2.
\end{aligned}$$

For $\frac{1}{2} < \nu < 1$, we can argue similar as before, however, ν is replaced by $1-\nu$ and a and b are interchanged. Note that $\lfloor 2^k(1-\nu)\rfloor = 2^k - \lfloor 2^k\nu\rfloor - 1$ if $2^k\nu$ is not an integer. If $2^k\nu$ is an integer, then $r_k = 0$, and hence, inequality (2.8) follows, too.

A similar argument, shows equality (2.9) when $\nu = \frac{t}{2^n}$. ∎

Remark 2.2. Note that the series appear in this theorem is a positive series with a finite upper bound. So it is convergent. This fact is also satisfied for all other series appearing in this note.

Interchanging the numbers a and b in inequality (2.8), we can state the following result for the Heinz mean.

Corollary 2.3. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla b \geqslant H_\nu(a,b) + \sum_{k=0}^{\infty} r_k\big[H_{\frac{m_k}{2^k}}(a,b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a,b) + H_{\frac{m_k+1}{2^k}}(a,b)\big].$$

In the following theorem, we state a reverse of Young inequality.

Theorem 2.4. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla_\nu b \leqslant a\sharp_\nu b + (\sqrt{a}-\sqrt{b})^2 - \sum_{k=0}^{\infty} r_k\big[\big(a^{\frac{m_k}{2^k}}b^{1-\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{\frac{m_k+1}{2^k}}b^{1-\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2. \tag{2.11}$$

Proof. By $a\sharp_\nu b + b\sharp_\nu a \geq 2\sqrt{ab}$ and inequality (2.8), we have

$$\begin{aligned}
(\sqrt{a}-\sqrt{b})^2 - a\nabla_\nu b &= b\nabla_\nu a - 2\sqrt{ab}\\
&\geq -a\sharp_\nu b + \sum_{k=0}^{\infty} r_k\big[\big(a^{\frac{m_k}{2^k}}b^{1-\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(a^{\frac{m_k+1}{2^k}}b^{1-\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2.
\end{aligned}$$

So the result follows. ∎

Corollary 2.5. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla b \leqslant H_\nu(a,b) + (\sqrt{a}-\sqrt{b})^2 - \sum_{k=0}^{\infty} r_k\big[H_{\frac{m_k}{2^k}}(a,b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a,b) + H_{\frac{m_k+1}{2^k}}(a,b)\big].$$

Remark 2.6. Replacing a and b by their squares in (2.8) and (2.11), respectively, we obtain

$$a^2\nabla_\nu b^2 \geqslant a^2\sharp_\nu b^2 + \sum_{k=0}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}}b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}}b^{\frac{m_k+1}{2^k}}\big]^2 \tag{2.12}$$

and

$$a^2\nabla_\nu b^2 \leqslant a^2\sharp_\nu b^2 + (a-b)^2 - \sum_{k=0}^{\infty} r_k\big[a^{\frac{m_k}{2^k}}b^{1-\frac{m_k}{2^k}} - a^{\frac{m_k+1}{2^k}}b^{1-\frac{m_k+1}{2^k}}\big]^2. \tag{2.13}$$

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The following two theorems, are useful to prove a version of these inequalities for the Hilbert-Schmidt norm of matrices.

Theorem 2.7. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$(a\nabla_\nu b)^2 \geqslant (a\sharp_\nu b)^2 + r_0^2(a-b)^2 + \sum_{k=1}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}}\big]^2. \tag{2.14}$$

Proof. By (2.12), we have

$$\begin{aligned}(a\nabla_\nu b)^2 - r_0^2(a-b)^2 &= a^2\nabla_\nu b^2 - r_0(a-b)^2 \\ &\geq (a\sharp_\nu b)^2 + \sum_{k=1}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}}\big]^2.\end{aligned}$$

Hence, the proof is complete. ◻

Theorem 2.8. *Let a, b be two positive real numbers and $\nu \in (0,1)$. Then*

$$(a\nabla_\nu b)^2 \leqslant (a\sharp_\nu b)^2 + (1-r_0)^2(a-b)^2 - \sum_{k=1}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}}\big]^2. \tag{2.15}$$

Proof. We have

$$\begin{aligned}&(a\nabla_\nu b)^2 - (1-r_0)^2(a-b)^2 \\ &\quad= a^2\nabla_\nu b^2 - (1-r_0)(a-b)^2 \\ &\quad\leqslant (a\sharp_\nu b)^2 + r_0(a-b)^2 \\ &\qquad - \sum_{k=0}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}}\big]^2 \\ &\qquad\qquad \text{by inequality (2.13)} \\ &\quad= (a\sharp_\nu b)^2 - \sum_{k=1}^{\infty} r_k\big[a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}}\big]^2,\end{aligned}$$

which leads to inequality (2.15). ◻

3. Related operator inequalities. To state the operator version of the inequalities obtained in Section 2, we need the following lemma.

Lemma 3.1. [5] *Let $X \in \mathbb{B}(\mathcal{H})$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in \sigma(X)$ (the spectrum of X). Then $f(X) \geqslant g(X)$.*

Next, we give the first result in this section, which is based on Theorem 2.1 and is a refinement of Theorem 1 in [13].

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive invertible operators and $\nu \in (0,1)$. Then,*

$$A\nabla_\nu B \geqslant A\sharp_\nu B + \sum_{k=0}^{\infty} r_k[A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}} B + A\sharp_{\frac{m_k+1}{2^k}} B]. \tag{3.16}$$

Proof. Choosing $a = 1$, in Theorem 2.1, we have

$$1-\nu+\nu b \geqslant b^\nu + \sum_{k=0}^{\infty} r_k\big[\big(b^{\frac{m_k}{2^k}}\big)^{\frac{1}{2}} - \big(b^{\frac{m_k+1}{2^k}}\big)^{\frac{1}{2}}\big]^2,$$

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for any $b > 0$. If $X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, then $\sigma(X) \subseteq (0, \infty)$. According to Lemma 3.1, we get

$$(1-\nu)I + \nu X \geqslant X^{\nu} + \sum_{k=0}^{\infty} r_k [X^{\frac{m_k}{2^k}} - 2X^{\frac{2m_k+1}{2^{k+1}}} + X^{\frac{m_k+1}{2^k}}].$$

Multiplying both sides by $A^{\frac{1}{2}}$, we obtain

$$A \nabla_{\nu} B \geqslant A \sharp_{\nu} B + \sum_{k=0}^{\infty} r_k [A \sharp_{\frac{m_k}{2^k}} B - 2A \sharp_{\frac{2m_k+1}{2^{k+1}}} B + A \sharp_{\frac{m_k+1}{2^k}} B].$$

This completes the proof. □

In the inequality (3.16), one can replace A and B by A^{-1} and B^{-1}, and take the inverse. One obtaines an operator inequality between the weighted geometric mean and the weighted harmonic mean, which represents a refinement of inequalities (30)–(34) in [13].

The following theorem is an operator version of Theorem 2.4 and is a refinement of Theorem 2 in [13].

Theorem 3.3. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive invertible operators and $\nu \in (0, 1)$.*

$$A \nabla_{\nu} B \leqslant A \sharp_{\nu} B + (A - 2A \sharp B + B) - \sum_{k=0}^{\infty} r_k [A \sharp_{\frac{m_k}{2^k}} B - 2A \sharp_{\frac{2m_k+1}{2^{k+1}}} B + A \sharp_{\frac{m_k+1}{2^k}} B].$$

Proof. By Theorem 2.4, using the same ideas as in the proof of Theorem 3.2, we can get the result. □

Corollary 3.4. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive invertible operators and $\nu \in (0, 1)$. Then*

$$A \nabla B \geqslant H_{\nu}(A, B) + \sum_{k=0}^{\infty} r_k [H_{\frac{m_k}{2^k}}(A, B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A, B) + H_{\frac{m_k+1}{2^k}}(A, B)].$$

and

$$A \nabla B \leqslant H_{\nu}(A, B) + (A - 2A \sharp B + B) - \sum_{k=0}^{\infty} r_k [H_{\frac{m_k}{2^k}}(A, B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A, B) + H_{\frac{m_k+1}{2^k}}(A, B)].$$

4. The Hilbert-Schmidt norm version. In this section, we obtain some inequalities for the Hilbert-Schmidt norm. Applying Theorem 2.7, we get the following theorem that is a refinement of the first inequality in (1.6) and (1.7).

Theorem 4.1. *Let $A, B, X \in \mathbb{M}_n$ such that A and B are two positive semidefinite matrices and $\nu \in (0, 1)$. Then*

$$\|A^{1-\nu} X B^{\nu}\|_2^2 + r_0^2 \|AX - XB\|_2^2 + \sum_{k=1}^{\infty} r_k \|A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}} X B^{\frac{m_k+1}{2^k}}\|_2^2 \leqslant \|(1-\nu)AX + \nu XB\|_2^2.$$

Proof. Since A and B are positive semidefinite $n \times n$ matrices, there exist unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U diag(\lambda_1, \ldots, \lambda_n) U^*$ and $B = V diag(\mu_1, \ldots, \mu_n) V^*$. Let $Y = U^* X V = (y_{ij})$. Then it's straightforward to check that

$$(1-\nu)AX + \nu XB = U[((1-\nu)\lambda_i + \nu \mu_j) \circ Y]V^*,$$

$$AX - XB = U[(\lambda_i - \mu_j) \circ Y]V^*,$$

$$A^{1-\nu} X B^{\nu} = U[(\lambda_i^{1-\nu} \mu_j^{\nu}) \circ Y]V^*$$

and

$$A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}} X B^{\frac{m_k+1}{2^k}} = U[(\lambda_i^{1-\frac{m_k}{2^k}} \mu_j^{\frac{m_k}{2^k}} - \lambda_i^{1-\frac{m_k+1}{2^k}} \mu_j^{\frac{m_k+1}{2^k}}) \circ Y]V^*.$$

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Utilizing the unitarily invariant property of $\|.\|_2$ and Theorem 2.7, we have

$$
\begin{aligned}
&\|(1-\nu)AX+\nu XB\|_2^2\\
&=\|((1-\nu)\lambda_i+\nu\mu_j)\circ Y\|_2^2\\
&=\sum_{i,j=1}^{n}((1-\nu)\lambda_i+\nu\mu_j)^2|y_{ij}|^2\\
&\geqslant\sum_{i,j=1}^{n}\left\{(\lambda_i^{1-\nu}\mu_j^{\nu})^2+r_0^2(\lambda_i-\mu_j)^2+\sum_{k=1}^{\infty}r_k(\lambda_i^{1-\frac{m_k}{2^k}}\mu_j^{\frac{m_k}{2^k}}-\lambda_i^{1-\frac{m_k+1}{2^k}}\mu_j^{\frac{m_k+1}{2^k}})^2\right\}|y_{ij}|^2\\
&=\sum_{i,j=1}^{n}(\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2+\sum_{i,j=1}^{n}r_0^2(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&\quad+\sum_{i,j=1}^{n}\left\{\sum_{k=1}^{\infty}r_k(\lambda_i^{1-\frac{m_k}{2^k}}\mu_j^{\frac{m_k}{2^k}}-\lambda_i^{1-\frac{m_k+1}{2^k}}\mu_j^{\frac{m_k+1}{2^k}})^2|y_{ij}|^2\right\}\\
&=\sum_{i,j=1}^{n}(\lambda_i^{1-\nu}\mu_j^{\nu})^2|y_{ij}|^2+\sum_{i,j=1}^{n}r_0^2(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&\quad+\sum_{k=1}^{\infty}\left\{\sum_{i,j=1}^{n}r_k(\lambda_i^{1-\frac{m_k}{2^k}}\mu_j^{\frac{m_k}{2^k}}-\lambda_i^{1-\frac{m_k+1}{2^k}}\mu_j^{\frac{m_k+1}{2^k}})^2|y_{ij}|^2\right\}\\
&=\|A^{1-\nu}XB^{\nu}\|_2^2+r_0^2\|AX-XB\|_2^2\\
&\quad+\sum_{k=1}^{\infty}r_k\|A^{1-\frac{m_k}{2^k}}XB^{\frac{m_k}{2^k}}-A^{1-\frac{m_k+1}{2^k}}XB^{\frac{m_k+1}{2^k}}\|_2^2.
\end{aligned}
$$

So, the proof is complete. ☐

In the last theorem, a refinement of the second inequality in (1.6) and (1.7) is given.

THEOREM 4.2. *Let $A, B, X \in \mathbb{M}_n$ such that A and B are two positive semidefinite matrices and $\nu \in (0, 1)$. Then*

$$\|(1-\nu)AX+\nu XB\|_2^2\leqslant\|A^{1-\nu}XB^{\nu}\|_2^2+(1-r_0)^2\|AX-XB\|_2^2-\sum_{k=1}^{\infty}r_k\|A^{1-\frac{m_k}{2^k}}XB^{\frac{m_k}{2^k}}-A^{1-\frac{m_k+1}{2^k}}XB^{\frac{m_k+1}{2^k}}\|_2^2.$$

Proof. By Theorem 2.8 and using the same idea as in the proof of Theorem 4.1, we can obtain the desired result. ☐

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