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SPN GRAPHS*

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Abstract. A simple graph G is an SPN graph if every copositive matrix having graph G is the sum of a positive semidefinite and nonnegative matrix. SPN graphs were introduced in [N. Shaked-Monderer. SPN graphs: When copositive = SPN. *Linear Algebra Appl.*, 509:82–113, 2016.], where it was conjectured that the complete subdivision graph of K_4 is an SPN graph. This conjecture is disproved, which in conjunction with results in the Shaked-Monderer paper show that a subdivision of K_4 is a SPN graph if and only if at most one edge is subdivided. It is conjectured that a graph is an SPN graph if and only if it does not have an F_5 minor, where F_5 is the fan on five vertices. To establish that the complete subdivision graph of K_4 is not an SPN graph, rank-1 completions are introduced and graphs that are rank-1 completable are characterized.

Key words. Copositive, SPN, F_5, Fan, Completion, Matrix, Graph.

AMS subject classifications. 15B48, 05C50, 15A83.

1. Introduction.

A real symmetric matrix A is *copositive* if $\mathbf{x}^T A\mathbf{x} \geq 0$ for every nonnegative vector $\mathbf{x}$. A matrix A is *SPN* if it is a sum of a positive semidefinite matrix and a symmetric nonnegative one. Every SPN matrix is copositive but not conversely. Determining whether a real symmetric matrix A is copositive is hard (in complexity terms this problem is co-NP complete [5]), while determining whether A is SPN is easier (it can be done by solving a semidefinite program[1]). Patterns of nonzero entries can play a role. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, the *graph of* A is $G(A) = ([n], E)$ where $ij \in E$ if and only if $i \neq j$ and a_{ij} is non-zero ($[n]$ denotes the set $\{1, \ldots, n\}$). It was suggested by Shaked-Monderer [6] to characterize all graphs G with the property that a symmetric matrix A with $G(A) = G$ is copositive if and only if it is SPN. Such a graph is called an *SPN graph*; a graph that does not have this property is called *non-SPN*. In [6], the study of SPN graphs was initiated and numerous results that specific graphs were or were not SPN were established (see also the corrigendum [7]). Some conjectures were made regarding the full characterization of SPN graphs. The following theorem is a main result of our paper, and it shows that one of these conjectures, [6, Conjecture 9.13], is false. (For definitions of graph theory terms and notations used in the paper, see Section 2.)

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[1] More explicitly, consider the semidefinite program

$$\begin{array}{ll} \text{minimize} & \langle 0, X\rangle \\ \text{subject to} & A = X + Y \\ & X \in \mathcal{PSD}_n \\ & Y \geq 0 \\ & X, Y \in \mathcal{S}_n \end{array}$$

where $\mathcal{S}_n$ is the space of $n \times n$ symmetric matrices, $\mathcal{PSD}_n$ is the cone of $n \times n$ positive semidefinite matrices, and $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathcal{S}_n$. If A is SPN, the optimal value of the problem is 0. If A is not SPN, the feasible set defined by the constraints is empty and the optimal value is $+\infty$.

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Theorem 1.1. *The complete subdivision $\overset{\circ}{\overleftrightarrow{K_4}}$ is a non-SPN graph.*

By [6, Theorem 9.6], every graph obtained by replacing a single edge in K_4 by a path is SPN, and by [6, Theorem 9.8], every graph obtained by subdividing between two to five edges of K_4, each at least once, is not SPN. Together with Theorem 1.1, this implies the following corollary.

Corollary 1.2. *A subdivision of K_4 is an SPN graph if and only if at most one of the original edges was subdivided.*

We make the following conjecture regarding the characterization of SPN graphs.

Conjecture 1.3. *The following are equivalent for a graph G:*

(a) G is SPN.
(b) G does not have the fan graph F_5 shown in Figure 1.1 as a minor.
(c) Every block of G is one of
- *an edge and its endpoints,*
- *a graph obtained from the complete graph on four vertices by subdividing a single edge any number of times (including not subdividing it), or*
- *any subdivision (including none) of a graph that consists of triangles sharing a common edge.*

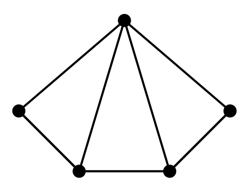

Figure 1.1. *The fan F_5.*

In Section 5, we show the equivalence and (b) and (c) in the conjecture and the implication (a)⇒(c). We also discuss there what remains to be done in order to establish the above conjectured characterization. Theorem 1.1 and its corollary are established in Section 4, using rank-1 completions. The latter are introduced in Section 3, where we characterize graphs G that have the property that every partial symmetric positive rank-1 matrix with specified entries corresponding to the edges of G can be completed to a symmetric positive rank-1 matrix.

2. Preliminaries. We assume the reader is familiar with basic graph theory notions, and concentrate on the terminology and notations used in this paper (see, for example, [2]). A (simple) *graph* G is a pair (V, E), where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges, i.e., two element subsets of V. The edge $\{i, j\}$ is often denoted by ij. A *complete graph* on n vertices has an edge between every pair of distinct vertices and is denoted by K_n. A graph $H = (V(H), E(H))$ is a *subgraph* of $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph G and $e = uv \in E(G)$, subdividing edge e creates a new graph G_e from G by adding a new vertex w adjacent to u and v and deleting e, so $G_e = (V(G) \cup \{w\}, (E(G) \setminus \{e\}) \cup \{uw, vw\})$. Any graph obtained by successively subdividing edges of G is a *subdivision* of G. The graph obtained from G by subdividing each edge once is denoted by $\overset{\circ}{\overleftrightarrow{G}}$. The *contraction* of an edge $e = uv$ of G is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of the following operations (in any order): Delete an isolated

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vertex; delete an edge; contract an edge. A vertex v of a graph G is a *cut vertex* of G if deleting v disconnects G. A *block* of G is a subgraph that has no cut vertex and is maximal with respect to this property. A graph is 2-connected if it has at least three vertices and does not have a cut vertex.

A signed graph is a graph of the form $\mathcal{G} = (V, E, \Sigma)$ where V is the set of vertices, E is the set of edges, and $\Sigma : E \to \{+, -\}$ signs the edges. The *signed graph of a symmetric matrix* A, denoted by $\mathcal{G}(A)$, is obtained from $G(A)$ by assigning the sign of a_{ij} to the edge ij. A signed subgraph $\mathcal{H}$ of a signed graph $\mathcal{G}$ is a subgraph $\mathcal{H}$ of $\mathcal{G}$ with $\Sigma(\mathcal{H})$ being the restriction of $\Sigma(\mathcal{G})$ to $E(\mathcal{H})$. In signed graph drawings, a dashed line denotes a negative edge and a solid line a positive edge. For a graph $G = (V, E)$ (or a signed graph $\mathcal{G} = (V, E, \Sigma)$) and $S \subseteq V$, $G[S] = (S, E[S])$ (or $\mathcal{G}[S] = (S, E[S], \Sigma[S])$) is the *induced* subgraph of G (or $\mathcal{G}$) on the vertices S, where $E[S] = \{uv \in E : u, v \in S\}$ (and $\Sigma[S]$ is the restriction of Σ to $E[S]$). For a signed graph $\mathcal{G}$ and such S, $G_-[S]$ denotes the graph on vertices S whose edges are the negative edges of $\mathcal{G}[S]$.

The space of $n \times n$ real symmetric matrices is denoted by $\mathcal{S}_n$. The set of all copositive matrices in $\mathcal{S}_n$ is denoted by $\mathcal{COP}_n$. This is a closed convex cone in $\mathcal{S}_n$, and it contains the closed convex cone $\mathcal{SPN}_n$ consisting of the SPN matrices of order n. The inclusion $\mathcal{SPN}_n \subseteq \mathcal{COP}_n$ is an equality if and only if $n \leq 4$. The cone $\mathcal{COP}_5$, as the first to differ from its subcone $\mathcal{SPN}_5$, has been studied in [1, 3]. As in [1], we define

$$S(\boldsymbol{\theta}) = \begin{bmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_4+\theta_5) & -\cos(\theta_5) \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\theta_1+\theta_5) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ \cos(\theta_4+\theta_5) & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ -\cos(\theta_5) & \cos(\theta_1+\theta_5) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{bmatrix}$$

for $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. For $\boldsymbol{\theta} > \mathbf{0}$ such that $\sum_{i=1}^5 \theta_i < \pi$, the matrices $S(\boldsymbol{\theta})$ are (extremal) *Hildebrand matrices* [3], $S(\mathbf{0})$ is the (extremal) *Horn matrix*, and for other $\boldsymbol{\theta} \geq \mathbf{0}$ with $\sum_{i=1}^5 \theta_i < \pi$, the matrix $S(\boldsymbol{\theta})$ is a copositive matrix that is not SPN [1].

The submatrix of the $n \times n$ matrix A in rows indexed by $R \subseteq [n]$ and columns indexed by $C \subseteq [n]$ is denoted by $A[R|C]$. When $R = C$, we write $A[R]$ for $A[R|R]$, and $A[R]$ is called a *principal submatrix* of A. We sometimes omit the set brackets of R and C in these notations, writing $A[2]$ instead of $A[\{2\}]$, for example. If $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and A is nonsingular, the *Schur complement* of A in M is $M/A = C - B^T A^{-1} B$; a permutation similarity is applied to define the Schur complement $A/A[S]$ for $S \subset [n]$.

We will use the following results from [6].

Lemma 2.1. [6, Lemma 3.3(b)] *Let $A = \begin{bmatrix} c & \mathbf{a}^T \\ \mathbf{a} & B \end{bmatrix}$ with $c > 0$ and $\mathbf{a} \leq 0$. Then $A \in \mathcal{SPN}_n$ (respectively, $A \in \mathcal{COP}_n$) if and only if $A/A[1] \in \mathcal{SPN}_{n-1}$ (respectively, $A/A[1] \in \mathcal{COP}_{n-1}$).*

Lemma 2.2. [6, Lemma 8.1] *Let $\mathcal{G}$ is a signed graph, and let $\widehat{\mathcal{G}}$ be a signed graph obtained by subdividing a negative edge of $\mathcal{G}$, replacing it by a path with two negative edges. Then $\mathcal{G}$ is SPN if and only if $\widehat{\mathcal{G}}$ is SPN.*

3. Rank-1 completion. In Section 4, we will use rank-1 completions (specifically, Theorem 3.6) in the proof that the complete subdivision of K_4 is not SPN. In this section, we define and characterize rank-1 completable graphs, in addition to proving Theorem 3.6.

A *partial matrix* is a matrix in which some entries specified and others are not (no entries specified or all entries specified are permitted). We use $A^?$ to denote a partial matrix, and a_{ij} to denote a specified entry

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of $A^?$; ? is used to denote an unspecified entry. A partial matrix is *symmetric* if whenever a_{ij} is specified, so is a_{ji} and $a_{ji} = a_{ij}$. For a graph G, a *partial matrix described by* G is a symmetric partial matrix that has the i, j-entry specified if and only if ij is an edge of G. Note that this is a different association of a graph and (partial) matrices than that used throughout the rest of the paper, where matrices are complete and a nonedge of G is associated with a *zero* off-diagonal entry of the matrix.

A symmetric partial matrix is a *partial rank*-1 *matrix* if every specified entry is positive and all diagonal entries are unspecified. An $n \times n$ matrix $B = [b_{ij}]$ is a *rank*-1 *completion* of a partial rank-1 matrix $A^?$ if $\operatorname{rank} B = 1$, B is symmetric, every entry of B is positive, and $b_{ij} = a_{ij}$ for every specified entry a_{ij} of $A^?$. A graph G is *rank*-1 *completable* if every partial rank-1 matrix described by G can be completed to a symmetric positive rank-1 matrix. Note that we omit "symmetric and positive" from the terminology for brevity, but keep in mind that any rank-1 completion is, by definition, a symmetric positive matrix. Recall that a real symmetric matrix A is *completely positive* (*CP*) if $A = BB^T$, where $B \geq 0$. The rank-1 CP matrices are exactly the rank-1 matrices that are symmetric and nonnegative, i.e., the matrices of the form $\mathbf{b}\mathbf{b}^T$ where $\mathbf{0} \neq \mathbf{b} \geq \mathbf{0}$. Furthermore, $B = \mathbf{b}\mathbf{b}^T$ is positive if and only if $\mathbf{b}$ is positive. The cones of copositive matrices and CP matrices are dual to each other, so it is not surprising that CP matrices are involved in the study of SPN graphs.

Remark 3.1. A single vertex is rank-1 completable by choosing any positive value. A disjoint union of rank-1 completable graphs is also rank-1 completable: If $\mathbf{a}\mathbf{a}^T$, $\mathbf{a}$ positive, is a rank-1 completion of $A^?$ and $\mathbf{b}\mathbf{b}^T$, $\mathbf{b}$ positive, is a rank-1 completion of $B^?$, then $\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^T$ is a rank-1 completion of $A^? \oplus B^?$. It is easy to see that the converse also holds: If $A^? \oplus B^?$ has a rank-1 completion, then both $A^?$ and $B^?$ have rank-1 completions.

Lemma 3.2. *Let $A^?$ be a partial rank-1 matrix. Then $A^?$ has a rank-1 completion with diagonal entries $d_1, \dots, d_n$ if and only if $d_1, \dots, d_n$ are positive and $d_i d_j = a_{ij}^2$ for all $i < j$ such that a_{ij} is specified.*

Proof. Suppose $B = [b_{ij}]$ is a rank-1 completion of $A^?$ with diagonal $d_1, \dots, d_n$. By definition, these diagonal elements are positive. Every principal 2×2 submatrix of B has rank at most one (in fact exactly one, since it is positive), so $d_i d_j - b_{ij}^2 = 0$ for every $i < j$, in particular for the specified entries.

Now suppose $d_i > 0$ for $i = 1, \dots, n$ and $d_i d_j = a_{ij}^2$ for all $i < j$ such that a_{ij} is specified. Define $B = [b_{ij}]$ by $b_{ij} = \sqrt{d_i d_j}$. Then $B = \mathbf{b}\mathbf{b}^T$ with $\mathbf{b} = [\sqrt{d_1}, \dots, \sqrt{d_n}]^T$ is a rank-1 completion of $A^?$ with diagonal $d_1, \dots, d_n$. □

Let G be a graph with n vertices and m edges, where the edges are labeled as $e_1, \dots, e_m$. The vertex-edge incidence matrix of G is the $n \times m$ matrix $N = [n_{ij}]$, where $n_{ij} = 1$ if vertex i is an endpoint of edge e_j and is zero otherwise.

Lemma 3.3. *Let G be a graph with n vertices and m edges, and let $A^?$ be a partial rank-1 matrix described by G. Let N be the vertex-edge incidence matrix of G with edges $e_1, \dots, e_m$ where $e_k = i_k j_k$, and let $\mathbf{a} = [\log a_{i_1 j_1}, \dots, \log a_{i_m j_m}]^T \in \mathbb{R}^m$. Then $A^?$ has a rank-1 completion with diagonal entries $d_1, \dots, d_n$ if and only if $\mathbf{x}^T = [\log d_1, \dots, \log d_n]^T$ is a solution to $\mathbf{x}^T N = 2\mathbf{a}^T$.*

Proof. The system of equations

$$d_{i_\ell} d_{j_\ell} = a_{i_\ell j_\ell}^2, \quad \ell = 1, \dots, m$$

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is equivalent to the system of equations

$$\log d_{i_\ell} + \log d_{j_\ell} = 2 \log a_{i_\ell j_\ell}, \quad \ell = 1, \dots, m,$$

which is equivalent to $\mathbf{x}^T N = 2\mathbf{a}^T$ for $\mathbf{x} = [\log d_1, \dots, \log d_n]^T$. ◻

A connected graph is a tree if and only if it has exactly $n-1$ edges. A connected graph is unicyclic if and only if it has exactly n edges (a *unicyclic* graph is a graph with exactly one cycle). A cycle is *odd* or *even* according as the number of vertices in the cycle is odd or even.

THEOREM 3.4. *A connected graph G is rank-1 completable if and only if G is a tree or G is unicyclic with an odd cycle.*

Proof. Let n be the number of vertices of G and let m be the number of edges of G. By Lemma 3.3, G is rank-1 completable if and only if $\mathbf{x}^T N = 2\mathbf{a}^T$ is consistent for every $\mathbf{a} \in \mathbb{R}^m$, which is true if and only if $\operatorname{rank} N = m$. It is well known that $\operatorname{rank} N = n - c$ where c is the number of of connected components that are bipartite (see, for example, [4, Facts 39.4.3 and 39.4.6]). Hence, $\mathbf{x}^T N = 2\mathbf{a}^T$ is consistent for every $\mathbf{a} \in \mathbb{R}^m$ if and only if $m = n - c$. Since G is connected, $m = n - 1$ if G is bipartite and $m = n$ otherwise. In the first case, G is a tree (which is indeed bipartite), and in the latter case G is unicyclic, and as it is non-bipartite, its cycle is odd. ◻

In combination with Remark 3.1, this theorem implies the following.

COROLLARY 3.5. *A graph G is rank-1 completable if and only if each component of G is a tree or unicyclic with an odd cycle.*

The next theorem will be used in Section 4. It shows a connection between the rank-1 completable graphs discussed in this section and the study of SPN (signed) graphs. Theorem 3.6 generalizes [6, Lemma 8.4]. See Figure 3.1 for illustrations of the graph transformation described in this theorem, generalizing the Λ-paw transformation described in [6]. In this figure, $\mathcal{H}$ is a non-SPN signed F_5 and $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ are non-SPN signed graphs that may be obtained by the generalized graph transformation in Theorem 3.6.

THEOREM 3.6. *Let $\mathcal{H} = (V, E, \Sigma)$ be a non-SPN signed graph. Let $S \subseteq V$. Let $\mathcal{G}$ be a signed graph obtained by adding a vertex v adjacent to each of the vertices in S by a negative edge, (possibly) deleting some of the negative edges of $\mathcal{H}[S]$, (possibly) changing some negative edges between vertices in S to positive edges, and adding a positive edge between each pair of vertices in S that are not adjacent in $\mathcal{H}$. If $H_-[S]$ is rank-1 completable, then $\mathcal{G}$ is non-SPN.*

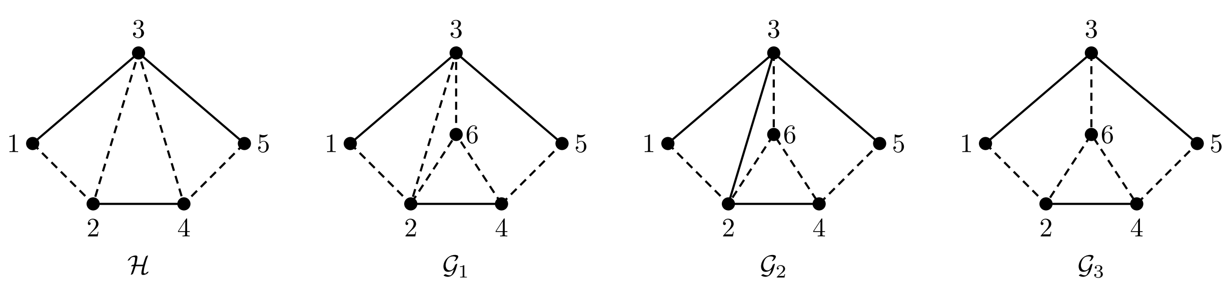

FIGURE 3.1. *Examples for Theorem 3.6 with $S = \{2, 3, 4\}$.*

Proof. Observe that $G_-[S] \subseteq H_-[S]$ and S is a clique in $H \cup G$ (where H and G are the underlying graphs of $\mathcal{H}$ and $\mathcal{G}$). Without loss of generality, assume $S = \{1, \dots, k\}$, and $V(\mathcal{H}) = [n]$. Let A be a

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copositive non-SPN matrix with $\mathcal{G}(A) = \mathcal{H}$, and let $B^?$ be a partial rank-1 CP $k \times k$ matrix defined as follows on $H_-[S]$: For an edge ij of $H_-[S]$ that is not an edge of $\mathcal{G}$, set $b_{ij} = -a_{ij}$. For an edge ij of $H_-[S]$ that is a negative edge of $\mathcal{G}$, choose b_{ij} such that $0 < b_{ij} < -a_{ij}$. For an edge ij of $H_-[S]$ that is a positive edge of $\mathcal{G}$, choose $b_{ij} > -a_{ij}$. By assumption, $B^?$ has a rank-1 completion $\mathbf{b}\mathbf{b}^T$, where $\mathbf{b} \in \mathbb{R}^k_+$ is a positive vector. Let $\mathbf{a} \in \mathbb{R}^n$ be the non-positive vector obtained by appending $n-k$ zeros to $-\mathbf{b}$. Consider the matrix

$$C = \begin{bmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{bmatrix}.$$

Then, $C/C[n+1] = A$ is a non-SPN copositive matrix. Therefore, C is a non-SPN copositive matrix by Lemma 2.1, and $\mathcal{G}(C) = \mathcal{G}$. ◻

4. Subdivisions of K_4. In this section, we show that the complete subdivision of K_4 is a non-SPN graph, thus proving Theorem 1.1. We begin with some tools.

LEMMA 4.1. *For $0 < \alpha, \beta, \gamma < \pi/2$, the partial positive semidefinite matrix*

$$A^? = \begin{bmatrix} 1 & -\cos(\alpha) & \cos(\alpha+\beta) & ? \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta+\gamma) \\ \cos(\alpha+\beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ ? & \cos(\beta+\gamma) & -\cos(\gamma) & 1 \end{bmatrix}$$

has a unique positive semidefinite completion obtained by setting $? = -\cos(\alpha+\beta+\gamma)$.

Proof. Since

$$\begin{bmatrix} 1 & -\cos(\alpha) & \cos(\alpha+\beta) & -\cos(\alpha+\beta+\gamma) \\ -\cos(\alpha) & 1 & -\cos(\beta) & \cos(\beta+\gamma) \\ \cos(\alpha+\beta) & -\cos(\beta) & 1 & -\cos(\gamma) \\ -\cos(\alpha+\beta+\gamma) & \cos(\beta+\gamma) & -\cos(\gamma) & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha+\beta) \\ -\cos(\alpha+\beta+\gamma) \end{bmatrix} \begin{bmatrix} 1 \\ -\cos(\alpha) \\ \cos(\alpha+\beta) \\ -\cos(\alpha+\beta+\gamma) \end{bmatrix}^T + \begin{bmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha+\beta) \\ -\sin(\alpha+\beta+\gamma) \end{bmatrix} \begin{bmatrix} 0 \\ -\sin(\alpha) \\ \sin(\alpha+\beta) \\ -\sin(\alpha+\beta+\gamma) \end{bmatrix}^T,$$

$A^?$ has a (rank-2) positive semidefinite completion. This is the unique positive semidefinite completion of $A^?$ because $A[\{1,2,3\}|\{4\}]$ must be orthogonal to $\ker A[1,2,3] = \operatorname{span}((\sin\beta, \sin(\alpha+\beta), \sin\alpha)^T)$ by the Column Inclusion Property of positive semidefinite matrices. ◻

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PROPOSITION 4.2. *The signed graph $\mathcal{H}_5$ shown in Figure 4.1 is non-SPN. Specifically, with $\theta_i > 0$ for $i = 1, 2, 3, 4$ and $\sum_{i=1}^{4} \theta_i < \frac{\pi}{2}$, the matrix*

$$A = \begin{bmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & 0 & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\pi/2+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ 0 & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2+\theta_1) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{bmatrix}, \tag{1}$$

is a non-SPN copositive matrix such that $\mathcal{G}(A) = \mathcal{H}_5$.

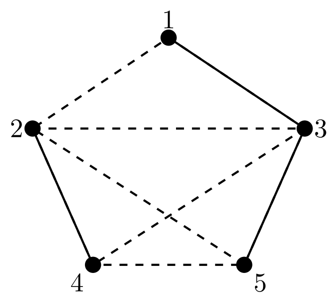

FIGURE 4.1. *The non-SPN signed graph $\mathcal{H}_5$.*

Proof. Observe that $A \geq S(\boldsymbol{\theta})$, where $\boldsymbol{\theta}^T = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, with $\theta_5 = \frac{\pi}{2}$ (A differs from $S(\boldsymbol{\theta})$ only in the $1, 4$ and $4, 1$ entries). Since the Hildebrand matrix $S(\boldsymbol{\theta})$ is copositive, A is copositive.

Suppose that the matrix A is SPN. Then, $A \geq P$ for some positive semidefinite matrix P. Each of the submatrices $A[i, i \dotplus 1, i \dotplus 2]$ is positive semidefinite of rank 2 for $i = 1, 2, 3, 5$. The vector $\mathbf{v}^{(i)} = (\sin\theta_{i\dotplus 1}, \sin(\theta_i + \theta_{i\dotplus 1}), \sin\theta_i)^T$ spans the nullspace of $A[i, i \dotplus 1, i \dotplus 2]$ (with $\theta_5 = \frac{\pi}{2}$). Since

$$0 = {\mathbf{v}^{(i)}}^T A[i, i \dotplus 1, i \dotplus 2]\mathbf{v}^{(i)} \geq {\mathbf{v}^{(i)}}^T P[i, i \dotplus 1, i \dotplus 2]\mathbf{v}^{(i)} \geq 0,$$

and $\mathbf{v}^{(i)}$ is a positive vector, we get that $P[i, i \dotplus 1, i \dotplus 2] = A[i, i \dotplus 1, i \dotplus 2]$, $i = 1, 2, 3, 5$. The only entries of P not contained in at least one of $P[i, i \dotplus 1, i \dotplus 2]$, $i = 1, 2, 3, 5$, are the 1,4 and 4,1 entries. Thus,

$$P = \begin{bmatrix} 1 & -\cos(\theta_1) & \cos(\theta_1+\theta_2) & p & 0 \\ -\cos(\theta_1) & 1 & -\cos(\theta_2) & \cos(\theta_2+\theta_3) & \cos(\pi/2+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos(\theta_2) & 1 & -\cos(\theta_3) & \cos(\theta_3+\theta_4) \\ p & \cos(\theta_2+\theta_3) & -\cos(\theta_3) & 1 & -\cos(\theta_4) \\ 0 & \cos(\pi/2+\theta_1) & \cos(\theta_3+\theta_4) & -\cos(\theta_4) & 1 \end{bmatrix}$$

for some $-1 \leq p \leq 1$. Since P is positive semidefinite, so is $P[2, 3, 4, 5]$. Therefore, $\cos(\pi/2 + \theta_1) = -\cos(\theta_2 + \theta_3 + \theta_4)$ by Lemma 4.1, contradicting $\sum_{i=1}^{4} \theta_i < \pi/2$ and $\theta_i > 0$ for $i = 1, 2, 3, 4$. Thus, no such positive semidefinite P exists and A is not SPN. ∎

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We now prove Theorem 1.1.

Proof of Theorem 1.1. Let $\mathcal{G}_i, i = 1, \ldots, 4$ be the graphs in Figure 4.2. Observe that $\mathcal{G}_4$ is a signing of $\overline{\overset{\circ}{K_4}}$. The graph $\mathcal{G}_1$ is $\mathcal{H}_5$, which is non-SPN by Proposition 4.2. We next show that $\mathcal{G}_2$ is non-SPN. Let A be a non-SPN matrix with graph $\mathcal{G}_1$, defined as in (1), where $\theta_i > 0$ for every $1 \leq i \leq 4$, $\sum_{i=1}^{4} \theta_i < \frac{\pi}{2}$, and

$$\sin(\theta_1) < \frac{\cos(\theta_2)\cos(\theta_4)}{\cos(\theta_3)}.$$

Let

$$\mathbf{a}^T = (0, -\frac{\sin(\theta_1)}{\cos(\theta_4)}, -\cos(\theta_3), -1, -\cos(\theta_4)).$$

Then

$$B = \begin{bmatrix} A + \mathbf{a}\mathbf{a}^T & \mathbf{a} \\ \mathbf{a}^T & 1 \end{bmatrix}$$

is a copositive matrix (as a sum of the copositive $A \oplus 0$ and a rank-1 positive semidefinite matrix). Since the off-diagonal entries in the 6th column of B are nonpositive, $A = B/B[6]$ is also copositive by Lemma 2.1. Since A is non-SPN, B is non-SPN. As $\mathcal{G}(B) = \mathcal{G}_2$, $\mathcal{G}_2$ is non-SPN.

Since $\mathcal{G}_2$ is non-SPN, Theorem 3.6 (applied to $\mathcal{H} = \mathcal{G}_2$ and $S = \{2, 3, 6\}$) implies that the signed graph $\mathcal{G}_3$ is also non-SPN. The signed graph $\mathcal{G}_4$ is obtained from the non-SPN $\mathcal{G}_3$ by subdividing negative edges, and therefore is also non-SPN by Lemma 2.2. □

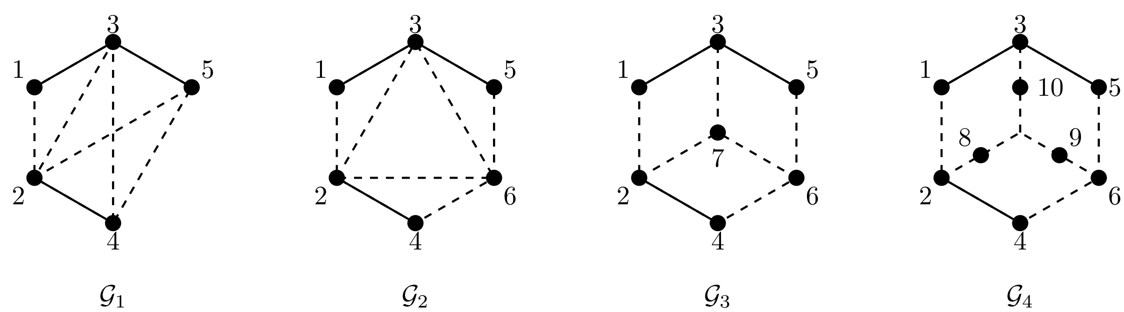

FIGURE 4.2. *Signed graphs used in the proof of Theorem 1.1.*

REMARK 4.3. By further replacing negative edges in $\mathcal{G}_4$ by longer paths of negative edges, it is easy to see that any subdivision of K_4 with all edges subdivided, each at least once, is non-SPN. As mentioned in the introduction, this, together with results from [6] proves Corollary 1.2.

5. Discussion and conjectures. In this section, we establish the equivalence of statements (b) and (c) in Conjecture 1.3, summarize what is known in support of this conjecture and what is still needed to establish it. We also state some weaker conjectures. Following the notation in [6], T_k is a graph on k vertices that consists of $k-2$ triangles sharing a common edge, DR_k denotes a K_4 with one edge subdivided $k-4$ times, and CD_6 is the graph shown in Figure 5.1.

Observe that F_5 is a minor of CD_6: It is obtained by contracting the edge uv in Figure 5.1. It is also a minor of any graph obtained from K_4 by subdividing at least two edges. The edge to be contracted depends on whether the two subdivided edges are incident or not, as shown in Figure 5.2. In each case contracting

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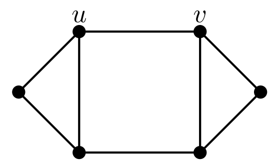

Figure 5.1. CD_6.

Figure 5.2. *Two-edge subdivisions of* K_4.

the edge uv yields an F_5. We use these observations to prove the equivalence of (b) and (c) in Conjecture 1.3 in the next theorem.

Theorem 5.1. *A graph G does not have the fan graph F_5 as a minor if and only if every block of G is one of*

- *a K_2,*
- *a DR_k, or*
- *any subdivision (including none) of a T_k.*

Proof. As F_5 is 2-connected, it is a minor of a graph G if and only if it is a minor of a (2-connected) block of G.

Suppose every block of G is one of a K_2, a DR_k, or a subdivision of a T_k. Clearly F_5 is not a minor of K_2. Deleting an edge or a vertex from a graph cannot raise the degree of any vertex in the graph. Contracting an edge incident with a vertex of degree two cannot produce a vertex of degree greater than the maximum of the degrees of its endpoints. If the base edge in a T_k is contracted, the resulting minor is not 2-connected. Hence, any 2-connected minor of a subdivision of T_k has at most two vertices of degree three or more, and in particular is not F_5, which has three such vertices. Contracting an edge in a DR_k yields one of a DR_{k-1}, a cycle, or a subdivision of T_4. Therefore, in a 2-connected minor of DR_k, the maximum degree of a vertex is at most three, and the minor cannot be F_5.

Now suppose F_5 is not a minor of G. Each block of G is either K_2 or is 2-connected. By the observations preceding this theorem, a 2-connected block of G cannot contain a subdivision of F_5, a subdivision of CD_6, or a subdivision of K_4, where at least two edges have been subdivided. It is shown in [6, Remark 9.11] that if a 2-connected graph contains no subdivision of F_5, no subdivision of CD_6, and no subdivision of K_4 where at least two edges have been subdivided, then this graph is either a subdivision of T_k, $k \geq 3$, or a DR_k, $k \geq 4$. ∎

The next result follows from [6, Theorem 9.10] and Corollary 1.2.

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Theorem 5.2. *Let G be an SPN graph. Then G does not contain the following subgraphs:*

(1) a subdivision of the fan F_5,
(2) a subdivision of CD_6,
(3) a subdivision of K_4 in which at least two edges were subdivided at least once each.

As in the proof of Theorem 5.1, [6, Remark 9.11] combined with Theorem 5.2 shows that the implication (a)⇒(c) of Conjecture 5.4 holds. It remains to prove that (c) (or (b)) implies that the graph is SPN. It is shown in [6] that DR_k and K_2 are SPN graphs, and that every block of G is SPN then so is G. Thus, if every subdivision of T_k is SPN, Conjecture 1.3 is true and SPN-graphs are determined. It is known that T_k is SPN for $k = 3, 4, 5$, and subdivisions of T_k are SPN for $k = 3, 4$ (see [6, 7]). Conjecture 1.3 is established for graphs on five vertices in [6, Theorem 7.3] and [7].

Finally, we list some weaker conjectures, in increasing order of strength, leading up to Conjecture 1.3, which is the strongest.

Conjecture 5.3. *Any subdivision of a non-SPN graph is non-SPN.*

Conjecture 5.3 has been established in the case that the subdivided edge corresponds to a negative entry in a realizing non-SPN matrix in [6, Lemma 8.1] (see Lemma 2.2). So it remains to consider the case that the corresponding entry is positive.

If ζ is a graph property, and G has property ζ, then we say G is a ζ graph (e.g., SPN graph). A graph property ζ is *minor-closed* if G is a ζ graph implies that H is a ζ graph for every minor H of G. If ζ is minor-closed, then ζ graphs are characterized by a finite set of forbidden minors [2, Section 12.7].

Conjecture 5.4. *Being an SPN graph is minor-closed, i.e., if H is a minor of an SPN graph G, then H is SPN.*

Since subgraphs of SPN graphs are SPN [6, Lemma 4.2], proving Conjecture 5.4 amounts to showing that the class of SPN graphs is closed under edge contractions. If Conjecture 5.4 were established, then the SPN graphs would be characterized by a finite set of forbidden minors, so Conjecture 1.3 would be that the only forbidden minor is F_5.

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