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ON SIGN PATTERN MATRICES THAT ALLOW OR REQUIRE ALGEBRAIC POSITIVITY*

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Abstract. A square matrix M with real entries is algebraically positive (AP) if there exists a real polynomial p such that all entries of the matrix $p(M)$ are positive. A square sign pattern matrix S allows algebraic positivity if there is an algebraically positive matrix M whose sign pattern is S. On the other hand, S requires algebraic positivity if matrix M, having sign pattern S, is algebraically positive. Motivated by open problems raised in a work of Kirkland, Qiao, and Zhan (2016) on AP matrices, all nonequivalent irreducible 3×3 sign pattern matrices are listed and classify into three groups (i) those that require AP, (ii) those that allow but not require AP, or (iii) those that do not allow AP. A necessary condition for an irreducible $n \times n$ sign pattern to allow algebraic positivity is also provided.

Key words. Algebraically positive matrices, Signed digraphs, Sign pattern matrices.

AMS subject classifications. 15B35, 15B48, 05C50.

1. Introduction. We denote the set of $m \times n$ real matrices by $M_{m\times n}$ and the set of all $n \times n$ matrices by M_n and the (i, j) entry of a matrix A by $(A)_{ij}$.

A matrix $A \in M_{m\times n}$ is *positive*, written $A > 0$, if all entries of A are positive. Several applications of positive matrices appear in various fields of study such as Markov chains in probability theory, population models, iterative methods in numerical analysis, economic models, epidomiology, low-dimensional topology, physics, and many more (for example, see [1, 4]). Some natural generalizations of the concept of matrix positivity have also been studied extensively such as matrix nonnegativity, matrix primitivity and eventual positivity. In [3], Kirkland, Qiao and Zhan presented another generalization called algebraic positivity. A matrix $A \in M_n$ is *algebraically positive* if $p(A) > 0$ for some real polynomial p. Using the Cayley-Hamilton theorem, it can be deduced that a matrix $A \in M_n$, where $n \geq 2$, is algebraically positive if and only if there are real numbers $k_1, \dots, k_{n-1}$ such that the off-diagonal entries of the following matrix are all positive:

$$k_1 A + \cdots + k_{n-1} A^{n-1}.$$

The authors of [3] used the Perron-Frobenius theorem to prove the following characterization of algebraically positive matrices.

Theorem 1.1 (Kirkland, Qiao, and Zhan, 2016). *A real matrix is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right positive eigenvectors.*

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A *sign pattern* is a matrix whose entries are from the set $\{0, +, -\}$. We define the *pattern class* of a sign pattern S, denoted by $Q(S)$, as the set

$$Q(S) = \{A \in M_{m,n} \mid \operatorname{sgn}((A)_{ij}) = (S)_{ij} \text{ for all } 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

of real matrices whose sign pattern is S. In this paper, we are interested in determining when a given square sign pattern matrix *allows algebraic positivity*, i.e., there exists an algebraically positive matrix in its sign pattern class; or *requires algebraic positivity*, i.e., all matrices in its sign pattern class are algebraically positive.

The following results from [3] will be useful in our study:

Theorem 1.2 (Kirkland, Qiao, and Zhan, 2016).

1. *Every algebraically positive matrix is irreducible.*
2. *If A is an irreducible real matrix all of whose off-diagonal entries are non-negative (or nonpositive), then A is algebraically positive.*
3. *If a sign pattern allows algebraic positivity, then every row and column contains a $+$, or every row and column contains a $-$.*

Recall that a matrix $A \in M_n$ is reducible if there exists a permutation matrix P such that PAP^T is of the form

$$PAP^T = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

for some $A_1 \in M_s$ such that $1 \leq s < n$. The reducibility of a matrix is easily determined by looking at its associated digraph. The *digraph* of A, denoted by $\Gamma(A)$, is the directed graph whose vertex set is $\{1, \ldots, n\}$ and whose edge set is $E_A = \{(i,j) \mid (A)_{ij} \neq 0\}$. It is known (see [2]) that a matrix is irreducible if and only if its digraph is strongly connected, i.e., for every pair of distinct vertices i and j, there is a directed path in $\Gamma(A)$ from i to j. Note that the digraph of A^T is obtained from the digraph of A by reversing the direction of all the edges. Meanwhile, if P is a permutation matrix, then the digraph of PAP^T is obtained from the digraph of A by permuting the labels of the vertices of $\Gamma(A)$. Thus, if $\Gamma(A)$ is irreducible, then so are $\Gamma(A^T)$ and $\Gamma(PAP^T)$. For the purpose of this study, we say that two digraphs are *equivalent* if one can be obtained from the other by (i) reversing the direction of all the edges, (ii) permuting the labels of the vertices, or (iii) doing both. In Appendix A, we list the 26 nonequivalent irreducible digraphs in with vertex set $\{1, 2, 3\}$.

Theorem 1.3. *If A is algebraically positive, then the following matrices are also algebraically positive:*

1. A^T*;*
2. $-A$*;*
3. PAP^T *for any permutation matrix* P*;*
4. $\beta A + \alpha I$ *for any* $\alpha \in \mathbb{R}$ *and any* $\beta \in \mathbb{R} \setminus \{0\}$*.*

Proof. Suppose p is a real polynomial such that $p(A) > 0$ and suppose P is a permutation matrix. Then $p(A^T) = p(A)^T > 0$ and $p(PAP^T) = Pp(A)P^T > 0$. Define $q(x) = p(\frac{1}{\beta}x - \frac{\alpha}{\beta})$, then $q(\beta A + \alpha I) = p(A) > 0$. In particular, if $\beta = -1$ and $\alpha = 0$, we have $q(-A) > 0$. ∎

For the purpose of our study, we say that two sign pattern matrices are *equivalent sign patterns* if one can be obtained from the other by (i) transposition; (ii) permutation similarity; (iii) negation; (iv) any combination of the first three transformations.

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2. Main results. The main result of this study gives a necessary condition for sign patterns that allow algebraic positivity.

THEOREM 2.1. *Suppose A is an irreducible $n \times n$ sign pattern matrix. We write $A = A_+ + A_-$, where A_+ contains only the nonnegative signs of A, while A_- contains only non-positive signs of A. Define $B_A = A_+ - A_-^T$. If B_A is reducible, then A does not allow algebraic positivity.*

Proof. Let $A \in M_n(\{+, -, 0\})$ and

$$(A_+)_{ij} = \begin{cases} + & \text{if } (A)_{ij} = + \\ 0 & \text{otherwise} \end{cases}, \qquad (A_-)_{ij} = \begin{cases} - & \text{if } (A)_{ij} = - \\ 0 & \text{otherwise} \end{cases}.$$

Define $B_A = A = A_+ - A_-^T$. Clearly, $B_A \in M_n(\{0, +\})$. Suppose B_A is reducible. Then, we can assume without loss of generality that

$$B_A = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix} \tag{2.1}$$

for some $B_1 \in M_s(\{0, +\})$, and for some $B_3 \in M_{n-s}(\{0, +\})$. Otherwise, we can replace A by PAP^T, where P is a permutation matrix such that PB_AP^T is in the said form. From (2.1), it will follow that A is of the form

$$A = \begin{bmatrix} C - D & E \\ -F & G - H \end{bmatrix}$$

for some $C, D \in M_s(\{0, +\})$, $G, H \in M_{n-s}(\{0, +\})$ and $E \in M_{s,n-s}(\{0, +\})$ and $F \in M_{n-s,s}(\{0, +\})$. Note also that the entries of E or F cannot all be zero since A is irreducible.

Suppose there exists an $n \times n$ matrix $X \in Q(A)$ such that X is algebraically positive. Then we can say that

$$X = \begin{bmatrix} X_C - X_D & X_E \\ -X_F & X_G - X_H \end{bmatrix} \in Q(A),$$

where $X_J \in Q(J)$ for $J = C, D, E, F, G, H$. By Theorem 1.1, X has a simple real eigenvalue, say $\lambda \in \mathbb{R}$, with corresponding positive left and right eigenvectors

$$\begin{bmatrix} u^T & v^T \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix} > 0, \text{ where } u, y \in M_{s\times 1} \text{ and } v, z \in M_{(n-s)\times 1}.$$

Note that

$$\begin{bmatrix} u^T & v^T \end{bmatrix} (X - \lambda I) = \begin{bmatrix} u^T(X_C - X_D - \lambda I) - v^T X_F & * \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$(X - \lambda I) \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} (X_C - X_D - \lambda I)y + X_E z \\ \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, $u^T(X_C - X_D - \lambda I) = v^T X_F$ and $(X_C - X_D - \lambda I)y = -X_E z$. Consider the relation

$$u^T(X_C - X_D - \lambda I)y = v^T X_F y = -u^T X_E z.$$

Since the entries of u^T, v^T, y, z are positive and X_E, X_F are nonzero nonnegative matrices, $v^T X_F y$ is a positive real number, while $-u^T X_E z$ is a negative real number. This gives a contradiction. Hence, no $X \in Q(A)$ is algebraically positive. $\square$

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Here is an example on how to utilize this result.

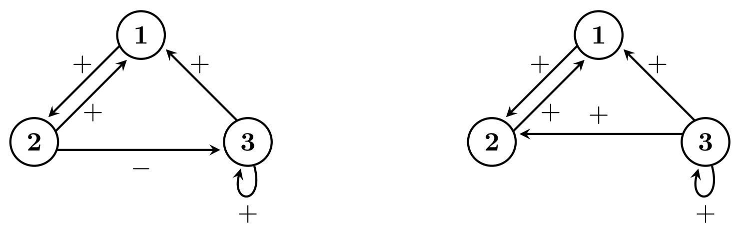
EXAMPLE 2.2. Consider the sign pattern A given below. We write A as follows, separating the positive (A_+) and the negative (A_-) signs.

$$A = \begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ + & 0 & + \end{bmatrix} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ + & 0 & + \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & - \\ 0 & 0 & 0 \end{bmatrix}.$$

We then write its B_A as follows, negating and transposing A_-.

$$B = \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ + & + & + \end{bmatrix} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ + & 0 & + \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & + & 0 \end{bmatrix}.$$

We can treat the negative signs of A as a reversal of the direction on the corresponding edges in the signed digraph of A to obtain the signed digraph of B_A as seen below. Note that B_A is reducible since its corresponding digraph is not strongly connected [2].



Signed Digraph of A Signed Digraph of B_A

Figure 1: Illustrates how to obtain the digraph of B_A from A, as described in Theorem 2.1.

We also note that the converse of Theorem 2.1 is not true, by providing a counterexample below.

EXAMPLE 2.3. Consider the sign pattern C below. By Theorem 1.2.3, C does not allow algebraic positivity. Now, we write C as follows:

$$C = \begin{bmatrix} 0 & - & 0 \\ - & 0 & + \\ + & 0 & + \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & + \\ + & 0 & + \end{bmatrix} + \begin{bmatrix} 0 & - & 0 \\ - & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, one can easily check that the matrix B_C below is irreducible.

$$B_C = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ + & 0 & + \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & + \\ + & 0 & + \end{bmatrix} + \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Recall that a matrix $X \in M_n$ is algebraically positive if and only if $X + \alpha I_n$ is algebraically positive for any $\alpha \in \mathbb{R}$. We now introduce the notion of a *scalar-shift subclass of a sign pattern matrix*. We say that a sign pattern B is a (scalar-shift) subclass of another sign pattern A, denoted by $B \trianglelefteq A$, if for any $X \in Q(B)$, there exists $\alpha \in \mathbb{R}$ such that the sign pattern of $X + \alpha I_n$ is equivalent to A.

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Example 2.4. Let $B = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ and $A = \begin{bmatrix} - & + \\ + & - \end{bmatrix}$. Then $B \trianglelefteq A$ but $A \ntrianglelefteq B$.

The following theorem follows directly from Theorem 1.3.4.

Theorem 2.5. *Suppose $A \trianglelefteq B$.*

1. *If B requires algebraic positivity, then so does A.*
2. *If A allows algebraic positivity, then so does B.*

Note that the converse of each statement in Theorem 2.5 does not hold, as shown in the following examples.

Example 2.6. Let

$$A = \begin{bmatrix} 0 & - & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix}, \quad B = \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & + \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}.$$

Clearly, $A, C \trianglelefteq B$.

1. Note that C requires algebraic positivity (see SP 12.4 in Appendix A), while B only allows but does not require algebraic positivity.
2. A does not allow algebraic positivity (see SP 3.3 in Appendix A) even though B allows algebraic positivity.

Finally, in Appendix A, we list down all nonequivalent irreducible 3×3 sign pattern matrices and determine whether they require, allow or do not allow algebraic positivity.

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Appendix A. Nonequivalent irreducible 3-by-3 sign pattern matrices. We will say that a given sign pattern matrix is (a) RAP if it requires algebraic positivity; (b) AAP if it does not require but allows algebraic positivity; (c) DNA if it does not allow algebraic positivity.

In this section, we wish to classify each possible 3×3 sign pattern matrix into one of the three types above. Since reducible matrices belong to the DNA category, we will only consider irreducible matrices. Moreover, we will only list nonequivalent sign pattern matrices. For efficiency, we list the nonequivalent digraphs and group them according to their edge count. Then for each digraph, we enumerate all corresponding nonequivalent sign pattern matrices.

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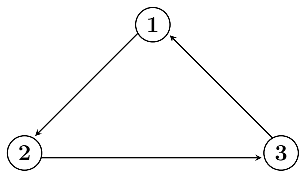
A. **Graphs with 3 directed edges**. Up to equivalence, there is only one strongly connected digraph with 3 edges and there are 2 nonequivalent sign pattern matrices having this digraph. We classify these sign patterns using Theorems 1.2.2 and 1.2.3 to, as indicated below.



Digraph 1

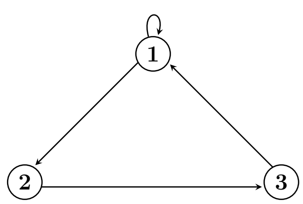
$$\begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 1.1: **RAP**

$$\begin{bmatrix} 0 & - & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 1.2: **DNA**

B. **Graphs with 4 directed edges**. There are three nonequivalent strongly connected digraphs with 4 directed edges. We list the nonequivalent sign pattern matrices corresponding to each digraph and except for SP 4.4, we use Theorems 1.2.2 and 1.2.3 to classify these sign patterns as indicated below.



Digraph 2

$$\begin{bmatrix} * & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

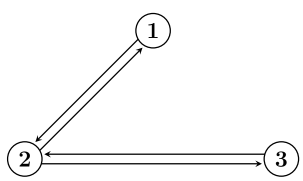
SP 2.1: **RAP**

$$\begin{bmatrix} * & - & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 2.2: **DNA**

$$\begin{bmatrix} * & + & 0 \\ 0 & 0 & - \\ + & 0 & 0 \end{bmatrix}$$

SP 2.3: **DNA**



Digraph 3

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$$

SP 3.1: **RAP**

$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}$$

SP 3.2: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix}$$

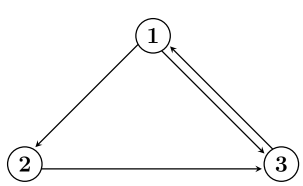
SP 3.3: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

SP 3.4: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & + \\ 0 & - & 0 \end{bmatrix}$$

SP 3.5: **DNA**



Digraph 4

$$\begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 4.1: **RAP**

$$\begin{bmatrix} 0 & + & * \\ 0 & 0 & + \\ - & 0 & 0 \end{bmatrix}$$

SP 4.2: **DNA**

$$\begin{bmatrix} 0 & - & * \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 4.3: **DNA**

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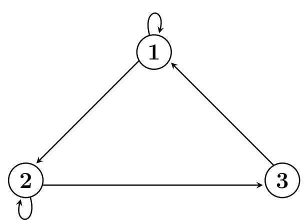
$$\begin{bmatrix} 0 & + & - \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & - & + \\ 0 & 0 & - \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & - & + \\ 0 & 0 & + \\ - & 0 & 0 \end{bmatrix}$$

SP 4.4: **RAP** SP 4.5: **DNA** SP 4.6: **DNA**

Suppose that the sign pattern of $A = [a_{ij}]$ is SP 4.4. Take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $\frac{k_1}{k_2} > \frac{a_{12}a_{23}}{-a_{13}}$ and $k_0 > -k_2a_{13}a_{31}$. Then $p(A) > 0$.

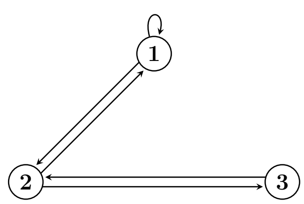
C. **Graphs with 5 directed edges**. There are six nonequivalent strongly connected digraphs with 5 directed edges. We list the nonequivalent sign pattern matrices corresponding to each digraph. Except for SP 6.4, SP 8.4, SP 9.2, SP 10.4, and the AAP sign patterns, we use Theorems 1.2.2, 1.2.3 and 2.1 to classify these sign patterns as indicated below.



Digraph 5

$$\begin{bmatrix} * & + & 0 \\ 0 & * & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ 0 & * & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & + & 0 \\ 0 & * & - \\ + & 0 & 0 \end{bmatrix}$$

SP 5.1: **RAP** SP 5.2: **DNA** SP 5.3: **DNA**



Digraph 6

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix} \qquad \begin{bmatrix} * & + & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

SP 6.1: **RAP** SP 6.2: **DNA** SP 6.3: **DNA**

$$\begin{bmatrix} + & - & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix} \qquad \begin{bmatrix} + & + & 0 \\ + & 0 & - \\ 0 & - & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

SP 6.4: **RAP** SP 6.5: **DNA** SP 6.6: **DNA**

Suppose that the sign pattern of $A = [a_{ij}]$ is SP 6.4. Take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1 > 0 > k_2$ and $\frac{k_1}{k_2} > -a_{11}$, $k_0 > \max\{-k_2(a_{11}^2 + a_{12}a_{21}) - k_1a_{11}, -k_2(a_{12}a_{21} + a_{31}a_{23}), -k_2a_{31}a_{23}\}$. Then $p(A) > 0$.

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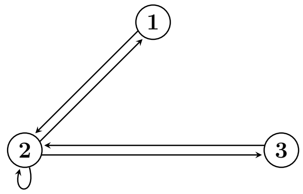


Digraph 7

$$\begin{bmatrix} 0 & + & 0 \\ + & * & + \\ 0 & + & 0 \end{bmatrix}$$

SP 7.1: **RAP**

$$\begin{bmatrix} 0 & - & 0 \\ + & * & + \\ 0 & + & 0 \end{bmatrix}$$

SP 7.2: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ - & * & + \\ 0 & + & 0 \end{bmatrix}$$

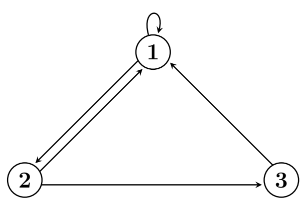
SP 7.3: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ + & * & - \\ 0 & + & 0 \end{bmatrix}$$

SP 7.4: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ + & * & + \\ 0 & - & 0 \end{bmatrix}$$

SP 7.5: **DNA**



Digraph 8

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 8.1: **RAP**

$$\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 8.2: **AAP**

$$\begin{bmatrix} * & - & 0 \\ * & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 8.3: **DNA**

$$\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ + & 0 & 0 \end{bmatrix}$$

SP 8.4: **RAP**

$$\begin{bmatrix} * & + & 0 \\ * & 0 & - \\ + & 0 & 0 \end{bmatrix}$$

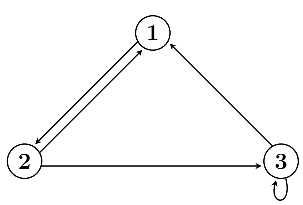
SP 8.5: **DNA**

$$\begin{bmatrix} * & + & 0 \\ * & 0 & + \\ - & 0 & 0 \end{bmatrix}$$

SP 8.6: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 8. If the sign pattern of A is

- SP 8.2, then A is AP if and only if $a_{11}a_{21} + a_{23}a_{31} > 0$. In particular, if we take $a_{11} = a_{12} = a_{23} = a_{31} = -a_{21} = 1$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = -a_{21} = 1$ and $a_{23} = a_{31} = 10$, then A is AP.
- SP 8.4, take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $\frac{k_1}{k_2} > -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}}$ and $k_0 > -k_1a_{11} - k_2(a_{11}^2 + a_{12}a_{21})$ so that $p(A) > 0$.



Digraph 9

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ + & 0 & * \end{bmatrix}$$

SP 9.1: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & 0 & + \end{bmatrix}$$

SP 9.2: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & 0 & - \end{bmatrix}$$

SP 9.3: **AAP**

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$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & + \\ + & 0 & * \end{bmatrix}$$
SP 9.4: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ + & 0 & * \end{bmatrix}$$
SP 9.5: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ - & 0 & + \end{bmatrix}$$
SP 9.6: **AAP**

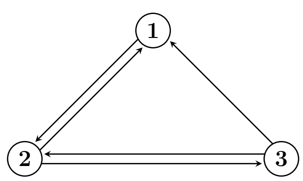
$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ + & 0 & * \end{bmatrix}$$
SP 9.7: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ - & 0 & + \\ + & 0 & + \end{bmatrix}$$
SP 9.8: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & - \\ + & 0 & * \end{bmatrix}$$
SP 9.9: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 9. If the sign pattern of A is

- SP 9.2, take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $\frac{k_1}{k_2} > \frac{a_{31}a_{23}}{-a_{21}}$ and $k_0 > -k_2a_{21}a_{12}$ so that $p(A) > 0$.
- SP 9.3, then A is AP if and only if $a_{21}a_{33} < a_{23}a_{31}$. In particular, if we take $a_{12} = a_{23} = a_{31} = -a_{21} = -a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{23} = a_{31} = 2$ and $a_{21} = a_{33} = -1$, then A is AP.
- SP 9.6, then A is AP if and only if $a_{21}a_{33} > a_{23}a_{31}$. In particular, if we take $a_{12} = -a_{23} = -a_{31} = a_{21} = a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{21} = a_{33} = 2$ and $a_{23} = a_{31} = -1$, then A is AP.



Digraph 10

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 10.1: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$$
SP 10.2: **AAP**

$$\begin{bmatrix} 0 & + & 0 \\ * & 0 & - \\ * & * & 0 \end{bmatrix}$$
SP 10.3: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & - & 0 \end{bmatrix}$$
SP 10.4: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$$
SP 10.5: **AAP**

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ - & + & 0 \end{bmatrix}$$
SP 10.6: **DNA**

$$\begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 10.7: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 10. If the sign pattern of A is

- SP 10.2, then A is AP if and only if $a_{21}a_{32}^2 < a_{12}a_{31}^2$. In particular, if we take $a_{12} = a_{21} = a_{23} = a_{31} = -a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{21} = a_{23} = a_{31} = 2$ and $a_{32} = -1$, then A is AP.
- SP 10.4, take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $\frac{k_1}{k_2} > \max\left\{\frac{a_{31}a_{12}}{-a_{32}}, \frac{a_{31}a_{23}}{-a_{21}}\right\}$ and $k_0 > -k_2(a_{12}a_{21} + a_{23}a_{32})$ so that $p(A) > 0$.

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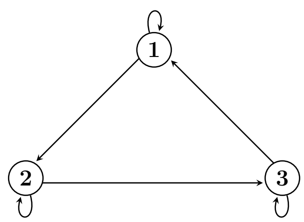
- SP 10.5, then A is AP if and only if $a_{23}a_{31}^2 < a_{21}^2a_{32}$ and $a_{21}a_{31}^2 < a_{21}a_{32}^2$. In particular, if we take $a_{12} = a_{21} = a_{23} = -a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{21} = a_{23} = a_{32} = 2$ and $a_{31} = -1$, then A is AP.

D. **Graphs with 6 directed edges**. There are eight nonequivalent strongly connected digraphs with 6 directed edges. We list the nonequivalent sign pattern matrices corresponding to each digraph and except for SP 12.4, SP 13.4, SP 14.4, SP 15.4, SP 17.4, SP 18.4, and the AAP sign patterns, we use Theorems 1.2.2, 1.2.3 and 2.1 to classify these sign patterns as indicated below.



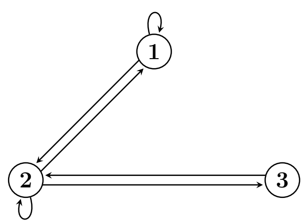
Digraph 11

$$\begin{bmatrix} * & + & 0 \\ 0 & * & + \\ + & 0 & * \end{bmatrix}$$

SP 11.1: **RAP**

$$\begin{bmatrix} * & - & 0 \\ 0 & * & + \\ + & 0 & * \end{bmatrix}$$

SP 11.2: **DNA**



Digraph 12

$$\begin{bmatrix} * & + & 0 \\ + & * & + \\ 0 & + & 0 \end{bmatrix}$$

SP 12.1: **RAP**

$$\begin{bmatrix} * & + & 0 \\ - & * & + \\ 0 & + & 0 \end{bmatrix}$$

SP 12.2: **DNA**

$$\begin{bmatrix} * & + & 0 \\ * & * & + \\ 0 & - & 0 \end{bmatrix}$$

SP 12.3: **DNA**

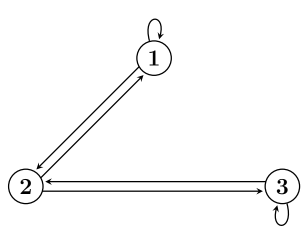
$$\begin{bmatrix} + & - & 0 \\ - & * & + \\ 0 & + & 0 \end{bmatrix}$$

SP 12.4: **RAP**

$$\begin{bmatrix} + & + & 0 \\ + & * & - \\ 0 & - & 0 \end{bmatrix}$$

SP 12.5: **DNA**

Suppose that the sign pattern of $A = [a_{ij}]$ is 12.4. Then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_2 < 0$, $-(a_{11} + a_{22}) < \frac{k_1}{k_2} < -a_{22}$ and k_0 be larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.



Digraph 13

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ 0 & + & * \end{bmatrix}$$

SP 13.1: **RAP**

$$\begin{bmatrix} + & + & 0 \\ + & 0 & - \\ 0 & - & + \end{bmatrix}$$

SP 13.2: **AAP**

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ 0 & - & * \end{bmatrix}$$

SP 13.3: **DNA**

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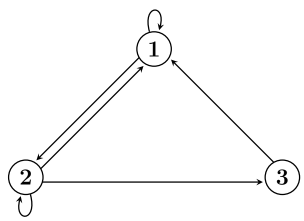
$$\begin{bmatrix} + & - & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ + & 0 & + \\ 0 & - & * \end{bmatrix} \qquad \begin{bmatrix} - & - & 0 \\ - & 0 & + \\ 0 & + & + \end{bmatrix}$$

SP 13.4: **RAP** SP 13.5: **DNA** SP 13.6: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 13. If the sign pattern of A is

- SP 13.2, then A is AP if and only if $a_{33} > a_{11}$. In particular, if we take $a_{11} = a_{12} = a_{21} = -a_{23} = -a_{32} = a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = a_{21} = -a_{23} = -a_{32} = 1$ and $a_{33} = 3$, then A is AP.
- SP 13.4, then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_2 < 0$, $-a_{11} < \frac{k_1}{k_2} < -a_{33}$ and k_0 be larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.



Digraph 14

$$\begin{bmatrix} * & + & 0 \\ + & * & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} + & + & 0 \\ - & + & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} - & + & 0 \\ - & + & + \\ + & 0 & 0 \end{bmatrix}$$

SP 14.1: **RAP** SP 14.2: **AAP** SP 14.3: **AAP**

$$\begin{bmatrix} - & + & 0 \\ - & - & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} + & - & 0 \\ - & + & + \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ + & * & + \\ + & 0 & 0 \end{bmatrix}$$

SP 14.4: **RAP** SP 14.5: **AAP** SP 14.6: **DNA**

$$\begin{bmatrix} * & + & 0 \\ + & * & - \\ + & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & 0 \\ + & * & + \\ - & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & + & 0 \\ + & * & - \\ - & 0 & 0 \end{bmatrix}$$

SP 14.7: **DNA** SP 14.8: **DNA** SP 14.9: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 14. If the sign pattern of A is

- SP 14.2, then A is AP if and only if $a_{11}a_{21}, a_{22}a_{21} > -a_{23}a_{31}$. In particular, if we take $a_{11} = a_{12} = -a_{21} = a_{22} = a_{23} = a_{31} = 1$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = a_{22} = a_{23} = a_{31} = 2$ and $a_{21} = -1$, then A is AP.
- SP 14.3, then A is AP if and only if $a_{22}a_{21} > -a_{23}a_{31}$. In particular, if we take $-a_{11} = a_{12} = -a_{21} = a_{22} = a_{23} = a_{31} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{22} = a_{23} = a_{31} = 2$ and $a_{11} = a_{21} = -1$, then A is AP.
- SP 14.4, then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $-(a_{11} + a_{22}) < \frac{k_1}{k_2} < -(a_{11} + a_{22}) - \frac{a_{23}a_{31}}{a_{21}}$ and k_0 be larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.
- SP 14.5, then A is AP if and only if $a_{11}a_{21}, a_{22}a_{21} < -a_{23}a_{31}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = a_{22} = a_{23} = a_{31} = 1$, then A is not AP. On the other hand, if we take $a_{11} = -a_{12} = a_{22} = a_{23} = a_{31} = 1$ and $a_{21} = -2$, then A is AP.

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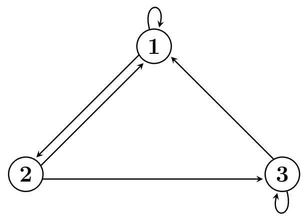

Digraph 15

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ + & 0 & * \end{bmatrix}$$

SP 15.1: **RAP**

$$\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ + & 0 & + \end{bmatrix}$$

SP 15.2: **AAP**

$$\begin{bmatrix} * & + & 0 \\ * & 0 & - \\ + & 0 & * \end{bmatrix}$$

SP 15.3: **DNA**

$$\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ + & 0 & + \end{bmatrix}$$

SP 15.4: **RAP**

$$\begin{bmatrix} + & + & 0 \\ - & 0 & + \\ + & 0 & - \end{bmatrix}$$

SP 15.5: **AAP**

$$\begin{bmatrix} * & - & 0 \\ + & 0 & + \\ + & 0 & * \end{bmatrix}$$

SP 15.6: **DNA**

$$\begin{bmatrix} - & + & 0 \\ - & 0 & + \\ + & 0 & - \end{bmatrix}$$

SP 15.7: **AAP**

$$\begin{bmatrix} + & + & 0 \\ + & 0 & - \\ - & 0 & + \end{bmatrix}$$

SP 15.8: **AAP**

$$\begin{bmatrix} * & + & 0 \\ * & 0 & + \\ - & 0 & * \end{bmatrix}$$

SP 15.9: **DNA**

$$\begin{bmatrix} - & + & 0 \\ + & 0 & - \\ - & 0 & + \end{bmatrix}$$

SP 15.10: **AAP**

$$\begin{bmatrix} * & + & 0 \\ + & 0 & - \\ - & 0 & - \end{bmatrix}$$

SP 15.11: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 15. If the sign pattern of A is

- SP 15.2, then A is AP if and only if $a_{11}a_{21} - a_{33}a_{21} > -a_{23}a_{31}$. In particular, if we take $a_{12} = -a_{21} = a_{23} = a_{31} = a_{33} = 1$ and $a_{11} = 2$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = a_{33} = 1$, then A is AP.
- SP 15.4, then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $-a_{11} < \frac{k_1}{k_2} < -a_{11} - \frac{a_{23}a_{31}}{a_{21}}$ and k_0 be larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.
- SP 15.5, then A is AP if and only if $a_{11}a_{21} - a_{33}a_{21} > -a_{23}a_{31}$. In particular, if we take $a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = -a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = a_{23} = a_{31} = 2$ and $a_{21} = a_{33} = -1$, then A is AP.
- SP 15.7, then A is AP if and only if $a_{33}a_{21} < a_{23}a_{31}$. In particular, if we take $-a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = -a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{12} = a_{23} = a_{31} = 2$ and $a_{11} = a_{21} = a_{33} = -1$, then A is AP.
- SP 15.8, then A is AP if and only if $a_{11}a_{21} - a_{33}a_{21} < -a_{23}a_{31}$. In particular, if we take $a_{11} = a_{12} = a_{21} = -a_{23} = -a_{31} = a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = a_{21} = -a_{23} = -a_{31} = 1$ and $a_{33} = 3$, then A is AP.
- SP 15.10, then A is AP if and only if $-a_{33}a_{21} < -a_{23}a_{31}$. In particular, if we take $-a_{11} = a_{12} = a_{21} = -a_{23} = -a_{31} = a_{33} = 1$, then A is not AP. On the other hand, if we take $-a_{11} = a_{12} = a_{21} = -a_{23} = -a_{31} = 1$ and $a_{33} = 2$, then A is AP.

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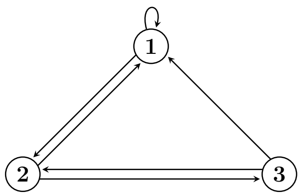

Digraph 16

$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ SP 16.1: **RAP**

$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$ SP 16.2: **AAP**

$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$ SP 16.3: **AAP**

$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$ SP 16.4: **AAP**

$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ + & - & 0 \end{bmatrix}$ SP 16.5: **AAP**

$\begin{bmatrix} + & - & 0 \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$ SP 16.6: **AAP**

$\begin{bmatrix} + & - & 0 \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$ SP 16.7: **AAP**

$\begin{bmatrix} + & - & 0 \\ - & 0 & + \\ - & + & 0 \end{bmatrix}$ SP 16.8: **AAP**

$\begin{bmatrix} - & - & 0 \\ * & 0 & + \\ * & * & 0 \end{bmatrix}$ SP 16.9: **DNA**

$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ - & - & 0 \end{bmatrix}$ SP 16.10: **DNA**

$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ - & + & 0 \end{bmatrix}$ SP 16.11: **DNA**

$\begin{bmatrix} * & - & 0 \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$ SP 16.12: **DNA**

$\begin{bmatrix} + & - & 0 \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ SP 16.13: **DNA**

$\begin{bmatrix} + & - & 0 \\ * & 0 & + \\ * & - & 0 \end{bmatrix}$ SP 16.14: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 16. If the sign pattern of A is

- SP 16.2, then A is AP if and only if $-\frac{a_{12}a_{31}}{a_{32}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}} < -a_{11} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{23} = a_{31} = a_{32} = 2$ and $a_{21} = -1$, then A is AP.
- SP 16.3, then A is AP if and only if $-\frac{a_{12}a_{31}}{a_{32}}, -a_{11} - \frac{a_{23}a_{31}}{a_{21}} < -a_{11} - \frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = -a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = a_{32} = 2$ and $a_{21} = -1$, then A is AP.
- SP 16.4, then A is AP if and only if $-a_{11} - \frac{a_{21}a_{32}}{a_{31}} < -\frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = -a_{31} = 1$ and $a_{32} = -3$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = a_{31} = 2$ and $a_{32} = -1$, then A is AP.
- SP 16.5, then A is AP if and only if $-a_{11}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}} < -\frac{a_{12}a_{31}}{a_{32}}, -a_{11} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = -a_{32} = 1$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{23} = a_{31} = 2$ and $a_{21} = a_{32} = -1$, then A is AP.

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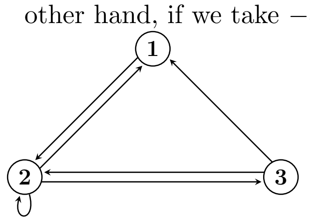
- SP 16.6, then A is AP if and only if $-a_{11}, -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}} < -\dfrac{a_{12}a_{31}}{a_{32}}, -a_{11} - \dfrac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{11} = -a_{12} = -a_{21} = a_{31} = a_{32} = 1$ and $a_{23} = 0.1$, then A is AP.
- SP 16.7, then A is AP if and only if $-a_{11}, -a_{11} - \dfrac{a_{21}a_{32}}{a_{31}} < -\dfrac{a_{12}a_{31}}{a_{32}}, -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $-a_{12} = a_{21} = a_{23} = -a_{31} = 1$, $a_{11} = 11$, and $a_{32} = 0.1$, then A is AP.
- SP 16.8, then A is AP if and only if $-a_{11}, -a_{11} - \dfrac{a_{21}a_{32}}{a_{31}}, -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}} < -\dfrac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = a_{23} = -a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $-a_{12} = -a_{21} = a_{23} = -a_{31} = 1$, $a_{11} = 11$, and $a_{32} = 0.1$, then A is AP.



Digraph 17

$$\begin{bmatrix} 0 & + & 0 \\ + & * & + \\ + & + & 0 \end{bmatrix}$$

SP 17.1: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ - & * & + \\ + & + & 0 \end{bmatrix}$$

SP 17.2: **AAP**

$$\begin{bmatrix} 0 & + & 0 \\ * & * & - \\ * & * & 0 \end{bmatrix}$$

SP 17.3: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ - & - & + \\ + & - & 0 \end{bmatrix}$$

SP 17.4: **RAP**

$$\begin{bmatrix} 0 & + & 0 \\ + & * & + \\ - & + & 0 \end{bmatrix}$$

SP 17.5: **AAP**

$$\begin{bmatrix} 0 & + & 0 \\ - & * & + \\ - & + & 0 \end{bmatrix}$$

SP 17.6: **DNA**

$$\begin{bmatrix} 0 & + & 0 \\ - & + & + \\ + & - & 0 \end{bmatrix}$$

SP 17.7: **AAP**

$$\begin{bmatrix} 0 & - & 0 \\ + & * & - \\ + & + & 0 \end{bmatrix}$$

SP 17.8: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 17. If the sign pattern of A is

- SP 17.2, then A is AP if and only if $-a_{22}, -\dfrac{a_{21}a_{32}}{a_{31}}, -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -a_{33} - \dfrac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{21} = -3$, then A is not AP. On the other hand, if we take $a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 2$ and $a_{21} = -1$, then A is AP.
- SP 17.4, then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $-a_{22} < \frac{k_1}{k_2} < -a_{22} - \max\{\frac{a_{23}a_{31}}{a_{21}}, \frac{a_{31}a_{12}}{a_{32}}\}$ and k_0 be larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.
- SP 17.5, then A is AP if and only if $-a_{22}, -a_{33} - \dfrac{a_{23}a_{31}}{a_{21}}, -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -\dfrac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$ and $a_{31} = -3$, then A is not AP. On the other hand, if we take $a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 2$ and $a_{31} = -1$, then A is AP.
- SP 17.7, then A is AP if and only if $-a_{22}, -\dfrac{a_{21}a_{32}}{a_{31}} < -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}}, -a_{33} - \dfrac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $a_{12} = -a_{21} = a_{23} = a_{31} = -a_{32} = 1$ and $a_{22} = -5$, then A is not AP. On the other hand, if we take $a_{12} = a_{22} = a_{23} = a_{31} = 2$ and $a_{21} = a_{32} = -1$, then A is AP.

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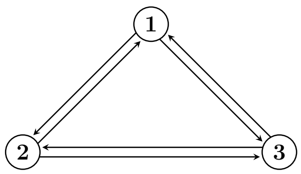


Digraph 18

$$\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 18.1: **RAP**

$$\begin{bmatrix} 0 & - & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 18.2: **AAP**

$$\begin{bmatrix} 0 & - & + \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$$
SP 18.3: **DNA**

$$\begin{bmatrix} 0 & - & + \\ + & 0 & - \\ - & + & 0 \end{bmatrix}$$
SP 18.4: **RAP**

$$\begin{bmatrix} 0 & - & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 18.5: **AAP**

$$\begin{bmatrix} 0 & - & - \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 18.6: **DNA**

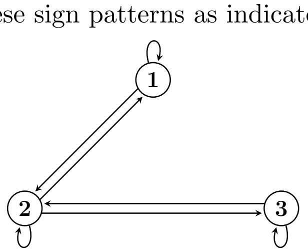
$$\begin{bmatrix} 0 & - & + \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 18.7: **AAP**

$$\begin{bmatrix} 0 & - & - \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 18.8: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 18. If the sign pattern of A is

- SP 18.2, then A is AP if and only if $-\frac{a_{12}a_{23}}{a_{13}}, -\frac{a_{23}a_{31}}{a_{21}}, -\frac{a_{21}a_{13}}{a_{23}}, -\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $-a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 2$ and $a_{12} = -1$, then A is AP.
- SP 18.4, then take $p(x) = x^2 + k_0$ such that $k_0 > -a_{12}a_{21} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32}$ and $k_0 > -a_{32}a_{23} - a_{13}a_{31}$ so that $p(A) > 0$.
- SP 18.5, then A is AP if and only if $-\frac{a_{12}a_{23}}{a_{13}}, -\frac{a_{21}a_{13}}{a_{23}}, -\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -\frac{a_{13}a_{32}}{a_{12}}, -\frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $-a_{12} = a_{13} = -a_{21} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{13} = a_{23} = a_{31} = a_{32} = 2$ and $a_{12} = a_{21} = -1$, then A is AP.
- SP 18.7, then A is AP if and only if $-\frac{a_{12}a_{23}}{a_{13}}, -\frac{a_{23}a_{31}}{a_{21}}, -\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -\frac{a_{13}a_{32}}{a_{12}}, -\frac{a_{21}a_{13}}{a_{23}}$. In particular, if we take $-a_{12} = a_{13} = a_{21} = -a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $a_{13} = a_{21} = a_{31} = a_{32} = 2$ and $a_{12} = a_{23} = -1$, then A is AP.

E. **Graphs with 7 directed edges**. There are five nonequivalent strongly connected digraphs with 7 directed edges. We list the nonequivalent sign pattern matrices corresponding to each digraph and except for SP 19.4, 20.4, and the AAP sign patterns, we use Theorems 1.2.2, 1.2.3 and 2.1 to classify these sign patterns as indicated below.



Digraph 19

$$\begin{bmatrix} * & + & 0 \\ + & * & + \\ 0 & + & * \end{bmatrix}$$
SP 19.1: **RAP**

$$\begin{bmatrix} + & - & 0 \\ - & * & + \\ 0 & + & + \end{bmatrix}$$
SP 19.2: **AAP**

$$\begin{bmatrix} * & - & 0 \\ + & * & * \\ 0 & * & * \end{bmatrix}$$
SP 19.3: **DNA**

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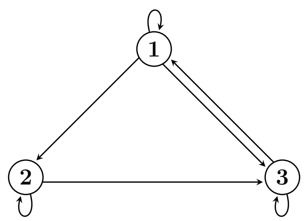
$$\begin{bmatrix} - & + & 0 \\ + & * & - \\ 0 & - & + \end{bmatrix} \qquad \begin{bmatrix} + & + & 0 \\ + & * & - \\ 0 & - & - \end{bmatrix}$$

SP 19.4: **RAP** SP 19.5: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 19. If the sign pattern of A is

- SP 19.2, then A is AP if and only if $a_{11} > a_{33}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = \pm a_{22} = a_{23} = a_{32} = a_{33} = 1$, then A is not AP. On the other hand, if we take $-a_{12} = -a_{21} = \pm a_{22} = a_{23} = a_{32} = a_{33} = 1$ and $a_{11} = 2$, then A is AP.
- SP 19.4, take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_2 < 0$, $-(a_{22} + a_{33}) < \frac{k_1}{k_2} < -(a_{11} + a_{22})$ and k_0 larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.



Digraph 20

$$\begin{bmatrix} * & + & + \\ 0 & * & + \\ + & 0 & * \end{bmatrix} \qquad \begin{bmatrix} + & + & - \\ 0 & + & + \\ + & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & + & * \\ 0 & * & - \\ * & 0 & * \end{bmatrix}$$

SP 20.1: **RAP** SP 20.2: **AAP** SP 20.3: **DNA**

$$\begin{bmatrix} - & + & - \\ 0 & + & + \\ + & 0 & - \end{bmatrix} \qquad \begin{bmatrix} * & + & - \\ 0 & - & + \\ + & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & + & + \\ 0 & * & + \\ - & 0 & * \end{bmatrix}$$

SP 20.4: **RAP** SP 20.5: **AAP** SP 20.6: **DNA**

$$\begin{bmatrix} + & + & - \\ 0 & + & + \\ - & 0 & + \end{bmatrix} \qquad \begin{bmatrix} * & - & + \\ 0 & + & - \\ + & 0 & * \end{bmatrix} \qquad \begin{bmatrix} - & + & - \\ 0 & + & + \\ - & 0 & * \end{bmatrix}$$

SP 20.7: **AAP** SP 20.8: **AAP** SP 20.9: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 20. If the sign pattern of A is

- SP 20.2, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33}, -a_{11} - a_{33} < -a_{11} - a_{33} - \dfrac{a_{12}a_{23}}{a_{13}}$. In particular, if we take $a_{12} = a_{22} = a_{23} = a_{31} = 1$, $a_{11} = \pm a_{33} = 2$, and $a_{13} = -5$, then A is not AP. On the other hand, if we take $a_{11} = a_{12} = -a_{13} = a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{22} = 3$, then A is AP.
- SP 20.4, then take $p(x) = k_2x^2 + k_1x + k_0$ such that $k_1, k_2 > 0$, $-(a_{11} + a_{33}) \leq \frac{k_1}{k_2} \leq -(a_{11} + a_{33}) - \frac{a_{12}a_{23}}{a_{13}}$ and k_0 is larger than all the diagonal entries of $-k_1A - k_2A^2$ so that $p(A) > 0$.
- SP 20.5, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33}, -a_{11} - a_{33} < -a_{11} - a_{33} - \dfrac{a_{12}a_{23}}{a_{13}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{13} = a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{22} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = -a_{22} = a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{13} = -0.1$, then A is AP.
- SP 20.7, then A is AP if and only if $-a_{11} - a_{33}, -a_{11} - a_{33} - \dfrac{a_{12}a_{23}}{a_{13}} < -a_{11} - a_{22}, -a_{22} - a_{33}$.

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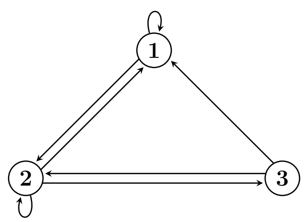
In particular, if we take $a_{11} = a_{12} = -a_{13} = a_{22} = a_{23} = -a_{31} = a_{33} = 1$, then A is not AP. On the other hand, if we take $\pm a_{12} = a_{22} = a_{23} = -a_{31} = 1$ and $a_{11} = -a_{13} = a_{33} = 5$, then A is AP.

- SP 20.8, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33} < -a_{11} - a_{33}, -a_{11} - a_{33} - \dfrac{a_{12}a_{23}}{a_{13}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{22} = -a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{13} = 0.1$, then A is not AP. On the other hand, if we take $\pm a_{11} = -a_{12} = a_{13} = -a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{22} = 5$, then A is AP.



Digraph 21

$\begin{bmatrix} * & + & 0 \\ + & * & + \\ + & + & 0 \end{bmatrix}$ SP 21.1: **RAP**

$\begin{bmatrix} * & + & 0 \\ - & * & + \\ + & + & 0 \end{bmatrix}$ SP 21.2: **AAP**

$\begin{bmatrix} * & + & 0 \\ + & * & + \\ + & - & 0 \end{bmatrix}$ SP 21.3: **AAP**

$\begin{bmatrix} * & + & 0 \\ + & * & + \\ - & + & 0 \end{bmatrix}$ SP 21.4: **AAP**

$\begin{bmatrix} + & - & 0 \\ - & * & + \\ + & + & 0 \end{bmatrix}$ SP 21.5: **AAP**

$\begin{bmatrix} + & - & 0 \\ + & * & + \\ - & + & 0 \end{bmatrix}$ SP 21.6: **AAP**

$\begin{bmatrix} * & + & 0 \\ - & * & + \\ + & - & 0 \end{bmatrix}$ SP 21.7: **AAP**

$\begin{bmatrix} - & + & 0 \\ + & - & - \\ - & + & 0 \end{bmatrix}$ SP 21.8: **AAP**

$\begin{bmatrix} - & + & 0 \\ + & * & - \\ + & - & 0 \end{bmatrix}$ SP 21.9: **AAP**

$\begin{bmatrix} * & - & 0 \\ + & * & + \\ + & + & 0 \end{bmatrix}$ SP 21.10: **DNA**

$\begin{bmatrix} * & + & 0 \\ + & * & - \\ + & + & 0 \end{bmatrix}$ SP 21.11: **DNA**

$\begin{bmatrix} * & - & 0 \\ + & * & - \\ + & + & 0 \end{bmatrix}$ SP 21.12: **DNA**

$\begin{bmatrix} * & - & 0 \\ + & * & + \\ + & - & 0 \end{bmatrix}$ SP 21.13: **DNA**

$\begin{bmatrix} * & + & 0 \\ - & * & - \\ + & + & 0 \end{bmatrix}$ SP 21.14: **DNA**

$\begin{bmatrix} * & + & 0 \\ - & * & + \\ - & + & 0 \end{bmatrix}$ SP 21.15: **DNA**

$\begin{bmatrix} * & + & 0 \\ + & * & + \\ - & - & 0 \end{bmatrix}$ SP 21.16: **DNA**

$\begin{bmatrix} - & - & 0 \\ - & * & + \\ + & + & 0 \end{bmatrix}$ SP 21.17: **DNA**

$\begin{bmatrix} - & - & 0 \\ + & * & + \\ - & + & 0 \end{bmatrix}$ SP 21.18: **DNA**

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$$\begin{bmatrix} + & + & 0 \\ + & * & - \\ - & + & 0 \end{bmatrix} \qquad \begin{bmatrix} - & + & 0 \\ + & + & - \\ - & + & 0 \end{bmatrix} \qquad \begin{bmatrix} + & + & 0 \\ + & * & - \\ + & - & 0 \end{bmatrix}$$

SP 21.19: **DNA** SP 21.20: **DNA** SP 21.21: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 21. If the sign pattern of A is

- SP 21.2, then A is AP if and only if $-a_{11}-a_{22}, -a_{22}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -a_{22}-\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{21} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{21} = -0.1$, then A is AP.
- SP 21.3, then A is AP if and only if $-a_{11}-a_{22}, -a_{22}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}} < -a_{22}-\frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11} = a_{12} = \pm a_{22} = a_{23} = a_{31} = -a_{32} = 1$ and $a_{21} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{31} = 1$ and $a_{32} = -0.1$, then A is AP.
- SP 21.4, then A is AP if and only if $-a_{11}-a_{22}, -a_{22}, -a_{22}-\frac{a_{12}a_{31}}{a_{32}}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}} < -a_{11}-\frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$ and $a_{31} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$ and $a_{31} = -0.1$, then A is AP.
- SP 21.5, then A is AP if and only if $-a_{11}-a_{22}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}} < -a_{22}, -a_{22}-\frac{a_{12}a_{31}}{a_{32}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $a_{11} = -a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{21} = -0.1$, then A is not AP. On the other hand, if we take $a_{11} = -a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{21} = -10$, then A is AP.
- SP 21.6, then A is AP if and only if $-a_{11}-a_{22}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}} < -a_{22}, -a_{22}-\frac{a_{12}a_{31}}{a_{32}}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $a_{11} = -a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$ and $a_{31} = -0.1$, then A is not AP. On the other hand, if we take $-a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$, $a_{11} = 10$, and $a_{31} = -5$, then A is AP.
- SP 21.7, then A is AP if and only if $-a_{22}, -a_{11}-a_{22}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}} < -a_{22}-\frac{a_{12}a_{31}}{a_{32}}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{21} = \pm a_{22} = a_{23} = -a_{32} = 1$ and $a_{31} = 0.1$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = -a_{21} = \pm a_{22} = a_{23} = -a_{32} = 1$ and $a_{31} = 10$, then A is AP.
- SP 21.8, then A is AP if and only if $-a_{22}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}} < -a_{11} - a_{22}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $-a_{11} = a_{21} = -a_{22} = -a_{23} = -a_{31} = a_{32} = 1$ and $a_{12} = 10$, then A is not AP. On the other hand, if we take $a_{21} = -a_{23} = -a_{31} = a_{32} = 1$, $a_{12} = -a_{22} = 10$, and $a_{11} = -3$, then A is AP.
- SP 21.8, then A is AP if and only if $-a_{22}, -a_{22}-\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-a_{22}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -a_{11}-a_{22}-\frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $-a_{11} = a_{12} = a_{21} = \pm a_{22} = -a_{23} = a_{31} = 1$ and $a_{32} = -0.1$, then A is not AP. On the other hand, if we take $-a_{11} = a_{12} = a_{21} = \pm a_{22} = -a_{23} = a_{31} = 1$ and $a_{32} = -10$, then A is AP.

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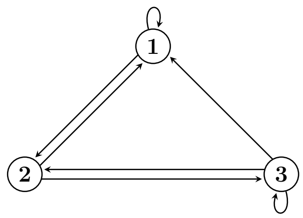

Digraph 22

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ + & + & * \end{bmatrix}$$
SP 22.1: **RAP**

$$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ + & + & * \end{bmatrix}$$
SP 22.2: **AAP**

$$\begin{bmatrix} * & + & 0 \\ + & 0 & + \\ - & + & * \end{bmatrix}$$
SP 22.3: **AAP**

$$\begin{bmatrix} - & - & 0 \\ - & 0 & + \\ + & + & - \end{bmatrix}$$
SP 22.4: **AAP**

$$\begin{bmatrix} + & - & 0 \\ - & 0 & + \\ + & + & * \end{bmatrix}$$
SP 22.5: **AAP**

$$\begin{bmatrix} + & - & 0 \\ + & 0 & + \\ - & + & * \end{bmatrix}$$
SP 22.6: **AAP**

$$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ + & - & * \end{bmatrix}$$
SP 22.7: **AAP**

$$\begin{bmatrix} * & - & 0 \\ + & 0 & + \\ + & + & * \end{bmatrix}$$
SP 22.8: **DNA**

$$\begin{bmatrix} * & - & 0 \\ + & 0 & - \\ + & + & * \end{bmatrix}$$
SP 22.9: **DNA**

$$\begin{bmatrix} * & - & 0 \\ + & 0 & + \\ + & - & * \end{bmatrix}$$
SP 22.10: **DNA**

$$\begin{bmatrix} * & + & 0 \\ - & 0 & + \\ - & + & * \end{bmatrix}$$
SP 22.11: **DNA**

$$\begin{bmatrix} - & - & 0 \\ - & 0 & + \\ + & + & + \end{bmatrix}$$
SP 22.12: **DNA**

$$\begin{bmatrix} - & - & 0 \\ + & 0 & + \\ - & + & * \end{bmatrix}$$
SP 22.13: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 22. If the sign pattern of A is

- SP 22.2, then A is AP if and only if $-a_{11}, -a_{33}, -a_{11} - a_{33} - \dfrac{a_{21}a_{32}}{a_{31}}, -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{21} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{21} = -0.1$, then A is AP.
- SP 22.3, then A is AP if and only if $-a_{11}, -a_{33}, -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}}, -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{33} - \dfrac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = a_{32} = \pm a_{33} = 1$ and $a_{31} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = a_{23} = a_{32} = \pm a_{33} = 1$ and $a_{31} = -0.1$, then A is AP.
- SP 22.4, then A is AP if and only if $-a_{11}, -a_{11} - \dfrac{a_{23}a_{31}}{a_{21}} < -a_{33}, -a_{11} - a_{33} - \dfrac{a_{21}a_{32}}{a_{31}}, -a_{33} - \dfrac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $-a_{11} = -a_{12} = -a_{21} = a_{23} = a_{31} = -a_{33} = 1$ and $a_{32} = 10$,

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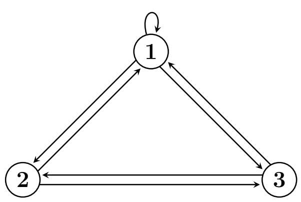
then A is not AP. On the other hand, if we take $-a_{11} = -a_{12} = -a_{21} = a_{23} = a_{31} = 1$, $a_{32} = 0.1$, and $a_{33} = -10$, then A is AP.

- SP 22.5, then A is AP if and only if $-a_{11}, -a_{11} - \frac{a_{23}a_{31}}{a_{21}} < -a_{33}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11} = -a_{12} = -a_{21} = a_{31} = \pm a_{33} = 1$ and $a_{23} = a_{32} = 10$, then A is not AP. On the other hand, if we take $-a_{12} = -a_{21} = a_{23} = a_{31} = \pm a_{33} = 1$, $a_{11} = 15$, and $a_{32} = 10$, then A is AP.
- SP 22.6, then A is AP if and only if $-a_{11}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}} < -a_{33}, -a_{11} - \frac{a_{23}a_{31}}{a_{21}}, -a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11} = -a_{12} = a_{21} = a_{23} = -a_{31} = \pm a_{33} = 1$ and $a_{32} = 10$, then A is not AP. On the other hand, if we take $a_{21} = a_{23} = \pm a_{33} = 1$, $a_{11} = 15$, $a_{31} = -5$ and $-a_{12} = a_{32} = 0.1$, then A is AP.
- SP 22.7, then A is AP if and only if $-a_{11}, -a_{33}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}} < -a_{11} - \frac{a_{23}a_{31}}{a_{21}}, -a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = 1$ and $-a_{32} = \pm a_{33} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{23} = a_{32} = -0.1$, then A is AP.



Digraph 23

$$\begin{bmatrix} * & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 23.1: **RAP**

$$\begin{bmatrix} * & - & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 23.2: **AAP**

$$\begin{bmatrix} * & + & + \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 23.3: **AAP**

$$\begin{bmatrix} * & - & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 23.4: **AAP**

$$\begin{bmatrix} * & + & + \\ + & 0 & - \\ + & - & 0 \end{bmatrix}$$
SP 23.5: **AAP**

$$\begin{bmatrix} * & - & + \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 23.6: **AAP**

$$\begin{bmatrix} * & - & + \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$$
SP 23.7: **AAP**

$$\begin{bmatrix} + & - & - \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 23.8: **AAP**

$$\begin{bmatrix} * & - & + \\ + & 0 & - \\ - & + & 0 \end{bmatrix}$$
SP 23.9: **AAP**

$$\begin{bmatrix} * & - & - \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$$
SP 23.10: **DNA**

$$\begin{bmatrix} * & - & + \\ + & 0 & + \\ + & - & 0 \end{bmatrix}$$
SP 23.11: **DNA**

$$\begin{bmatrix} * & - & + \\ - & 0 & - \\ + & + & 0 \end{bmatrix}$$
SP 23.12: **DNA**

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$$\begin{bmatrix} * & - & - \\ + & 0 & - \\ + & + & 0 \end{bmatrix} \qquad \begin{bmatrix} * & - & + \\ + & 0 & + \\ - & - & 0 \end{bmatrix} \qquad \begin{bmatrix} - & - & - \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$

SP 23.13: **DNA** SP 23.14: **DNA** SP 23.15: **DNA**

Suppose that the digraph of $A=[a_{ij}]\in M_3(\mathbb{R})$ is Digraph 23. If the sign pattern of A is

- SP 23.2, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -\frac{a_{13}a_{21}}{a_{23}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}$, and $-\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11}=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=1$ and $a_{12}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=1$ and $a_{12}=-0.1$, then A is AP.
- SP 23.3, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -\frac{a_{13}a_{21}}{a_{23}}$. In particular, if we take $\pm a_{11}=a_{12}=a_{13}=a_{21}=a_{31}=a_{32}=1$ and $a_{23}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{12}=a_{13}=a_{21}=a_{31}=a_{32}=1$ and $a_{23}=-0.1$, then A is AP.
- SP 23.4, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -\frac{a_{13}a_{21}}{a_{23}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11}=a_{13}=-a_{21}=a_{23}=a_{31}=a_{32}=1$ and $a_{12}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{13}=a_{23}=a_{31}=a_{32}=1$ and $a_{12}=a_{21}=-0.1$, then A is AP.
- SP 23.5, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}} < -\frac{a_{13}a_{21}}{a_{23}}, -\frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11}=a_{12}=a_{13}=a_{21}=a_{31}=-a_{32}=1$ and $a_{23}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{12}=a_{13}=a_{21}=a_{31}=1$, $a_{23}=-0.1$, and $a_{32}=-0.3$, then A is AP.
- SP 23.6, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{12}a_{31}}{a_{32}} < -\frac{a_{13}a_{21}}{a_{23}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11}=a_{13}=a_{21}=a_{31}=a_{32}=1$ and $a_{12}=a_{23}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{13}=a_{21}=a_{31}=a_{32}=1$, $a_{12}=-0.1$, and $a_{23}=-0.3$, then A is AP.
- SP 23.7, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -\frac{a_{13}a_{21}}{a_{23}}, -\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11}=a_{13}=a_{21}=a_{23}=a_{32}=1$ and $a_{12}=a_{31}=-10$, then A is not AP. On the other hand, if we take $\pm a_{11}=a_{13}=a_{21}=a_{23}=a_{32}=1$, $a_{12}=-0.1$, and $a_{31}=-0.3$, then A is AP.
- SP 23.8, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}} < -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{13}a_{21}}{a_{23}}, -\frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11}=a_{12}=-a_{13}=a_{23}=a_{31}=a_{32}=1$ and $a_{21}=-10$, then A is not AP. On the other hand, if we take $-a_{12}=-a_{13}=a_{23}=a_{31}=a_{32}=1$, $a_{11}=20$, and $a_{21}=-10$, then A is AP.
- SP 23.9, then A is AP if and only if $-a_{11}-\frac{a_{12}a_{23}}{a_{13}}, -a_{11}-\frac{a_{23}a_{31}}{a_{21}}, -\frac{a_{12}a_{31}}{a_{32}} < -a_{11}-\frac{a_{21}a_{32}}{a_{31}}, -\frac{a_{13}a_{21}}{a_{23}}, -a_{11}-\frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $-a_{12}=a_{13}=a_{21}=-a_{23}=-a_{31}=a_{32}1$ and $\pm a_{11}=10$, then A is not AP. On the other hand, if we take $\pm a_{11}=-a_{12}=a_{13}=a_{21}=-a_{31}=a_{32}1$ and $a_{23}=10$, then A is AP.

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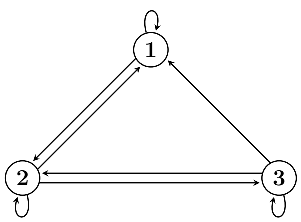
F. **Graphs with 8 directed edges**. There are two nonequivalent strongly connected digraphs with 8 directed edges. We list the nonequivalent sign pattern matrices corresponding to each digraph and except for the AAP sign patterns, we use Theorems 1.2.2, 1.2.3 and 2.1 to classify these sign patterns as indicated below.



Digraph 24

$$\begin{bmatrix} * & + & 0 \\ + & * & + \\ + & + & * \end{bmatrix}$$
SP 24.1: **RAP**

$$\begin{bmatrix} * & + & 0 \\ - & * & + \\ + & + & * \end{bmatrix}$$
SP 24.2: **AAP**

$$\begin{bmatrix} * & + & 0 \\ + & * & + \\ - & + & * \end{bmatrix}$$
SP 24.3: **AAP**

$$\begin{bmatrix} - & - & 0 \\ - & * & + \\ + & + & - \end{bmatrix}$$
SP 24.4: **AAP**

$$\begin{bmatrix} + & - & 0 \\ - & * & + \\ + & + & * \end{bmatrix}$$
SP 24.5: **AAP**

$$\begin{bmatrix} - & - & 0 \\ + & - & + \\ - & + & - \end{bmatrix}$$
SP 24.6: **AAP**

$$\begin{bmatrix} + & - & 0 \\ + & * & + \\ - & + & * \end{bmatrix}$$
SP 24.7: **AAP**

$$\begin{bmatrix} * & + & 0 \\ - & * & + \\ + & - & * \end{bmatrix}$$
SP 24.8: **AAP**

$$\begin{bmatrix} * & - & 0 \\ + & * & + \\ + & + & * \end{bmatrix}$$
SP 24.9: **DNA**

$$\begin{bmatrix} * & - & 0 \\ + & * & - \\ + & + & * \end{bmatrix}$$
SP 24.10: **DNA**

$$\begin{bmatrix} * & - & 0 \\ + & * & + \\ + & - & * \end{bmatrix}$$
SP 24.11: **DNA**

$$\begin{bmatrix} * & + & 0 \\ - & * & + \\ - & + & * \end{bmatrix}$$
SP 24.12: **DNA**

$$\begin{bmatrix} - & - & 0 \\ - & * & + \\ + & + & + \end{bmatrix}$$
SP 24.13: **DNA**

$$\begin{bmatrix} - & - & 0 \\ + & - & + \\ - & + & + \end{bmatrix}$$
SP 24.14: **DNA**

$$\begin{bmatrix} - & - & 0 \\ + & + & + \\ - & + & * \end{bmatrix}$$
SP 24.15: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 24. If the sign pattern of A is

- SP 24.2, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33}, -a_{11} - a_{33} - \dfrac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \dfrac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = a_{12} = \pm a_{22} = a_{23} = a_{31} = 1$ and $-a_{21} = a_{32} = \pm a_{33} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{21} = -0.1$, then A is AP.
- SP 24.3, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33}, -a_{11} - a_{22} - \dfrac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} - \dfrac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{33} - \dfrac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = a_{21} = \pm a_{22} = a_{23} = a_{32} = \pm a_{33} = 1$ and $a_{12} = -a_{31} = 10$, then A is not AP. On the other hand, if we take

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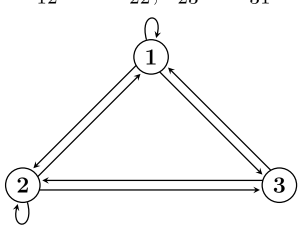
$\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = \pm a_{33} = 1$ and $a_{31} = -0.1$, then A is AP.

- SP 24.4, then A is AP if and only if $-a_{11} - a_{22}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}} < -a_{22} - a_{33}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $-a_{11} = -a_{12} = -a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = -a_{33} = 1$, then A is not AP. On the other hand, if we take $-a_{11} = -a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$, $a_{21} = -10$, and $a_{33} = -2$, then A is AP.
- SP 24.5, then A is AP if and only if $-a_{11} - a_{22}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}} < -a_{22} - a_{33}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11} = -a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{21} = -0.1$, then A is not AP. On the other hand, if we take $-a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$, $a_{21} = -10$, and $a_{11} = 2$, then A is AP.
- SP 24.6, then A is AP if and only if $-a_{11} - a_{22}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}} < -a_{22} - a_{33}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $-a_{11} = -a_{12} = a_{21} = -a_{22} = a_{23} = -a_{31} = a_{32} = -a_{33} = 1$, then A is not AP. On the other hand, if we take $-a_{11} = -a_{12} = a_{21} = a_{23} = 1$, $a_{22} = a_{31} = -10$, $a_{32} = 12$, and $a_{33} = 2$, then A is AP.
- SP 24.7, then A is AP if and only if $-a_{11} - a_{22}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}} < -a_{22} - a_{33}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $a_{11} = -a_{12} = a_{21} = \pm a_{22} = a_{23} = -a_{31} = a_{32} = \pm a_{33} = 1$, then A is not AP. On the other hand, if we take $a_{21} = \pm a_{22} = a_{23} = a_{32} = \pm a_{33} = 1$, $a_{31} = -10$, $a_{12} = -0.1$, and $a_{11} = 3$, then A is AP.
- SP 24.8, then A is AP if and only if $-a_{11} - a_{22}, -a_{22} - a_{33}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}} < -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{21} = a_{23} = a_{31} = 1$, $\pm a_{22} = -a_{32} = 10$, and $\pm a_{33} = 20$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = \pm a_{22}, a_{23} = a_{31} = \pm a_{33} = 1$ and $a_{21} = a_{32} = -0.1$, then A is AP.



Digraph 25

$$\begin{bmatrix} * & + & + \\ + & * & + \\ + & + & 0 \end{bmatrix}$$
SP 25.1: **RAP**

$$\begin{bmatrix} * & - & + \\ + & * & + \\ + & + & 0 \end{bmatrix}$$
SP 25.2: **AAP**

$$\begin{bmatrix} * & + & - \\ + & * & + \\ + & + & 0 \end{bmatrix}$$
SP 25.3: **AAP**

$$\begin{bmatrix} * & - & + \\ - & * & + \\ + & + & 0 \end{bmatrix}$$
SP 25.4: **AAP**

$$\begin{bmatrix} * & + & - \\ + & * & + \\ - & + & 0 \end{bmatrix}$$
SP 25.5: **AAP**

$$\begin{bmatrix} * & - & + \\ + & * & - \\ + & + & 0 \end{bmatrix}$$
SP 25.6: **AAP**

$$\begin{bmatrix} * & + & + \\ + & * & - \\ - & + & 0 \end{bmatrix}$$
SP 25.7: **AAP**

$$\begin{bmatrix} * & - & + \\ + & * & - \\ - & + & 0 \end{bmatrix}$$
SP 25.8: **AAP**

$$\begin{bmatrix} + & - & - \\ - & * & + \\ + & + & 0 \end{bmatrix}$$
SP 25.9: **AAP**

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$$\begin{bmatrix} + & - & - \\ + & * & + \\ - & + & 0 \end{bmatrix}$$

SP 25.10: **AAP**

$$\begin{bmatrix} * & - & - \\ + & * & + \\ + & + & 0 \end{bmatrix}$$

SP 25.11: **DNA**

$$\begin{bmatrix} * & + & + \\ + & * & + \\ - & - & 0 \end{bmatrix}$$

SP 25.12: **DNA**

$$\begin{bmatrix} * & - & - \\ + & * & - \\ + & + & 0 \end{bmatrix}$$

SP 25.13: **DNA**

$$\begin{bmatrix} * & - & - \\ + & * & + \\ + & - & 0 \end{bmatrix}$$

SP 25.14: **DNA**

$$\begin{bmatrix} - & - & - \\ - & * & + \\ + & + & 0 \end{bmatrix}$$

SP 25.15: **DNA**

$$\begin{bmatrix} - & - & - \\ + & * & + \\ - & + & 0 \end{bmatrix}$$

SP 25.16: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 25. If the sign pattern of A is

- SP 25.2, then A is AP if and only if $-a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}}, -a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{12} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{12} = -0.1$, then A is AP.
- SP 25.3, then A is AP if and only if $-a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}}, -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}} < -a_{11} - \frac{a_{12}a_{23}}{a_{13}}$. In particular, if we take $\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{13} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{13} = -0.1$, then A is AP.
- SP 25.4, then A is AP if and only if $-a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{13} = -a_{21} = \pm a_{22} = a_{31} = a_{32} = 1$ and $a_{23} = 2$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$ and $a_{12} = a_{21} = -0.1$, then A is AP.
- SP 25.5, then A is AP if and only if $-a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}} < -a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = a_{12} = -a_{13} = a_{21} = \pm a_{22} = -a_{31} = a_{32} = 1$ and $a_{23} = 2$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$ and $a_{13} = a_{31} = -0.1$, then A is AP.
- SP 25.6, then A is AP if and only if $-a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{13} = a_{21} = \pm a_{22} = -a_{23} = a_{32} = 1$ and $a_{31} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{31} = a_{32} = 1$ and $a_{12} = a_{23} = -0.1$, then A is AP.
- SP 25.7, then A is AP if and only if $-a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}}$. In particular, if we take $\pm a_{11} =$

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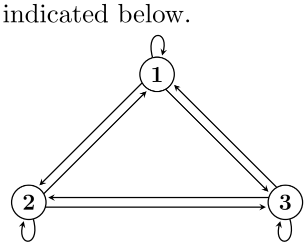
$a_{13} = a_{21} = \pm a_{22} = -a_{23} = -a_{31} = a_{32} = 1$ and $a_{12} = 2$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{12} = a_{13} = a_{21} = \pm a_{22} = a_{32} = 1$ and $a_{23} = a_{31} = -0.1$, then A is AP.

- SP 25.8, then A is AP if and only if $-a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}}$. In particular, if we take $-a_{12} = a_{13} = a_{21} = -a_{23} = -a_{31} = a_{32} = 1$, $\pm a_{11} = 5$, and $\pm a_{22} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{32} = 1$ and $a_{12} = a_{23} = a_{31} = -0.1$, then A is AP.
- SP 25.9, then A is AP if and only if $-a_{11} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $a_{11} = -a_{12} = -a_{13} = -a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $-a_{12} = -a_{13} = \pm a_{22} = a_{23} = a_{31} = a_{32} = 1$, $a_{11} = 10$, and $a_{21} = -5$, then A is AP.
- SP 25.10, then A is AP if and only if $-a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - \frac{a_{13}a_{21}}{a_{23}}, -a_{22} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - \frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $a_{11} = -a_{12} = -a_{13} = a_{21} = \pm a_{22} = a_{23} = -a_{31} = a_{32} = 1$, then A is not AP. On the other hand, if we take $-a_{12} = -a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{32} = 1$, $a_{11} = 10$, and $a_{31} = -2$, then A is AP.

G. **Graphs with 9 directed edges**. There is only one nonequivalent strongly connected digraph with 9 directed edges. We list the nonequivalent sign pattern matrices corresponding to the digraph and except for the AAP sign patterns, we use Theorems 1.2.2, 1.2.3 and 2.1 to classify these sign patterns as indicated below.



Digraph 26

$$\begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix}$$
SP 26.1: **RAP**

$$\begin{bmatrix} * & - & + \\ + & * & + \\ + & + & * \end{bmatrix}$$
SP 26.2: **AAP**

$$\begin{bmatrix} * & - & + \\ + & * & + \\ + & - & * \end{bmatrix}$$
SP 26.3: **DNA**

$$\begin{bmatrix} * & - & + \\ - & * & + \\ + & + & * \end{bmatrix}$$
SP 26.4: **AAP**

$$\begin{bmatrix} * & - & + \\ + & * & - \\ + & + & * \end{bmatrix}$$
SP 26.5: **AAP**

$$\begin{bmatrix} * & - & - \\ + & * & - \\ + & + & * \end{bmatrix}$$
SP 26.6: **DNA**

$$\begin{bmatrix} * & - & + \\ + & * & - \\ - & + & * \end{bmatrix}$$
SP 26.7: **AAP**

$$\begin{bmatrix} * & - & + \\ - & * & - \\ + & + & - \end{bmatrix}$$
SP 26.8: **AAP**

$$\begin{bmatrix} * & - & + \\ - & - & - \\ + & + & + \end{bmatrix}$$
SP 26.9: **DNA**

Suppose that the digraph of $A = [a_{ij}] \in M_3(\mathbb{R})$ is Digraph 26. If the sign pattern of A is

- SP 26.2, then A is AP if and only if $-a_{11} - a_{33} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} -$

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$\frac{a_{13}a_{21}}{a_{23}}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}$. In particular, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{12} = -10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{12} = -0.1$, then A is AP.

- SP 26.4, then A is AP if and only if $-a_{11} - a_{33} - \frac{a_{12}a_{23}}{a_{13}}, -a_{22} - a_{33} - \frac{a_{13}a_{21}}{a_{23}}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}$. In particular, if we take $\pm a_{11} = -a_{12} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{13} = -a_{21} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = \pm a_{22} = a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{12} = a_{21} = -0.1$, then A is AP.
- SP 26.5, then A is AP if and only if $-a_{11} - a_{33} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{22} - a_{33} - \frac{a_{13}a_{21}}{a_{23}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{13} = a_{21} = \pm a_{22} = -a_{23} = a_{31} = a_{32} = \pm a_{33} = 1$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{31} = a_{32} = \pm a_{33} = 1$ and $a_{12} = a_{23} = -0.1$, then A is AP.
- SP 26.7, then A is AP if and only if $-a_{11} - a_{33} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}} < -a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{22} - a_{33} - \frac{a_{13}a_{21}}{a_{23}}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{13} = a_{21} = -a_{23} = -a_{31} = a_{32} = 1$, $\pm a_{11} = 1$, and $\pm a_{33} = 10$, then A is not AP. On the other hand, if we take $\pm a_{11} = a_{13} = a_{21} = \pm a_{22} = a_{32} = \pm a_{33} = 1$ and $a_{12} = a_{23} = a_{31} = -0.1$, then A is AP.
- SP 26.8, then A is AP if and only if $-a_{11} - a_{22} - \frac{a_{13}a_{32}}{a_{12}}, -a_{22} - a_{33} - \frac{a_{13}a_{21}}{a_{23}}, -a_{11} - a_{22} - \frac{a_{23}a_{31}}{a_{21}} < -a_{11} - a_{33} - \frac{a_{12}a_{23}}{a_{13}}, -a_{11} - a_{33} - \frac{a_{21}a_{32}}{a_{31}}, -a_{22} - a_{33} - \frac{a_{12}a_{31}}{a_{32}}$. In particular, if we take $\pm a_{11} = -a_{12} = a_{13} = -a_{21} = \pm a_{22} = -a_{23} = a_{31} = a_{32} = -a_{33} = 1$,then A is not AP. On the other hand, if we take $\pm a_{11} = -a_{12} = -a_{21} = \pm a_{22} = -a_{23} = a_{31} = a_{32} = 1$, $a_{13} = 5$, and $a_{33} = -10$, then A is AP.