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MAPS PRESERVING NORMS OF GENERALIZED WEIGHTED QUASI-ARITHMETIC MEANS OF INVERTIBLE POSITIVE OPERATORS*

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Abstract. In this paper, the problem of describing the structure of transformations leaving norms of generalized weighted quasi-arithmetic means of invertible positive operators invariant is discussed. In a former result of the authors, this problem was solved for weighted quasi-arithmetic means, and here the corresponding result is generalized by establishing its solution under certain mild conditions. It is proved that in a quite general setting, generalized weighted quasi-arithmetic means on self-adjoint operators are not monotone in their variables which is an interesting property. Moreover, the relation of these means with the Kubo-Ando means is investigated and it is shown that the common members of the classes of these types of means are weighted arithmetic means.

Key words. Invertible positive operators, Generalized weighted quasi-arithmetic means, Kubo-Ando means, Nonlinear preservers.

AMS subject classifications. 47A64, 47B49.

1. Introduction and statement of the main results. In [2, Theorem 1], under certain conditions, the general forms of maps on the set of complex positive definite matrices of a fixed size preserving a norm of a weighted quasi-arithmetic mean were described. The main aim of this paper is to extend that result for generalized weighted quasi-arithmetic means. The motivation of reaching this goal comes from a private discussion with Zoltán Daróczy in which he proposed us to investigate the problem studied in that theorem also in the case of the latter means. Originally, they were defined only for real numbers by Matkowski in [7, Definition 2/(2)] (we remark that the paper [1] was the first article in which their notion appeared). However, they have the advantage that their definition can be extended to the operator setting using their formula for numbers. Before introducing the corresponding extension, we collect some basic notions and notation that will be used throughout the paper.

Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} > 1$. We will denote by $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})_{sa}$ the C^*-algebra of all bounded linear operators on H with unit I and the vector space of the self-adjoint elements in $\mathcal{L}(\mathcal{H})$, respectively. For any set $D \subset \mathbb{R}$, the symbol $\mathcal{L}(\mathcal{H})_{sa}^D$ stands for the collection of all operators in $\mathcal{L}(\mathcal{H})_{sa}$ with spectra in D. An element $A \in \mathcal{L}(\mathcal{H})$ is termed positive if $\langle Ax, x\rangle \geq 0$ is satisfied by every vector $x \in \mathcal{H}$. The usual order $\leq$ on $\mathcal{L}(\mathcal{H})_{sa}$ is defined by $A \leq B$ if $B - A$ is positive $(A, B \in \mathcal{L}(\mathcal{H})_{sa})$. The symbols $\mathcal{L}(\mathcal{H})_+$ and $\mathcal{L}(\mathcal{H})_{++}$ stand for the set of positive and invertible positive operators in $\mathcal{L}(\mathcal{H})$, respectively. Observe that $\mathcal{L}(\mathcal{H})_+ = \mathcal{L}(\mathcal{H})_{sa}^{[0,\infty[}$ and $\mathcal{L}(\mathcal{H})_{++} = \mathcal{L}(\mathcal{H})_{sa}^{]0,\infty[}$.

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We fix a number $n \in \mathbb{N} \setminus \{1\}$, an interval $D \subset \mathbb{R}$ and continuous functions $f_i : D \to \mathbb{R}$ $(i = 1, \ldots, n)$ which are monotone in the same sense and not simultaneously constant on any nontrivial subinterval of D. The (operator theoretical version of the) generalized weighted quasi-arithmetic mean generated by $f_1, \ldots, f_n$ is defined by the equality

$$M_{f_1,\ldots,f_n}(A_1, \ldots, A_n) = (f_1 + \cdots + f_n)^{-1}(f_1(A_1) + \cdots + f_n(A_n))$$

for all operators $A_i \in \mathcal{L}(\mathcal{H})_{sa}^D$ $(i = 1, \ldots, n)$. Observe that $f_1 + \cdots + f_n$ is a continuous strictly monotone function with range containing $f_1(D) + \cdots + f_n(D)$, and using these properties, one can check that $M_{f_1,\ldots,f_n}(A_1, \ldots, A_n)$ is a well-defined element of $\mathcal{L}(\mathcal{H})_{sa}^D$ $(A_1, \ldots, A_n \in \mathcal{L}(\mathcal{H})_{sa}^D)$. In the special case where $n = 2$; $f_1 = wf, f_2 = (1 - w)f$ with a continuous injective function $f\colon D \to \mathbb{R}$ and a number $w \in [0, 1]$, the last displayed formula defines the (2-variable weighted) quasi-arithmetic means. For a brief introduction to them and to operator means in general and for some references on this topic, the reader can consult the paper [2]. The most fundamental means of the form $M_{f_1,\ldots,f_n}$ are the weighted arithmetic means, whose generating functions are $f_1 = w_1 \mathrm{id}_D, \ldots, f_n = w_n \mathrm{id}_D$, where $w_1, \ldots, w_n \geq 0$ are numbers with sum 1.

Since $M_{f_1,\ldots,f_n}$ is an operation on $\mathcal{L}(\mathcal{H})_{sa}^D$, it is a natural problem to describe the structure of homomorphisms with respect to it. Unfortunately, as it is explained below the formulation of *Problem A* in [2], in the case of quasi-arithmetic means – which are trivially seen to be means of the form $M_{f_1,\ldots,f_n}$ – those maps do not have any regular structure. Despite this fact, we may have hope for a regular form in the case of maps which preserve not $M_{f_1,\ldots,f_n}$ itself, but some numerical function, e.g. a norm of it. We remark that for a fundamental class of means of positive operators, the Kubo-Ando means, transformations preserving a norm of one of them were studied, e.g. in [3, 4, 10]. In the case of 2-variable weighted quasi-arithmetic means, the structure of preservers for a unitary invariant norm of such an operation was investigated in [2]. Below, we present our first two results in which, under certain conditions, the general form of maps on $\mathcal{L}(\mathcal{H})_{++}$ leaving a norm of $M_{f_1,\ldots,f_n}$ invariant is described. In order to do so, we recall that a norm $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ is termed unitary invariant if $N(UAV) = N(A)$ for all elements $A \in \mathcal{L}(\mathcal{H})$ and unitary operators U, V on $\mathcal{H}$. Our first result reads as follows.

Theorem 1.1. *Assume that* $\dim \mathcal{H} < \infty$ *and let* $f_1, \ldots, f_n\colon]0, \infty[\to \mathbb{R}$ *be continuous bijections which are monotone in the same sense and* $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ *be a unitary invariant norm. If* $\phi\colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ *is a bijective map satisfying*

$$N(M_{f_1,\ldots,f_n}(\phi(A_1), \ldots, \phi(A_n))) = N(M_{f_1,\ldots,f_n}(A_1, \ldots, A_n)) \tag{1.1}$$

for all $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{H})_{++}$*, then there is a unitary or an antiunitary operator* U *on* $\mathcal{H}$ *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

Our next result follows which shows that the previous one holds also in the case where the common range of $f_1, \ldots, f_n$ is not $\mathbb{R}$, but $]0, \infty[$.

Theorem 1.2. *Suppose that* $\dim \mathcal{H} < \infty$ *and let* $f_1, \ldots, f_n\colon]0, \infty[\to]0, \infty[$ *be continuous decreasing bijections and* $N\colon \mathcal{L}(\mathcal{H}) \to \mathbb{R}$ *be a unitary invariant norm . If* $\phi\colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ *is a bijection satisfying* (1.1) *for all* $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{H})_{++}$*, then there is a unitary or an antiunitary operator* U *on* $\mathcal{H}$ *such that* ϕ *is of the form*

$$\phi(A) = UAU^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}).$$

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We remark that the above results generalize [2, Theorem 1]. In the next part of the paper, we investigate monotonicity properties of $M_{f_1,\dots,f_n}$. We point out that there are several other problems which are frequently studied concerning means of real numbers, e.g. the problem of homogeneity and equality. In those problems, certain conditions hold for all values of the variables of the means under consideration. We can study their counterparts for the operator mean $M_{f_1,\dots,f_n}$ and apply those conditions to scalar operators in $\mathcal{L}(\mathcal{H})_{sa}^D$. In this way, we infer that they hold also for $M_{f_1,\dots,f_n}$ as a scalar mean. Therefore, the investigation of the mentioned problems in the case of the operator mean $M_{f_1,\dots,f_n}$ can be reduced to their study in the setting of generalized weighted quasi-arithmetic means of real numbers. However, this is not the case with monotonicity. As for the latter property of $M_{f_1,\dots,f_n}$, we have the result below. To formulate it, we recall that a map ϕ between subsets of $\mathcal{L}(\mathcal{H})_{sa}$ is called monotone if $A \leq B$ implies $\phi(A) \leq \phi(B)$ for all elements A, B in the domain of ϕ.

THEOREM 1.3. *Let $D \subset \mathbb{R}$ be a closed interval which is not bounded from above and $f_i : D \to \mathbb{R}$ $(i = 1, \dots, n)$ be continuous functions which are not simultaneously constant on any nontrivial subinterval of D. Furthermore, assume that for some number $j \in \{1, \dots, n\}$, the equality $|\lim_{x\to\infty} f_j(x)| = \infty$ holds, moreover one has that $f_1, \dots, f_n$ are increasing and $f_1 + \dots + f_n$ is strictly concave or that $f_1, \dots, f_n$ are decreasing and $f_1 + \dots + f_n$ is strictly convex. Then $M_{f_1,\dots,f_n}$ is not monotone in its jth variable.*

It is important to highlight the fact that $M_{f_1,\dots,f_n}$ as a mean on the set D of real numbers is clearly monotone in each of its variables and by this theorem, in the multidimensional case under quite general conditions on $f_1, \dots, f_n$, the operation $M_{f_1,\dots,f_n}$ is not monotone in any of its variables. This striking contrast shows that the properties of $M_{f_1,\dots,f_n}$ can be very much different in the one- and the multidimensional settings.

In our last result, we determine the common members of the classes of generalized weighted quasi-arithmetic means and of Kubo-Ando means on $\mathcal{L}(\mathcal{H})_+$. The latter means are defined as follows (see [5]). A binary operation $\sigma\colon \mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+ \to \mathcal{L}(\mathcal{H})_+$ is a Kubo-Ando mean if it has the next properties. For each elements $A, B, C, D \in \mathcal{L}(\mathcal{H})_+$ and sequences $(A_n), (B_n)$ in $\mathcal{L}(\mathcal{H})_+$:

(i) $I\sigma I = I$;
(ii) if $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$;
(iii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iv) if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\sigma B_n \downarrow A\sigma B$.

Here, the symbol $\downarrow$ stands for monotone decreasing convergence in the strong operator topology. Concerning Kubo-Ando means, we will need the following notions. A real-valued function f defined on a nontrivial interval D is termed d-monotone (or monotone of order d) if for each pair A, B of self-adjoint operators on a d-dimensional complex Hilbert space whose spectra are in D, one has $f(A) \leq f(B)$ in the case $A \leq B$. If this condition holds for each number $d \in \mathbb{N}$, then we say that f is operator monotone. We call f d-concave in the case where, for any operators A, B satisfying the above properties, the inequality

$$f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha) f(B) \quad (\alpha \in [0,1])$$

is fulfilled.

We see from the proof of [5, Theorem 3.2] that for a Kubo-Ando mean σ and a scalar $t > 0$ the operator $I\sigma(tI)$ is scalar. Therefore, we can define a function $f_\sigma\colon]0,\infty[\to [0,\infty[$, called the generating function of σ, with the property

$$f_\sigma(t)I = I\sigma(tI) \quad (t > 0).$$

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The cited proof also shows that if $d = \dim \mathcal{H} < \infty$, then f_σ is d-monotone and it is operator monotone in the case $\dim \mathcal{H} = \infty$. Moreover,

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2} B A^{-1/2}) A^{1/2} \tag{1.2}$$

for all $A, B \in \mathcal{L}(\mathcal{H})_{++}$. We deduce that the generating function of a Kubo-Ando mean on $\mathcal{L}(\mathcal{H})_+$ is monotone of an order greater than 1.

The most basic Kubo-Ando means are the weighted arithmetic means which are, as we have noted above, also generalized weighted quasi-arithmetic means. Now our last result follows in which we establish that – under very general conditions – among operations on $\mathcal{L}(\mathcal{H})_+$, only weighted arithmetic means can have this property.

Theorem 1.4. *A map $M : \mathcal{L}(\mathcal{H})_+ \times \mathcal{L}(\mathcal{H})_+ \to \mathcal{L}(\mathcal{H})_+$ is a generalized weighted quasi-arithmetic mean with injective generating functions $f_1, f_2 : [0, \infty[\to \mathbb{R}$ and also a Kubo-Ando mean if and only if it is a weighted arithmetic mean with positive weights.*

2. Proofs. In this section, we are going to present the verifications of the results in the introduction. We begin this work with the first one.

Proof of Theorem 1.1. By inserting equal operators $A_2, \ldots, A_n$ in (1.1), it can be seen that without loss of generality, we may and do assume that $n = 2$. Consider the maps $\psi_1, \psi_2 : \mathcal{L}(\mathcal{H})_{sa} \to \mathcal{L}(\mathcal{H})_{sa}$ given by

$$\psi_i(A) = f_i(\phi(f_i^{-1}(A))) \quad (A \in \mathcal{L}(\mathcal{H})_{sa},\ i = 1, 2).$$

Then ψ_1 and ψ_2 are bijective transformations and possess the property that

$$N((f_1 + f_2)^{-1}(\psi_1(A_1) + \psi_2(A_2))) = N((f_1 + f_2)^{-1}(A_1 + A_2)) \tag{2.3}$$

for all operators $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{sa}$. Observe that by the conditions of Theorem 1.1, the function $g = (f_1 + f_2)^{-1} : \mathbb{R} \to]0, \infty[$ is continuous, strictly monotone and there is an element $\alpha_0 \in \{-\infty, \infty\}$ for which $\lim_{\alpha \to \alpha_0} g(\alpha) = 0$. Assume that g is increasing. Then by [2, Lemma 2], for operators $A, B \in \mathcal{L}(\mathcal{H})_{sa}$ given arbitrarily one has

$$\begin{aligned} A \le B &\iff N(g(A + X)) \le N(g(B + X)) \ \forall\ X \in \mathcal{L}(\mathcal{H})_{sa} \\ &\iff N(g(\psi_1(A) + \psi_2(X))) \le N(g(\psi_1(B) + \psi_2(X))) \ \forall\ X \in \mathcal{L}(\mathcal{H})_{sa} \\ &\iff N(g(\psi_1(A) + Y)) \le N(g(\psi_1(B) + Y)) \ \forall\ Y \in \mathcal{L}(\mathcal{H})_{sa} \\ &\iff \psi_1(A) \le \psi_1(B). \end{aligned}$$

Similarly, we obtain that $\psi_2(A) \le \psi_2(B)$ exactly when $A \le B$. These conclusions can be reached using the latter argument also in the case where g is decreasing (then one can apply [2, Lemma 2] for the function $g \circ (-\mathrm{id}_{\mathbb{R}})$ instead of g). We infer that ψ_1, ψ_2 are order automorphisms of $\mathcal{L}(\mathcal{H})_{sa}$, meaning that they are bijective and preserve the order in both directions. Such transformations of $\mathcal{L}(\mathcal{H})_{sa}$ are described in [9, Theorem 2]. Applying that result to ψ_1, ψ_2, we deduce that there are invertible linear or conjugate-linear operators T_1, T_2 on $\mathcal{H}$ and elements $Y_1, Y_2 \in \mathcal{L}(\mathcal{H})_{sa}$ such that

$$\psi_1(A) = T_1 A T_1^* + Y_1, \quad \psi_2(A) = T_2 A T_2^* + Y_2 \quad (A \in \mathcal{L}(\mathcal{H})_{sa}). \tag{2.4}$$

Substituting these forms of ψ_1, ψ_2 to the equation (2.3), we get

$$N(g(T_1 A_1 T_1^* + T_2 A_2 T_2^* + (Y_1 + Y_2))) = N(g(A_1 + A_2)) \quad (A_1, A_2 \in \mathcal{L}(\mathcal{H})_{sa}).$$

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Inserting $A_1 = 0$ in this equality and using the argument in [2] from equation (10) until the end of the proof of Lemma 3 (for $g \circ (-\mathrm{id}_{\mathbb{R}})$ in the case where g is decreasing), we conclude that T_2 is unitary or antiunitary and $Y_1 + Y_2 = 0$. Similarly, we obtain that T_1 is also such an operator. Then the last displayed equality implies

$$\begin{gathered} N(g(T_2^* T_1 A_1 T_1^* T_2 + A_2)) = N(T_2^* g(T_1 A_1 T_1^* + T_2 A_2 T_2^*) T_2) \\ = N(g(T_1 A_1 T_1^* + T_2 A_2 T_2^*)) = N(g(A_1 + A_2)) \quad (A_1, A_2 \in \mathcal{L}(\mathcal{H})_{sa}). \end{gathered}$$

Now assume that g is increasing and perform the substitution $A_2 = \alpha(P - I)$ in this chain of relations with an arbitrary rank-one projection P on $\mathcal{H}$ and number $\alpha \in \mathbb{R}$. Then by [2, Lemma 1], it is very easy to see that tending to ∞ with α, the limits of the expressions before the first and after the last equality sign are $N(P)g(\operatorname{Tr} T_2^* T_1 A_1 T_1^* T_2 P)$ and $N(P)g(\operatorname{Tr} A_1 P)$, respectively. The chain in question shows that they are the same, so the injectivity of g yields that $\operatorname{Tr} T_2^* T_1 A_1 T_1^* T_2 P = \operatorname{Tr} A_1 P$. This easily implies the equality $\langle T_2^* T_1 A_1 T_1^* T_2 u, u\rangle = \langle A_1 u, u\rangle$ for any unit vector u in the range of P. Since P was arbitrary, we conclude that $T_2^* T_1 A_1 T_1^* T_2 = A_1$, i.e., $T_1 A_1 T_1^* = T_2 A_1 T_2^*$ $(A_1 \in \mathcal{L}(\mathcal{H})_{sa})$. A similar argument can be used to verify that the latter relation holds also in the case where g is decreasing (then [2, Lemma 1] should be applied for $g \circ (-\mathrm{id}_{\mathbb{R}})$ instead of g).

Referring to the conclusions in the last two paragraphs and to (2.4), we observe that

$$\psi_1(A) = T_1 A T_1^* + Y_1, \ \psi_2(A) = T_1 A T_1^* - Y_1 \quad (A \in \mathcal{L}(\mathcal{H})_{sa})$$

with a unitary-antiunitary operator T_1, or, equivalently,

$$\begin{gathered} T_1 f_1^{-1}(f_1(A) + Z_1) T_1^* = f_1^{-1}(T_1 f_1(A) T_1^* + Y_1) = \phi(A) \\ = f_2^{-1}(T_1 f_2(A) T_1^* - Y_1) = T_1 f_2^{-1}(f_2(A) - Z_1) T_1^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}), \end{gathered} \tag{2.5}$$

where $Z_1 = T_1^* Y_1 T_1$. This means that $f_1^{-1}(f_1(A) + Z_1) = f_2^{-1}(f_2(A) - Z_1)$, yielding that $(f_2 \circ f_1^{-1})(f_1(A) + Z_1) = f_2(A) - Z_1$ for all $A \in \mathcal{L}(\mathcal{H})_{++}$, which gives us that

$$(f_2 \circ f_1^{-1})(A + Z_1) = (f_2 \circ f_1^{-1})(A) - Z_1 \quad (A \in \mathcal{L}(\mathcal{H})_{sa}).$$

After performing the substitution $A = xI$ in this equation, we infer that

$$(f_2 \circ f_1^{-1})(x + z) = (f_2 \circ f_1^{-1})(x) - z$$

for all eigenvalues z of Z_1 and numbers $x \in \mathbb{R}$. The functions f_1, f_2 are monotone in the same sense, and hence, $f_2 \circ f_1^{-1}$ is increasing, which property together with the latter equality forces z to be 0. Since it was an arbitrary eigenvalue of Z_1, we deduce that $Z_1 = 0$, and, referring to (2.5), we then see that ϕ is of the desired form. Now the proof is complete. ∎

In what follows, we are going to show only the sketch of the verification of the second result, since it is very similar to that of the first one.

Sketch of the proof of Theorem 1.2. We may and do assume that $n = 2$. Define the maps $\psi_1, \psi_2\colon \mathcal{L}(\mathcal{H})_{++} \to \mathcal{L}(\mathcal{H})_{++}$ just as in the proof of Theorem 1.1. Then ψ_1 and ψ_2 are bijective and satisfy the equation (2.3) for all operators $A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}$. Observe that $g = (f_1 + f_2)^{-1}$ is a continuous strictly decreasing selfmap of $]0, \infty[$ for which $\lim_{x \to \infty} g(x) = 0$. Using the argument in the first paragraph of the previous proof for $g \circ (-\mathrm{id}_{]-\infty, 0[})$ instead of g, we obtain that ψ_1 and ψ_2 are order automorphisms of $\mathcal{L}(\mathcal{H})_{++}$.

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In [8, Theorem 1], Molnár described the structure of all such transformations. Due to that result, we obtain that there are invertible linear or conjugate-linear operators T_1, T_2 on $\mathcal{H}$ such that

$$\psi_1(A) = T_1 A T_1^*, \quad \psi_2(A) = T_2 A T_2^* \quad (A \in \mathcal{L}(\mathcal{H})_{++}). \tag{2.6}$$

Then by (2.3) and the latter conclusion

$$N(g(T_1 A_1 T_1^* + T_2 A_2 T_2^*)) = N(g(A_1 + A_2)) \quad (A_1, A_2 \in \mathcal{L}(\mathcal{H})_{++}).$$

Plugging $A_2 = (1/k)I$ $(k \in \mathbb{N})$ in this equality and taking the limit $k \to \infty$, we see that

$$N(g(T_1 A T_1^*)) = N(g(A)) \quad (A \in \mathcal{L}(\mathcal{H})_{++}),$$

i.e.,

$$N((g \circ (-\mathrm{id}_{]-\infty,0[}))(T_1 A T_1^*)) = N((g \circ (-\mathrm{id}_{]-\infty,0[}))(A))$$

for all operators $A \in -\mathcal{L}(\mathcal{H})_{++}$. Using the argument given in the last paragraph of the proof of [2, Lemma 3], we arrive at the conclusion that T_1 is unitary or antiunitary. This fact together with (2.6) and the definition of ψ_1 gives us that

$$\phi(A) = f_1^{-1}(T_1 f_1(A) T_1^*) = T_1 f_1^{-1}(f_1(A)) T_1^* = T_1 A T_1^* \ (A \in \mathcal{L}(\mathcal{H})_{++})$$

completing the proof. □

Now we are going to verify the third result.

Proof of Theorem 1.3. Observe that if any of the functions f_i $(i = 1, \ldots, n)$ is constant, then $M_{f_1,\ldots,f_n}$ is an $n-1$ variable generalized weighted quasi-arithmetic mean whose generating functions satisfy the conditions of Theorem 1.3. Moreover, clearly $M_{-f_1,\ldots,-f_n} = M_{f_1,\ldots,f_n}$, so by considering $-f_i$ instead of f_i $(i = 1, \ldots, n)$, regarding the conditions of that result w.l.o.g., we may and do assume that $f_1, \ldots, f_n$ are nonconstant increasing functions. Thus, by those hypotheses, $f_1 + \cdots + f_n$ is strictly concave and $\lim_{x \to \infty} f_j(x) = \infty$.

Now assume on the contrary that the conclusion of Theorem 1.3 does not hold, i.e., $M_{f_1,\ldots,f_n}$ is monotone in its jth variable. Define

$$g = f_1 + \cdots + f_{j-1} + f_{j+1} + \cdots + f_n.$$

Then the latter assumption immediately implies that the map $(f_j + g)^{-1}(g(A) + f_j(.))$ is monotone for any $A \in \mathcal{L}(\mathcal{H})_{sa}^D$. Let $A \in \mathcal{L}(\mathcal{H})_{sa}^D$ be an arbitrary element. By the conditions, there is a number $\alpha \in \mathbb{R}$ such that $D = [\alpha, \infty[$. It is trivial that, since $\lim_{x \to \infty} f_j(x) = \infty$ and f_j is increasing and continuous, $f_j(D) = [f_j(\alpha), \infty[$ and then one can check that

$$\{f_j(X) \mid X \in \mathcal{L}(\mathcal{H})_{sa}^D\} = \{Y \in \mathcal{L}(\mathcal{H})_{sa} \mid f_j(\alpha I) \leq Y\}.$$

Moreover, obviously $\alpha I \leq B$ $(B \in \mathcal{L}(\mathcal{H})_{sa}^D)$, therefore the previous observations give us that

$$(f_j + g)^{-1}(g(A) + f_j(\alpha I)) \leq (f_j + g)^{-1}(Z)$$

for all operators $Z \in \mathcal{L}(\mathcal{H})_{sa}$ satisfying $g(A) + f_j(\alpha I) \leq Z$. In the terminology of [11], this means that $(f_j + g)^{-1}$ is locally monotone at $g(A) + f_j(\alpha I)$. In [11, Theorem 1], the self-adjoint elements in a C^*-algebra at which a given strictly convex increasing function defined on an open interval which is not bounded from

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above are characterized as the central ones, i.e., those which commute with every member of that algebra. It is mentioned before that result that it is true also in the case where the interval in question is not open, provided that the considered function is continuous. On the other hand, since $f_j(x) \to \infty$ $(x \to \infty)$ and f_j, g are increasing and continuous, $\lim_{x\to\infty}(f_j+g)(x) = \infty$ implying that the function $(f_j+g)^{-1}$ also has these properties and it is defined on an interval which is not bounded from above. Observe that $(f_j+g)^{-1}$ is strictly convex, too due to the strict concavity of f_j+g. By the previous discussion, $g(A)+f_j(\alpha I)$ is a central element of $\mathcal{L}(\mathcal{H})$.

Such operators are well-known to be the scalar ones and then it follows that $g(A) \in \mathbb{R}I$. On the other hand, observe that g, being the sum of nonconstant increasing functions, is not constant, and thus, there exist numbers $x_1, x_2 \in D$ for which $g(x_1) \neq g(x_2)$. Now by picking a nontrivial projection $P \in \mathcal{L}(\mathcal{H})_{sa}$ and setting the arbitrary element $A \in \mathcal{L}(\mathcal{H})_{sa}^D$ to be $x_1P + x_2(I-P)$, we obtain a nonscalar operator $g(A) = g(x_1)P + g(x_2)(I-P)$. To sum up, our assumption has led to a contradiction, implying the statement of Theorem 1.3. ∎

We finish this section with the verification of the last result.

Proof of Theorem 1.4. By the introduction, any weighted arithmetic mean with positive weigths is clearly a mean of those types appearing in that result. Now assume that M is a mean of each of those kinds. Then there is a function $g\colon [0,\infty[\to \mathbb{R}$ such that $M = \sigma_g = M_{f_1,f_2}$ and g is monotone of a certain order $d > 1$. Let $x, y > 0$ be arbitrary numbers. By evaluating σ_g, M_{f_1,f_2} at the point (xI, yI) and using (1.2), it follows that

$$xg\left(\frac{y}{x}\right) = M_{f_1,f_2}(x,y). \tag{2.7}$$

Clearly, the left-hand side of this equation is homogeneous in (x,y), and hence, so is the other one, which clearly yields that it is a homogeneous generalized weighted quasi-arithmetic mean on $]0,\infty[$ with generating functions $f_1|_{]0,\infty[}, f_2|_{]0,\infty[}$. Such quantities are characterized in [7, Theorem 3]. Applying that result, we obtain the existence of numbers $a, b, c, d, p \in \mathbb{R}$ for which $p \neq 0$, $ac > 0$ and we have $f_1(x) = a\log x + b$, $f_2(x) = c\log x + d$ or $f_1(x) = ax^p + b$, $f_2(x) = cx^p + d$ $(x > 0)$. Due to the continuity of f_1, f_2 and to the relation $ac > 0$, the former case is excluded and $p > 0$. Moreover, the last two equalities hold for $x = 0$, and then by substituting $x = 1$ in (2.7) and using the notation $\alpha = a/(a+c)$, it follows that $g(y) = (\alpha + (1-\alpha)y^p)^{1/p}$. The result [6, Theorem 2.1] tells us that any d-monotone function on $]0,\infty[$ is concave, as it is concave of order $[d/2]$, thus so is $g|_{]0,\infty[}$. It is also twice differentiable, therefore we infer that

$$g''(x) = \alpha(1-\alpha)(p-1)x^{p-2}(\alpha + (1-\alpha)x^p)^{\frac{1}{p}-2} \leq 0 \quad (x > 0),$$

which, since $\alpha \in]0,1[$, yields $p \leq 1$. By what we have proved so far, we see that $0 < p \leq 1$. Now assume $p < 1$. Then, since $ac > 0$, $0 < p < 1$, it is obvious that f_1, f_2 would satisfy the conditions of Theorem 1.3, so it would apply and we would get that M_{f_1,f_2} is not monotone increasing in its variables. However, it is a Kubo-Ando mean, thus property (ii) in the definition of such means holds for it. It follows that $p < 1$ is untenable, and thus, we conclude that $p = 1$, and therefore, M_{f_1,f_2} is the arithmetic mean with the positive weights $\alpha, 1-\alpha$. The proof of Theorem 1.4 is complete. ∎

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