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THE INVERSE EIGENVALUE PROBLEM FOR LESLIE MATRICES*

LUCA BENVENUTI†

Abstract. The Nonnegative Inverse Eigenvalue Problem (NIEP) is the problem of determining necessary and sufficient conditions for a list of n complex numbers to be the spectrum of an entry–wise nonnegative matrix of dimension n. This is a very difficult and long standing problem and has been solved only for $n \leq 4$. In this paper, the NIEP for a particular class of nonnegative matrices, namely Leslie matrices, is considered. Leslie matrices are nonnegative matrices, with a special zero–pattern, arising in the Leslie model, one of the best known and widely used models to describe the growth of populations. The lists of nonzero complex numbers that are subsets of the spectra of Leslie matrices are fully characterized. Moreover, the minimal dimension of a Leslie matrix having a given list of three numbers among its spectrum is provided. This result is partially extended to the case of lists of $n > 2$ real numbers.

Key words. Nonnegative inverse eigenvalue problem, Nonnegative matrix, Leslie matrix, Polyhedral proper cone.

AMS subject classifications. 15A29, 15A18.

1. Introduction.

The Nonnegative Inverse Eigenvalue Problem (NIEP) is the problem of determining necessary and sufficient conditions for a list of n complex numbers to be the spectrum of an entry–wise nonnegative matrix of dimension n. The NIEP is a very difficult and long standing problem: It was firstly stated, in a simplified version, by Kolmogorov in 1937 [13] and it is still unsolved for lists of $n \geq 5$ numbers. More specifically, the NIEP for lists of $n = 3$ and $n = 4$ real numbers was solved by Perfect in 1952 [19] and by Loewy and London in 1978 [17], respectively. The NIEP for lists of $n = 3$ complex numbers was solved by Loewy and London [17], while, in 2007, Torre-Mayo et al. [23] solved the problem for lists of $n = 4$ complex numbers by means of inequalities on the coefficients of the polynomial whose roots are the numbers of the list (the NIEP for $n = 4$ was first solved in 1998 by Laffey and Meehan [18], from the viewpoint of Newton's identities and list's power sums). The solution in [23] was recently exploited by Benvenuti [4] in order to find conditions on the location of the numbers of the list in the complex plane.

When considering a list that is not the spectrum of any nonnegative matrix, one may ask whether the list can be a subset of the spectrum of a nonnegative matrix, i.e., some numbers can be appended to the list thus obtaining a larger list which is the spectrum of a nonnegative matrix. If no conditions are placed upon the appended numbers, then the problem becomes trivial since it suffices to append a single, sufficiently large, positive number to the list in order to obtain a list which is the spectrum of a nonnegative matrix[1] [10].

A more intriguing problem arises if only the addition of zeros is allowed. This problem corresponds to the characterization of the nonzero spectra of nonnegative matrices and turns out to be quite different from and more tractable than the NIEP [6]. In particular, necessary and sufficient conditions for a list with no zeros to be the nonzero part of the spectrum of a nonnegative matrix were given in [6, 15]. It is worth noting

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†Dipartimento di Ingegneria informatica, automatica e gestionale "A. Ruberti", Sapienza Università di Roma, Roma, Italy (luca.benvenuti@uniroma1.it).

[1] This result holds when the list of complex numbers is closed under complex conjugation.

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that the number of zeros that need to be added may be very large and hard to be estimated. Some estimates of this number have been recently given in some particular cases [8, 14].

In this paper, the NIEP for a particular class of nonnegative matrices, namely Leslie matrices, is considered. A Leslie matrix is a nonnegative matrix defined in the Leslie model [16], one of the most heavily used models in population ecology. This is a discrete–time model of an age–structured population which describes the development, mortality, and reproduction of the individuals of a population.

As shown in [12], there is no loss of generality in considering the NIEP for *non–singular row stochastic Leslie matrices*, i.e., nonnegative matrices of the form

$$L = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{1.1}$$

with $a_0 > 0$ and $\sum_{i=0}^{n-1} a_i = 1$. Note that a non-singular Leslie matrix is similar to the companion matrix via a diagonal transformation.

In [12], Kirkland characterized the regions L_n of points in the complex plane which can serve as characteristic roots of n-dimensional row stochastic Leslie matrices[2]. Representations of the regions L_n for $n = 3, \ldots, 8$ can be found in [3].

In this paper, the NIEP for Leslie matrices is considered. In particular, as stated above, there is no loss of generality in considering only *non-singular row stochastic Leslie matrices* and, consequently, only lists of nonzero complex numbers.

A list of n nonzero complex numbers will be said to be $\mathcal{L}$-realizable if there exists a non-singular row stochastic Leslie matrix of order n having the given list as spectrum.

Moreover, the problem of characterizing the lists of complex numbers that are subsets of the spectra of Leslie matrices is addressed. It is worth noting that, when considering Leslie matrices, this last problem is quite different from that for nonnegative matrices. In fact, as discussed above, if a given list is not the spectrum of any Leslie matrix, then appending zeros to the list does not change the situation at all. Moreover, in general, as will be clear later, it is not sufficient to append a single number to the list in order to obtain a list which is the spectrum of a Leslie matrix.

2. Conditions for $\mathcal{L}$-realizability. It is well known that non-negativity constraints on the entries of a matrix impose limitations on the location of its eigenvalues. When considering a non-singular row stochastic Leslie matrix, these limitations are determined not only by non-negativity of the matrix but also by its special structure. In more detail, the characteristic polynomial of a non-singular row stochastic Leslie matrix L of the form (1.1) is equal to

$$p_L(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$$

[2]The sets L_n considered in [12], to be precise, are the sets of the non-positive eigenvalues of row stochastic Leslie matrices, that is, the eigenvalue $\lambda = 1$ is not included in the sets.

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so that, from non-negativity of the coefficients a_i's and Descartes' rule of signs, it has exactly one positive root that, from the stochasticity of the matrix, is equal to 1. Moreover, a non-singular Leslie matrix is irreducible, and therefore, further limitations on the location of its eigenvalues are provided by the Perron-Frobenius theorem for irreducible nonnegative matrices [9, 21]. These conditions, together with the basic necessary conditions coming from the fact that a nonnegative matrix has real entries and nonnegative trace (non-negativity of the moments, closure under complex conjugation, ...) [11] , form the necessary conditions for a list of nonzero complex numbers to be the spectrum of a non-singular row stochastic Leslie matrix.

The following theorem, proved in [20], shows that, when considering lists of real numbers, the above described necessary conditions turn out to be also sufficient:

Theorem 2.1. *A list $\Lambda = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$ of $n > 2$ nonzero real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is $\mathcal{L}$-realizable if and only if the following set of conditions holds:*

1) $\lambda_1 = 1$;
2) $\lambda_1 + \lambda_2 + \cdots + \lambda_n \geq 0$;
3) $-1 < \lambda_i < 0$ *for* $i = 2, \ldots, n$.

Finally, some minor results can be stated for small lists of complex numbers. For example, when considering lists of three complex numbers, necessary conditions allow us to rewrite the generic list as

$$\Lambda = \{1, \lambda_2, \bar{\lambda}_2\} \tag{2.2}$$

with $\lambda_2 \notin \mathbb{R}$ such that $|\lambda_2| \leq 1$. Hence, the necessary and sufficient condition for $\mathcal{L}$-realizability of the list (2.2) is that

$$\lambda_2 \in L_3.$$

This follows immediately from the definition of the set L_3 itself.

Consider now lists of four complex numbers. In this case, necessary conditions allow us to rewrite the generic list as

$$\Lambda = \{1, \lambda_2, \lambda_3, \bar{\lambda}_3\} \tag{2.3}$$

with $-1 \leq \lambda_2 < 0$ and $\lambda_3 \notin \mathbb{R}$ such that $|\lambda_3| \leq 1$.

The next theorem provides the necessary and sufficient conditions for $\mathcal{L}$-realizability of the list (2.3).

Theorem 2.2. *The list $\Lambda = \{1, \lambda_2, \lambda_3, \bar{\lambda}_3\}$ of four complex numbers is the spectrum of a non-singular row stochastic Leslie matrix if and only if the following set of conditions holds:*

1) $-1 \leq \lambda_2 < 0$;
2) $\lambda_3 = \alpha + i\omega$ *is such that:*

$$-\frac{1+\lambda_2}{2} \leq \alpha \leq -\frac{\lambda_2\,(1+\lambda_2)}{2\,(1+\lambda_2+\lambda_2^2)},$$

$$0 < \omega^2 \leq -\lambda_2 - 2\alpha\,(1+\lambda_2) - \alpha^2 \quad \textit{for} \;\; -\frac{1+\lambda_2}{2} \leq \alpha \leq 0$$

and

$$-\frac{2\lambda_2\alpha}{1+\lambda_2} - \alpha^2 \leq \omega^2 \leq -\lambda_2 - 2\alpha\,(1+\lambda_2) - \alpha^2 \quad \textit{for} \;\; 0 \leq \alpha \leq -\frac{\lambda_2\,(1+\lambda_2)}{2\,(1+\lambda_2+\lambda_2^2)}.$$

Proof. Theorem immediately follows by imposing non negativity on the coefficients of the polynomial whose roots are the numbers of the list Λ. ∎

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3. Embedding lists to achieve $\mathcal{L}$-realizability. When the conditions of Theorem 2.1 are not satisfied, then there does not exist a non-singular row stochastic Leslie matrix of dimension n having the list Λ of n real numbers as spectrum. However, the list may be a subset of the spectrum of some "bigger" non-singular row stochastic Leslie matrix. Consider for example, the list of three real numbers

$$\Lambda = \{1, -0.5, -0.7\}. \tag{3.4}$$

Since the sum of all the numbers of the list is negative, then there does not exist a non-singular row stochastic Leslie matrix of dimension three having the list as spectrum. Actually, there does not even exist a nonnegative matrix of dimension three having the list Λ as spectrum, for the same reason. Nevertheless, the list is a subset of the spectra of the two following non-singular row stochastic Leslie matrices

$$L_1 = \begin{bmatrix} 0 & 0 & \frac{4}{2000} & \frac{1373}{2000} & \frac{623}{2000} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{35196}{200000} & \frac{116777}{200000} & \frac{48027}{200000} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

whose eigenvalues are

$$\sigma(L_1) = \{1, -0.5, -0.7, 0.1 \pm 0.938i\} \quad \text{and} \quad \sigma(L_2) = \{1, -0.5, -0.7, 0.494 \pm 0.828i, -0.394 \pm 0.764i\}.$$

The next theorem completely characterizes the lists of nonzero complex numbers that are subsets of the spectra of non-singular row stochastic Leslie matrices.

THEOREM 3.1. *A list $\Lambda = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$ of n nonzero complex numbers is a subset of the spectrum of a non-singular row stochastic Leslie matrix of dimension $N \geq n$ if and only if one of the following sets of conditions holds:*

a1) $|\lambda_i| < 1$ for all i;
a2) all the real numbers λ_i of the list, if any, are negative;

or

b1) $|\lambda_i| \leq 1$ for all i and $|\lambda_i| = 1$ for some i;
b2) all the numbers λ_i of unit modulus appear only once in the list and are among the r–th roots of unity for some positive integer r;
b3) taking the minimal value of r, no number of the list such that $|\lambda_i| < 1$ has an argument which is an integer multiple of $2\pi/r$.

Proof. (Sufficiency)

Case a. Suppose the polynomial

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

has no real positive roots and all its roots are less than one in modulus. Hence, from Lemma 4 in [22], the polynomial $(x - 1) \cdot p(x)$ divides a polynomial $q(x)$ of the form

$$q(x) = x^N - a_{N-1}x^{N-1} - \cdots - a_1 x - a_0 \tag{3.5}$$

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in which all the coefficients $a_1, \ldots, a_{N-1}$ are nonnegative and $a_0 > 0$. Moreover, since $q(1) = 0$, then

$$a_0 + a_1 + \cdots + a_{N-1} = 1.$$

Consequently, the row stochastic Leslie matrix

$$L = \begin{bmatrix} a_{N-1} & a_{N-2} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} \tag{3.6}$$

for which $p_L(x) = q(x)$, is a non-singular matrix having all the numbers of the list Λ among its eigenvalues.

Case b. Consider the polynomial

$$p(x) = \begin{cases} (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) & \text{if } \exists \lambda_i | \lambda_i = 1, \\ (x-1)(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) & \text{otherwise.} \end{cases}$$

This polynomial satisfies all the conditions (1)–(4) of condition (C) of Theorem 1 in [5]. As a consequence, it divides a polynomial $q(x)$ of the form (3.5) such that $q(1) = 0$ and in which all the coefficients $a_1, \ldots, a_{N-1}$ are nonnegative and $a_0 > 0$. Hence, the matrix L in (3.6) is a non-singular row stochastic Leslie matrix with all the numbers of the list Λ among its eigenvalues.

(Necessity) Since the list Λ can be embedded in a larger list Λ', which is the spectrum of a non-singular row stochastic Leslie matrix, then it must not contradict the necessary conditions for $\mathcal{L}$-realizability of the list Λ'. Hence, conditions *a1* and *a2* as well as conditions *b1*, *b2* and *b3* follow from the necessary conditions described at the beginning of Section 2. ☐

The next corollary immediately follows from the previous theorem in the case where the list contains only real numbers.

Corollary 3.2. *A list $\Lambda = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$ of $n > 2$ nonzero real numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is a subset of the spectrum of a non-singular row stochastic Leslie matrix of dimension $N \geq n$ if and only if one of the following sets of conditions holds:*

a) $-1 < \lambda_i < 0$ for all i;

or

b1) $\lambda_1 = 1$;
b2) $-1 < \lambda_i < 0$ for $i = 2, \ldots, n$.

For the list (3.4), for example, conditions b of Corollary 3.2 are satisfied and indeed, there exist non-singular row stochastic Leslie matrices having the numbers of the list among their eigenvalues. It is then interesting to evaluate the minimum number of elements that is necessary to append to a given list in order to achieve $\mathcal{L}$-realizability of the extended list. To gain insight into this problem the following definitions and lemmas are needed.

A set $\mathcal{C} \in \mathbb{R}^n$ is said to be a *cone* provided that $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha \geq 0$. A cone $\mathcal{C}$ is said to be *solid* if it contains an open ball of $\mathbb{R}^n$ and it is said to be *pointed* if $\mathcal{C} \cap -\mathcal{C} = \{0\}$. A cone which is closed, convex,

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solid and pointed is said to be a *proper cone*. A cone $\mathcal{C}$ is said to be *polyhedral*, or *finitely generated*, if it is expressible as the intersection of a finite family of closed half–spaces. Every polyhedral cone $\mathcal{C} \in \mathbb{R}^n$ is convex and closed and can be represented by a matrix $K \in \mathbb{R}^{n\times m}$, with nonzero columns, as the set of all nonnegative linear combinations of the columns of K, that is, $\mathcal{C} = \text{cone}\,(K)$. The matrix K is called a generating matrix for the cone. A generating matrix is said to be a *minimal generating matrix* for $\mathcal{C}$ in case its columns are non-negatively independent. In this case, the columns are the extreme rays of the cone. Given the polyhedral cones $\mathcal{C}_1 = \text{cone}\,(K_1)$ and $\mathcal{C}_2 = \text{cone}\,(K_2)$, the polyhedral cone generated by the matrix $[K_1\ K_2]$ is denoted by $\mathcal{C}_1 + \mathcal{C}_2$.

Given a list $\Lambda = \{\lambda_1, \lambda_2\ \ldots, \lambda_n\}$ of n nonzero complex numbers, closed under complex conjugation, let us associate to it the convex cone

$$\mathcal{C}\,(\Lambda) = \sum_{i=1}^{\infty} \mathcal{C}_i,$$

where the sets $\mathcal{C}_i$ are the polyhedral cones

$$\mathcal{C}_i = \text{cone}\left(\begin{bmatrix} b & Ab & A^2b & \cdots & A^{i-1}b\end{bmatrix}\right)$$

in which the matrix $A \in \mathbb{R}^{n\times n}$ is the real Jordan form of the matrices having the minimal polynomial equal to

$$p_A(x) = (x-\lambda_1)\,(x-\lambda_2)\cdots(x-\lambda_n)$$

and $b \in \mathbb{R}^n$ is the vector of all ones. In more detail, the matrix A is a block diagonal matrix where each block A_k has the form

$$A_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix} \in \mathbb{R}^{n_k\times n_k}$$

if λ_k is a real number appearing n_k times in the list, or

$$A_k = \begin{bmatrix} \alpha_k & \omega_k & 1 & 0 & & & \\ -\omega_k & \alpha_k & 0 & 1 & & 0 & \\ & & \ddots & & \ddots & & \\ & & & & \ddots & 1 & 0 \\ & & & & & 0 & 1 \\ & 0 & & & & \alpha_k & \omega_k \\ & & & & & -\omega_k & \alpha_k \end{bmatrix} \in \mathbb{R}^{2n_k\times 2n_k}$$

if the complex conjugate pair of numbers $\alpha_k \pm i\omega_k$ appears n_k times in the list. It is worth noting that, since the cone $\mathcal{C}\,(\Lambda)$ consists of the sum of an infinite number of polyhedral cones, then it may be the case that it is not a polyhedral cone. Moreover, by definition,

$$\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \subseteq \cdots$$

and if $\mathcal{C}_N = \mathcal{C}_{N+1}$, then

$$\mathcal{C}\,(\Lambda) = \mathcal{C}_i, \quad \forall i \geq N.$$

Theorem 3.1 and the results presented in [1] allow the following lemma to be proved:

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LEMMA 3.3. *A list $\Lambda = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$ of n nonzero complex numbers, closed under complex conjugation and containing the number 1 exactly once, is a subset of the spectrum of a non-singular row stochastic Leslie matrix if and only if the cone $\mathcal{C}(\Lambda)$, associated to the list, is a polyhedral proper cone.*

Proof. Since the list Λ contains the number 1 exactly once, then, from Corollaries 3.3 and 3.7 in [1], it follows that the cone $\mathcal{C}(\Lambda)$ is a polyhedral proper cone in $\mathbb{R}^n$ if and only if the conditions b of Theorem 3.1 hold. In particular, when considering Corollaries 3.3 and 3.7 one has to set $\omega_F = 1$ and $\deg \omega_F = 1$. Hence, the lemma remains proved noting that, when the list Λ contains the number 1, the conditions b of Theorem 3.1 are satisfied if and only if the list Λ is a subset of the spectrum of a non-singular row stochastic Leslie matrix of dimension $N \geq n$. ∎

REMARK 3.4. Note that, since being a polyhedral proper cone is a property invariant to non-singular linear transformations, then the result in Lemma 3.3 remains true even when the cone $\mathcal{C}(\Lambda)$ is defined on the basis of any reachable pair (A, b) for which the spectrum of the matrix A is equal to the list Λ.

The following lemma links the numbers of extremal vectors of the cone $\mathcal{C}(\Lambda)$ with the minimal number of elements to append to the list Λ in order to achieve $\mathcal{L}$-realizability.

LEMMA 3.5. *Given a list $\Lambda = \{\lambda_1, \lambda_2 \ldots, \lambda_n\}$ of n nonzero complex numbers, closed under complex conjugation and containing the number 1 exactly once, the minimal dimension of a non-singular row stochastic Leslie matrix, having the list Λ among its spectrum, coincides with the number of columns of any minimal generating matrix of the cone $\mathcal{C}(\Lambda)$.*

Proof. Denote by $n_{\mathcal{C}} \geq n$ the number of columns of a minimal generating matrix for $\mathcal{C}(\Lambda)$ and by $n_L \geq n$ the minimal dimension of a (non-singular) row stochastic Leslie matrix having a spectrum containing the list Λ. Consider then a non-singular row stochastic Leslie matrix L of minimal dimension n_L such that the list Λ is a subset of its spectrum. Then

$$p_L(x) = x^{n_L} - a_{n_L-1}x^{n_L-1} - \cdots - a_1 x - a_0,$$

where all the coefficients $a_1, \ldots, a_{n_L-1}$ are nonnegative, $a_0 > 0$ and $\sum_{i=0}^{n_L-1} a_i = 1$. Since the list Λ is a subset of the spectrum of L, then the matrix A, having the list Λ as spectrum, is such that $p_L(A) = 0$, that is,

$$A^{n_L} = a_0 I + a_1 A + \cdots + a_{n_L-1} A^{n_L-1}.$$

By multiplying both sides of the above equation by the vector b, one has

$$A^{n_L} b = a_0 b + a_1 Ab + \cdots + a_{n_L-1} A^{n_L-1} b,$$

that is, the vector $A^{n_L} b$ is a convex combination of the vectors b, Ab, $\ldots$, $A^{n_L-1}b$. Hence, the vector $A^{n_L} b$ belongs to the cone $\mathcal{C}_{n_L}$ so that

$$\mathcal{C}_{n_L} = \mathcal{C}_{n_L+1} = \mathcal{C}(\Lambda).$$

As a consequence, $n_{\mathcal{C}} \leq n_L$.

Assume now the cone $\mathcal{C}(\Lambda)$ to be a polyhedral proper cone having a minimal generating matrix K of dimension $n_{\mathcal{C}}$. Hence, by construction,

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{n_{\mathcal{C}}} = \mathcal{C}(\Lambda),$$

that is, the vector $A^{n_{\mathcal{C}}} b$ belongs to the cone $\mathcal{C}_{n_{\mathcal{C}}}$ and can be expressed as a convex combination of the vectors b, Ab, $\ldots$, $A^{n_{\mathcal{C}}-1}b$. This can be written as follows:

$$\mathcal{C}(\Lambda) = \text{cone}(K) = \text{cone}\left(\begin{bmatrix} b & Ab & \cdots & A^{n_{\mathcal{C}}-1}b \end{bmatrix}\right)$$

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and

$$AK = KM$$

with

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & a_{n_{\mathcal{C}}-1} \end{bmatrix} \in \mathbb{R}^{n_{\mathcal{C}} \times n_{\mathcal{C}}}$$

in which all the coefficients $a_1, \ldots, a_{n_{\mathcal{C}}-1}$ are nonnegative, $a_0 > 0$ and $\sum_{i=0}^{n_{\mathcal{C}}-1} a_i = 1$. Note that, if $a_0 = 0$, then $\mathcal{C}(\Lambda) = \mathcal{C}_{n_{\mathcal{C}}-1}$ thus contradicting minimality of $n_{\mathcal{C}}$.

Let $T \in \mathbb{R}^{n_{\mathcal{C}} \times n_{\mathcal{C}}}$ be the transformation matrix such that $J = T^{-1}MT$ is the real Jordan canonical form of M. Then, setting $K' = KT$, the following equalities hold true:

$$AK' = AKT = KMT = KTT^{-1}MT = KTJ = K'J. \tag{3.7}$$

Note that, since the matrix K is full rank, then the matrix T can always be chosen in such a way that the matrix K_1' consisting of the first n columns of K' is a full rank matrix. Consequently, equation (3.7) can be written as follows:

$$A \begin{bmatrix} K_1' & K_2' \end{bmatrix} = \begin{bmatrix} K_1' & K_2' \end{bmatrix} \begin{bmatrix} J_1 & * \\ 0 & J_2 \end{bmatrix}$$

and, in particular,

$$AK_1' = K_1' J_1.$$

Hence, $p_A(x) = p_{J_1}(x)$. Considering then a non-singular row stochastic Leslie matrix of the form (3.6) and of dimension $n_{\mathcal{C}}$, since

$$p_L(x) = p_M(x),$$

then

$$p_L(x) = p_M(x) = p_J(x) = p_{J_1}(x) \cdot p_{J_2}(x) = p_A(x) \cdot p_{J_2}(x).$$

Hence, there exists a non-singular row stochastic Leslie matrix L of dimension $n_{\mathcal{C}}$ having the list Λ among its spectrum. As a consequence, $n_L \leq n_{\mathcal{C}}$, and this concludes the proof. $\square$

Lemmas 3.3 and 3.5 allow the results presented in [2] to be used to determine the minimal dimension of a non-singular row stochastic Leslie matrix having a given list of three real numbers among its spectrum. The next theorem follows immediately from (Case B) of Theorem 5 in [2]:

THEOREM 3.6. *Consider a list $\Lambda = \{1, \lambda_2, \lambda_3\}$ of three real numbers such that $-1 < \lambda_3 \leq \lambda_2 < 0$. Then, the minimal dimension of a non-singular row stochastic Leslie matrix having the given list among its spectrum is the minimum odd number $n_L \geq 3$ for which the following holds:*

$$\begin{array}{ll} \lambda_3^{n_L} - \lambda_2^{n_L} + \lambda_2^{n_L-2}\left(1 - \lambda_3^{n_L}\right) + \lambda_3^{n_L-2}\left(\lambda_2^{n_L} - 1\right) \geq 0 & if\ \lambda_2 \neq \lambda_3, \\ n_L \lambda^{n_L-1}\left(\lambda^{n_L-2} - 1\right) + \left(n_L - 2\right)\lambda^{n_L-3}\left(1 - \lambda^{n_L}\right) \geq 0 & if\ \lambda_2 = \lambda_3 = \lambda. \end{array}.$$

REMARK 3.7. It is worth noting that the conditions of Theorem 3.6 reduce to condition 2 of Theorem 2.1 when $n_L = n = 3$. In fact, in this case, the following hold:

$$\lambda_3^3 - \lambda_2^3 + \lambda_2\left(1 - \lambda_3^3\right) + \lambda_3\left(\lambda_2^3 - 1\right) = (1 - \lambda_2)(1 - \lambda_3)(\lambda_2 - \lambda_3)(1 + \lambda_2 + \lambda_3)$$

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and

$$3\lambda^2(\lambda-1)+\left(1-\lambda^3\right)=(\lambda-1)^2(1+2\lambda).$$

Considering for example the list of three real numbers $\Lambda=\{1,-0.5,-0.7\}$, the minimum odd number for which the condition of Theorem 3.6 does hold is $n_L=5$.

The previous result can be partially extended to the case of lists of n real numbers as follows:

THEOREM 3.8. *Consider an $\mathcal{L}$-realizable list $\Lambda=\{1,\lambda_2,\dots,\lambda_n\}$ of $n>2$ nonzero real numbers. Then, the list $\Lambda=\{1,\lambda_2-\delta,\dots,\lambda_n-\delta\}$ is $\mathcal{L}$-realizable for*

$$0\le\delta\le\bar{\delta}=\frac{1+\lambda_2+\cdots+\lambda_n}{n}$$

and there exists a positive value $\Delta>\bar{\delta}$ such that for

$$\bar{\delta}<\delta<\Delta$$

the minimal dimension of a non-singular row stochastic Leslie matrix having the given list among its spectrum is $n_L=n+2$.

Proof. Since the list is $\mathcal{L}$-realizable, then, in view of Theorem 2.1 and Lemmas 3.3 and 3.5, the cone $\mathcal{C}(\Lambda)$ associated to the list is a polyhedral proper cone with n extreme rays. Hence,

$$\mathcal{C}(\Lambda)=\text{cone}\left(\begin{bmatrix}b & Ab & A^2b & \cdots & A^{n-1}b\end{bmatrix}\right)$$

and the vector A^nb can be expressed as a convex combination of the extreme rays of the cone, that is,

$$A^nb=a_0b+a_1Ab+\cdots+a_{n-1}A^{n-1}b. \tag{3.8}$$

Moreover, all the coefficients $a_0,\dots,a_{n-2}$ are positive while a_{n-1} is nonnegative. In fact,

$$a_0=|\lambda_2|\cdot|\lambda_3|\cdots|\lambda_n|>0,$$

$$a_{n-1}=1+\lambda_2+\cdots+\lambda_n\ge 0$$

and, for $h=2,\dots,n-1$,

$$\begin{aligned}a_{n-h}&=\sum_{2\le i_1<i_2<\cdots<i_{h-1}\le n}|\lambda_{i_1}|\cdot|\lambda_{i_2}|\cdots|\lambda_{i_{h-1}}|-\sum_{2\le i_1<i_2<\cdots<i_h\le n}|\lambda_{i_1}|\cdot|\lambda_{i_2}|\cdots|\lambda_{i_h}|\\&\ge(|\lambda_2|+|\lambda_3|+\cdots+|\lambda_n|)\cdot\sum_{2\le i_1<i_2<\cdots<i_{h-1}\le n}|\lambda_{i_1}|\cdot|\lambda_{i_2}|\cdots|\lambda_{i_{h-1}}|-\sum_{2\le i_1<i_2<\cdots<i_h\le n}|\lambda_{i_1}|\cdot|\lambda_{i_2}|\cdots|\lambda_{i_h}|>0\end{aligned}$$

since all the negative terms coming from the second summation cancel out with the corresponding terms coming from the product[3]. Let us now consider the list $\Lambda(\delta)=\{1,\lambda_2-\delta,\dots,\lambda_n-\delta\}$ with increasing values for δ, starting from $\delta=0$. If $\delta\le\bar{\delta}$, then the conditions of Theorem 2.1 do hold so that the list is $\mathcal{L}$-realizable. In fact:

$$1+(\lambda_2-\delta)+\cdots+(\lambda_n-\delta)=1+\lambda_2+\cdots+\lambda_n-n\delta\ge 0,$$

$$\lambda_i-\delta\le\lambda_i<0$$

[3]The expression of the coefficients a_i's can be derived using Viète's formulas. In these formulas, the indices i_k's are written in increasing order to ensure that each sub-product of $|\lambda_{i_k}|$'s is used exactly once.

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and

$$\delta \leq 1 + \lambda_2 + \cdots + \lambda_n < 1 + \lambda_i,$$

that is, $\lambda_i - \delta > -1$. In particular, when $\delta = \bar{\delta}$, the list $\Lambda(\bar{\delta})$ sums to zero. In this case, $a_0, \ldots, a_{n-2} > 0$ and $a_{n-1} = 0$, and equation (3.8) reduces to

$$A^n b = a_0 b + a_1 Ab + \cdots + a_{n-2} A^{n-2} b. \tag{3.9}$$

Hence, the vector $A^n b$ lies on a face of dimension $n-1$ of the cone $\mathcal{C}\left(\Lambda(\bar{\delta})\right)$. Note that also the vector $A^{n+1}b$ lies on a face of dimension $n-1$; in fact, by multiplying both sides of equation (3.9) by A, one has

$$A^{n+1} b = a_0 Ab + a_1 A^2 b + \cdots + a_{n-2} A^{n-1} b.$$

On the other hand, by multiplying both sides of equation (3.9) by A^2 and using equation (3.9) for $A^n b$, one has

$$\begin{aligned} A^{n+2} b &= (a_0 a_{n-2}) b + (a_1 a_{n-2}) Ab + \sum_{i=2}^{n-2} (a_i a_{n-2} + a_{i-2}) A^i b + a_{n-3} A^{n-1} b \\ &= a'_0 b + a'_1 Ab + \cdots + a'_{n-1} A^{n-1} b, \end{aligned}$$

where all the coefficients $a'_0, \ldots, a'_{n-1}$ are positive. Therefore, the vector $A^{n+2}b$ lies in the interior of the cone $\mathcal{C}\left(\Lambda(\bar{\delta})\right)$ (see the Lemma in [7]). As soon as $\delta > \bar{\delta}$, the vectors $A^n b$ and $A^{n+1} b$ do not belong to cone $\left(\left[b \; Ab \; A^2 b \; \cdots \; A^{n-1} b\right]\right)$ anymore while the vector $A^{n+2} b$ still lies in its interior. Hence, the cone $\mathcal{C}\left(\Lambda(\delta)\right)$ turns out to have $n+2$ extreme rays, that is,

$$\mathcal{C}\left(\Lambda(\delta)\right) = \text{cone}\left(\left[b \; Ab \; A^2 b \; \cdots \; A^{n-1} b \; A^n b \; A^{n+1} b\right]\right),$$

and this concludes the proof. ☐

Consider for example, the list of four real numbers

$$\Lambda(\delta) = \{1, -0.1 - \delta, -0.2 - \delta, -0.4 - \delta\}.$$

In view of Theorem 2.1, the list is $\mathcal{L}$-realizable for $0 \leq \delta \leq \bar{\delta} = 0.1$. As soon as $\delta > \bar{\delta}$, the list is not $\mathcal{L}$-realizable anymore but it is a subset of the spectrum of the following six dimensional row stochastic Leslie matrix:

$$L = \begin{bmatrix} 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{3.10}$$

whose coefficients, depending on δ, assume the values shown in Figure 1. As the figure makes clear, the list is a subset of the spectrum of the matrix L in (3.10) until $\delta \leq \Delta = 0.3047$. When $\delta = \Delta$, the coefficients a_2 and a_3 are equal to zero so that the vector $A^6 b$ lies on the three dimensional face of the cone $\mathcal{C}(\Lambda(\Delta))$ defined by the vectors b, Ab and $A^4 b$. Similarly, the vector $A^7 b$ lies on a three dimensional face of the cone,

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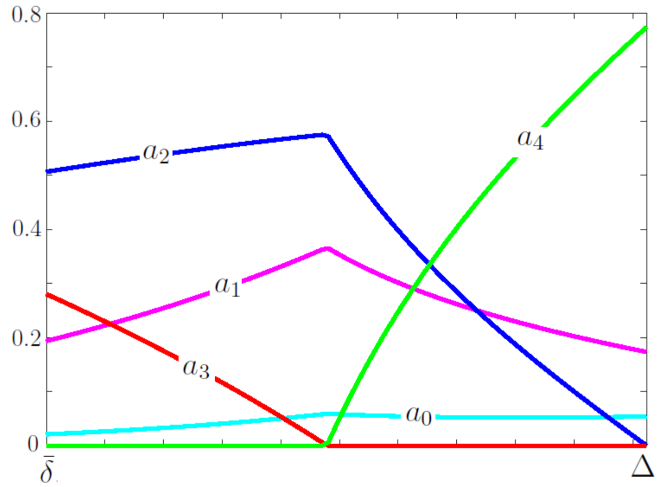

FIGURE 1. *Coefficients a_0, a_1, a_2, a_3 and a_4 of the matrix L in (3.10) for different values of δ, when $\bar{\delta} \leq \delta \leq \Delta$.*

that is, that defined by the vectors Ab, A^2b and A^5b. On the other hand, the vector A^8b lies in the interior of the cone. In fact, it can be written as

$$A^8b = 0.0471b + 0.1496Ab + 0.0268A^2b + 0.0865A^3b + 0.6088A^4b + 0.0812A^5b.$$

Hence, as soon as $\delta > \Delta$, the cone $\mathcal{C}(\Lambda(\delta))$ turns out to have 8 extreme rays. Several computations suggest that the same situation considered in the proof of Theorem 3.8, and shown in this example, occurs for different values of n and for increasing values of δ. In particular, if for a given value of the λ_i's the cone $\mathcal{C}(\Lambda)$ has N extreme rays, then, when decreasing the value of the λ_i's, the vector A^Nb approaches the face of dimension $n-1$ of the cone $\mathcal{C}(\Lambda)$ defined by the extreme rays b, Ab, $\ldots$, $A^{n-3}b$ and $A^{N-2}b$. At some point, the vector A^Nb, as well as the vector $A^{N+1}b$, lie on a face of dimension $n-1$ of the cone while the vector $A^{N+2}b$ is in its interior. Hence, a further decrease of the λ_i's determines the cone $\mathcal{C}(\Lambda)$ to have $N+2$ extreme rays, that is,

$$\mathcal{C}(\Lambda) = \operatorname{cone}\left(\left[b \ Ab \ A^2b \ \cdots \ A^{N-1}b \ A^Nb \ A^{N+1}b\right]\right).$$

The values of the λ_i's for which this occurs can be computed by imposing that the vector A^Nb is a convex combination of the extreme rays b, Ab, $\ldots$, $A^{n-3}b$ and $A^{N-2}b$. In this case, the following holds:

$$\det\left(\left[b \ Ab \ \cdots \ A^{n-3}b \ A^{N-2}b \ A^Nb\right]\right) = 0.$$

The above considerations suggest the following conjecture:

CONJECTURE 1. *Consider a list $\Lambda = \{1, \lambda_2 \ldots, \lambda_n\}$ of $n > 2$ nonzero real numbers such that $-1 < \lambda_i < 0$ for all i. Then, the minimal dimension n_L of a non-singular row stochastic Leslie matrix having the given list among its spectrum is the minimum number $n_L = n + 2k$, $k = 1, 2, \ldots$, for which the following holds:*

$$\det\left(\left[b \ Ab \ \cdots \ A^{n-3}b \ A^{n_L-2}b \ A^{n_L}b\right] \cdot \left[b \ Ab \ \cdots \ A^{n-1}b\right]\right) \geq 0. \tag{3.11}$$

REMARK 3.9. It is worth noting that condition (3.11) in Conjecture 1 reduces to that of Theorem 3.6 when $n = 3$, and to condition 2 of Theorem 2.1 when $n_L = n = 3$.

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When considering lists of three complex numbers, the next result immediately follows from the definition of the sets L_n:

THEOREM 3.10. *Consider a list $\Lambda = \{1, \alpha + i\omega, \alpha - i\omega\}$ of three complex numbers such that $\omega \neq 0$ and $\alpha^2 + \omega^2 \leq 1$. Then, the minimal dimension n_L of a non-singular row stochastic Leslie matrix having the given list among its spectrum is the minimum number n_L for which the following holds:*

$$\alpha \pm i\omega \in L_{n_L}.$$

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