Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

GENERALIZATION OF REAL INTERVAL MATRICES TO OTHER FIELDS*

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Abstract. An interval matrix is a matrix whose entries are intervals in $\mathbb{R}$. This concept, which has been broadly studied, is generalized to other fields. Precisely, a rational interval matrix is defined to be a matrix whose entries are intervals in $\mathbb{Q}$. It is proved that a (real) interval $p \times q$ matrix with the endpoints of all its entries in $\mathbb{Q}$ contains a rank-one matrix if and only if it contains a rational rank-one matrix, and contains a matrix with rank smaller than $\min\{p,q\}$ if and only if it contains a rational matrix with rank smaller than $\min\{p,q\}$; from these results and from the analogous criterions for (real) inerval matrices, a criterion to see when a rational interval matrix contains a rank-one matrix and a criterion to see when it is full-rank, that is, all the matrices it contains are full-rank, are deduced immediately. Moreover, given a field K and a matrix $\boldsymbol{\alpha}$ whose entries are subsets of K, a criterion to find the maximal rank of a matrix contained in $\boldsymbol{\alpha}$ is described.

Key words. Interval matrices, Rank, Rational realization.

AMS subject classifications. 15A99, 15A03, 65G40.

1. Introduction. Let $p, q \in \mathbb{N} \setminus \{0\}$; a $p \times q$ interval matrix $\boldsymbol{\alpha}$ is a $p \times q$ matrix whose entries are intervals in $\mathbb{R}$; we usually denote the entry i,j, $\boldsymbol{\alpha}_{i,j}$, by $[\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$. A $p \times q$ matrix A with entries in $\mathbb{R}$ is said contained in a $p \times q$ interval matrix $\boldsymbol{\alpha}$ if $a_{i,j} \in \boldsymbol{\alpha}_{i,j}$ for any i,j. There is a wide literature about interval matrices and the rank of the matrices they contain. In this paper, we generalize the concept of interval matrix to other fields and we start the study of the range of the rank of the contained matrices. Before sketching our results, we illustrate shortly some of the literature on interval matrices and the rank of the contained matrices and we say also some words on partial matrices and on the matrices with a given sign pattern; these research fields are connected with the theory of interval matrices.

Two of the most famous theorems on interval matrices are Rohn's theorems on full-rank interval matrices. We say that a $p \times q$ interval matrix $\boldsymbol{\alpha}$ has full rank if and only if all the matrices contained in $\boldsymbol{\alpha}$ have rank equal to $\min\{p,q\}$. For any $p \times q$ interval matrix $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$, let $\mathrm{mid}(\boldsymbol{\alpha})$, $\mathrm{rad}(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}|$ be respectively the midpoint, the radius and the modulus of $\boldsymbol{\alpha}$, that is, the $p \times q$ matrices such that

$$\mathrm{mid}(\boldsymbol{\alpha})_{i,j} = \frac{\underline{\alpha}_{i,j} + \overline{\alpha}_{i,j}}{2}, \quad \mathrm{rad}(\boldsymbol{\alpha})_{i,j} = \frac{\overline{\alpha}_{i,j} - \underline{\alpha}_{i,j}}{2} \quad \text{and} \quad |\boldsymbol{\alpha}|_{i,j} = \max\{|\underline{\alpha}_{i,j}|, |\overline{\alpha}_{i,j}|\}$$

for any i,j. The following theorem characterizes full-rank *square* interval matrices.

THEOREM 1. (Rohn, [14]) *Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times p$ interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i,j. Let $Y_p = \{-1,1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x. Then $\boldsymbol{\alpha}$ is a full-rank interval matrix if and only if, for each $x, y \in Y_p$,*

$$\det\Big(\mathrm{mid}(\boldsymbol{\alpha})\Big)\, \det\Big(\mathrm{mid}(\boldsymbol{\alpha}) - T_x\, \mathrm{rad}(\boldsymbol{\alpha})\, T_y\Big) > 0.$$

*Received by the editors on December 18, 2018. Accepted for publication on May 21, 2019. Handling Editor: Michael Tsatsomeros.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

See [14] and [15] for other characterizations. The following theorem characterizes full-rank $p \times q$ interval matrices; see [16], [17] and [20].

THEOREM 2. (Rohn) *A $p \times q$ interval matrix $\boldsymbol{\alpha}$ with $p \geq q$ has full rank if and only if the system of inequalities*

$$|\mathrm{mid}(\boldsymbol{\alpha})\, x| \leq \mathrm{rad}(\boldsymbol{\alpha})\, |x|, \quad x \in \mathbb{R}^q$$

has only the trivial solution $x = 0$.

A research area which can be connected with the theory of interval matrices is the one of the partial matrices: let K be a field; a partial matrix over K is a matrix where only some of the entries are given and they are elements of K; a completion of a partial matrix is a specification of the unspecified entries. The problem of determining the maximal and the minimal rank of the completions of a partial matrix has been widely studied. In particular, in [5], Cohen, Johnson, Rodman and Woerdeman determined the maximal rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices; see also [4] for the proof. The problem of a theoretical characterization of the minimal rank of the completions of a partial matrix seems more difficult and it has been solved only in some particular cases. We quote also the paper [8], where a criterion to say if a partial matrix has a completion of rank 1 is established.

In [19], we generalized Theorem 1 to general closed interval matrices, that is, matrices whose entries are closed connected nonempty subsets of $\mathbb{R}$; obviously, the notion of general closed interval matrices generalizes the one of partial matrices and the one of interval matrices.

Also for interval matrices, the problem of determining the miminal rank of the matrices contained in a given interval matrix seems much more difficult than the problem of determining the maximal rank. We recall that in [18], we determined the maximum rank of the matrices contained in a given interval matrix and we gave a theoretical characterization of interval matrices containing at least a matrix of rank 1. Precisely the last result is the following (where the word "reduced" means that every column and every row has at least one entry not containing 0).

THEOREM 3. *Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ reduced interval matrix with $p, q \geq 2$ and $0 \leq \underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. There exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = 1$ if and only if, for any $h \in \mathbb{N}$ with $2 \leq h \leq 2^{\min\{p,q\}-1}$, for any $i_1, \ldots, i_h \in \{1, \ldots, p\}$, for any $j_1, \ldots, j_h \in \{1, \ldots, q\}$ and for any permutation σ on h elements, we have that*

$$\underline{\alpha}_{i_1,j_1} \cdots \underline{\alpha}_{i_h,j_h} \leq \overline{\alpha}_{i_1,j_{\sigma(1)}} \cdots \overline{\alpha}_{i_h,j_{\sigma(h)}}. \tag{1.1}$$

In the previous paper [6], the authors studied the complexity of an algorithm to decide if an interval matrix contains a rank-one matrix and proved that the problem is NP-complete.

Finally, we quote another research area which can be related to partial matrices, to interval matrices and, more generally, to general interval matrices: the one of the matrices with a given sign pattern; let Q be a $p \times q$ matrix with entries in $\{+, -, 0\}$; we say that $A \in M(p \times q, \mathbb{R})$ has sign pattern Q if, for any i, j, we have that $a_{i,j}$ is positive (respectively, negative or zero) if and only if $Q_{i,j}$ is $+$ (respectively, $-$ or 0). Obviously, the set of the matrices with a given sign pattern can be thought as a general interval matrices whose entries are from $\{(0, +\infty), (-\infty, 0), [0]\}$. There are several papers studying the minimal and maximal rank of the matrices with a given sign pattern, see for instance [1], [2], [10] and [21]. In particular, in [1] and

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A publication of the International Linear Algebra Society
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[2], the authors proved that the minimum rank of the real matrices with a given sign pattern is realizable by a rational matrix in case this minumum is at most 2 or at least $\min\{p, q\} - 2$.

As we have already said, in this paper, we generalize the concept of interval matrices to other fields. We define a *rational interval matrix* to be a matrix whose entries are intervals in $\mathbb{Q}$; we prove that a (real) interval $p \times q$ matrix with the endpoints of all its entries in $\mathbb{Q}$ contains a rank-one matrix if and only if contains a rational rank-one matrix and contains a matrix with rank smaller than $\min\{p, q\}$ if and only if it contains a rational matrix with rank smaller than $\min\{p, q\}$; from these results and from Theorem 1 and Theorem 3 we deduce immediately a criterion to see when a rational interval matrix contains a rank-one matrix and a criterion to see when it is full-rank, that is, all the matrices it contains are full-rank, see Section 3. Moreover, in Remark 13, we observe that from the papers [3] and [9] we can deduce that it is not true that, for any r, if an interval matrix with the endpoints of all its entries in $\mathbb{Q}$ contains a rank-r real matrix, then it contains a rank-r rational matrix. Finally, given a field K, we define a *subset matrix over* K to be a matrix whose entries are nonempty subsets of K and we describe a criterion to find the maximal rank of a matrix contained in a subset matrix (see Section 4).

2. Notation and first remarks.

• Let $\mathbb{R}_{>0}$ be the set $\{x \in \mathbb{R}|\ x > 0\}$ and let $\mathbb{R}_{\geq 0}$ be the set $\{x \in \mathbb{R}|\ x \geq 0\}$; we define analogously $\mathbb{R}_{<0}$ and $\mathbb{R}_{\leq 0}$. We denote by $\mathbb{I}$ the set $\mathbb{R} - \mathbb{Q}$.

• Throughout the paper let $p, q \in \mathbb{N} \setminus \{0\}$.

• Let Σ_p be the set of the permutations on $\{1, \dots, p\}$. For any $\sigma \in \Sigma_p$, we denote the sign of the permutation σ by $\epsilon(\sigma)$.

• For any ordered multiset $J = (j_1, \dots, j_r)$, a *multiset permutation* $\sigma(J)$ of J is an ordered arrangement of the multiset $\{j_1, \dots, j_r\}$, where each element appears as often as it does in J.

• For any field K, let $M(p \times q, K)$ denote the set of the $p \times q$ matrices with entries in K. For any $A \in M(p \times q, K)$, let $\mathrm{rk}(A)$ denote the rank of A and let $A^{(j)}$ be the j-th column of A.

• For any vector space V over a field K and any $v_1, \dots, v_k \in V$, let $\langle v_1, \dots, v_k\rangle$ be the span of $v_1, \dots, v_k$.

• Let $\boldsymbol{\alpha}$ be a $p \times q$ subset matrix over a field K. Given a matrix $A \in M(p \times q, K)$, we say that $A \in \boldsymbol{\alpha}$ if and only if $a_{i,j} \in \boldsymbol{\alpha}_{i,j}$ for any i, j. We define

$$\mathrm{mrk}(\boldsymbol{\alpha}) = \min\{\mathrm{rk}(A)|\ A \in \boldsymbol{\alpha}\},$$

$$\mathrm{Mrk}(\boldsymbol{\alpha}) = \max\{\mathrm{rk}(A)|\ A \in \boldsymbol{\alpha}\}.$$

We call them respectively *minimal rank* and *maximal rank* of $\boldsymbol{\alpha}$. Moreover, we define

$$\mathrm{rkRange}(\boldsymbol{\alpha}) = \{\mathrm{rk}(A)|\ A \in \boldsymbol{\alpha}\};$$

we call it the *rank range* of $\boldsymbol{\alpha}$.

We say that an entry of $\boldsymbol{\alpha}$ is *degenerate* if its cardinality is 1.

REMARK 4. Let $\boldsymbol{\alpha}$ be a subset matrix over a field K. Observe that

$$\mathrm{rkRange}(\boldsymbol{\alpha}) = [\mathrm{mrk}(\boldsymbol{\alpha}), \mathrm{Mrk}(\boldsymbol{\alpha})] \cap \mathbb{N}.$$

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The proof is identical to the one of the case of interval matrices in [18] (Remark 3).

We refer to some classical books on interval analysis, such as [11], [12] and [13] for the definition of sum and multiplication of two intervals. In particular, for any interval α in $\mathbb{R}$ and any interval β either in $\mathbb{R}_{>0}$ or in $\mathbb{R}_{<0}$, we define $\frac{\alpha}{\beta}$ to be the set $\left\{ \frac{a}{b} \middle|\, a \in \alpha,\ b \in \beta \right\}$. Obviously, we can give analogous definitions for intervals in $\mathbb{Q}$.

Definition 5. Let $\boldsymbol{\alpha}$ be an interval matrix (respectively, a rational interval matrix). We say that another interval matrix (respectively, rational interval matrix) $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ by an *elementary row operation* if it is obtained from $\boldsymbol{\alpha}$ by one of the following operations (where we are considering interval arithmetic):

I) interchanging two rows,

II) multiplying a row by a nonzero real number (respectively, rational number),

III) adding to a row the multiple of another row by a real number (respectively, rational number).

In an analogous way, we may define *elementary column operations*.

Remark 6.

• Obviously, the operations of the first two kinds give an equivalence relation, but if we consider also the third kind we do not get an equivalence relation.

• Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ be two interval matrices (respectively, two rational interval matrices), such that $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ by elementary row (or column) operations. Then, obviously,

$$\mathrm{rkRange}(\boldsymbol{\alpha}) \subseteq \mathrm{rkRange}(\boldsymbol{\alpha}'). \tag{2.2}$$

Moreover, if $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ only by elementary row (or column) operations of kind I or II, we have the equality in (2.2).

Remark 7. Let $\boldsymbol{\alpha}$ be a (rational) interval matrix. If $\boldsymbol{\alpha}'$ is the (rational) interval matrix obtained from $\boldsymbol{\alpha}$ by deleting the columns and the rows such that all their entries contain 0, we have that $\mathrm{mrk}(\boldsymbol{\alpha}) = \mathrm{mrk}(\boldsymbol{\alpha}')$. Obviously, the analogous statement for Mrk does not hold.

3. Some results on rational interval matrices.

Theorem 8. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i, j. If there exists $A \in \alpha$ with $rk(A) < q$, then there exists $B \in \alpha \cap M(p \times q, \mathbb{Q})$ with $rk(B) < q$.*

Proof. We can suppose that $A^{(q)} \in \langle A^{(1)}, \ldots, A^{(q-1)} \rangle$; let $A^{(q)}$ be equal to

$$c_1 A^{(1)} + \cdots + c_{q-1} A^{(q-1)}$$

for some $c_1, \ldots, c_{q-1} \in \mathbb{R} \setminus \{0\}$. Up to swapping rows and columns, we can also suppose $c_1, \ldots, c_r \in \mathbb{Q}$, $c_{r+1}, \ldots, c_{q-1} \in \mathbb{I}$, $a_{1,q}, \ldots, a_{k,q} \in \mathbb{I}$ and $a_{k+1,q}, \ldots, a_{p,q} \in \mathbb{Q}$.

Finally, we can easily suppose that $a_{i,j} \in \mathbb{Q}$ for any $i = 1, \ldots, k$ and $j = 1, \ldots, q-1$; in fact: for any $i \in \{1, \ldots, k\}$, let $Z(i) = \{j \in \{1, \ldots, q-1\} |\ a_{i,j} \in \mathbb{I}\}$; if for some $i \in \{1, \ldots, k\}$ the set $Z(i)$ is nonempty, we have that for any $j \in Z(i)$ the entry $\boldsymbol{\alpha}_{i,j}$ is nondegenerate (since has rational endpoints and contains $a_{i,j}$,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

which is irrational); so there exist neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$ for any $j \in Z(i)$ such that

$$\sum_{j \in \{1,\dots,q-1\} \setminus Z(i)} c_j a_{i,j} + \sum_{j \in Z(i)} c_j U_{i,j} \subset \boldsymbol{\alpha}_{i,q}$$

(observe that, for $i = 1, \dots, k$, the entry $\boldsymbol{\alpha}_{i,q}$ is nondegenerate, since it has rational endpoints and contains $a_{i,q}$ which is irrational); hence, for any $j \in Z(i)$, we can change the entry $a_{i,j}$ into an element $\tilde{a}_{i,j}$ of $U_{i,j} \cap \mathbb{Q}$ and the entry $a_{i,q}$ into

$$\sum_{j \in \{1,\dots,q-1\} \setminus Z(i)} c_j a_{i,j} + \sum_{j \in Z(i)} c_j \tilde{a}_{i,j};$$

in this way, we get again a matrix with the last column in the span of the first $q-1$ columns; moreover, each of the first k rows of the matrix we have obtained has the first $q-1$ entries rational. So we can suppose that $a_{i,j} \in \mathbb{Q}$ for any $i = 1, \dots, k$ and $j = 1, \dots, q-1$.

For any $i = k+1, \dots, p$, we define:

$R_i = \{j \in \{1, \dots, r\} |\ a_{i,j} \in \mathbb{Q}\}$,

$N_i = \{j \in \{1, \dots, r\} |\ a_{i,j} \in \mathbb{I}\}$,

$\tilde{R}_i = \{j \in \{r+1, \dots, q-1\} |\ a_{i,j} \in \mathbb{Q}\}$,

$\tilde{N}_i = \{j \in \{r+1, \dots, q-1\} |\ a_{i,j} \in \mathbb{I}\}$.

Moreover, define

$$X = \left\{ i \in \{k+1, \dots, p\} |\ N_i \cup \tilde{N}_i = \emptyset \right\}.$$

• Let $i \in \{1, \dots, k\}$. For any $j = r+1, \dots, q-1$, there exists a neighbourhood V_j^i of c_j such that

$$\sum_{j=1,\dots,r} c_j a_{i,j} + \sum_{j=r+1,\dots,q-1} V_j^i a_{i,j} \subset \boldsymbol{\alpha}_{i,q}. \tag{3.3}$$

• Let $i \in \{k+1, \dots, p\} \setminus X$. By definition of the set X, there exists $\overline{\jmath}(i) \in N_i \cup \tilde{N}_i$. We consider neighbourhoods V_j^i of c_j contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$ for any $j \in \{r+1, \dots, q-1\}$ and neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$ for any $j \in N_i \cup \tilde{N}_i \setminus \{\overline{\jmath}(i)\}$ such that

$$-\frac{1}{c_{\overline{\jmath}(i)}} \left[\sum_{j \in N_i \setminus \{\overline{\jmath}(i)\}} c_j U_{i,j} + \sum_{j \in \tilde{N}_i} V_j^i U_{i,j} + \sum_{j \in \tilde{R}_i} V_j^i a_{i,j} + \sum_{j \in R_i} c_j a_{i,j} - a_{i,q} \right] \subset \boldsymbol{\alpha}_{i,\overline{\jmath}(i)} \tag{3.4}$$

if $c_{\overline{\jmath}(i)} \in \mathbb{Q}$ (i.e., $\overline{\jmath}(i) \in \{1, \dots, r\}$) and

$$-\frac{1}{V^i_{\overline{\jmath}(i)}} \left[\sum_{j \in N_i} c_j U_{i,j} + \sum_{j \in \tilde{N}_i \setminus \{\overline{\jmath}(i)\}} V_j^i U_{i,j} + \sum_{j \in \tilde{R}_i} V_j^i a_{i,j} + \sum_{j \in R_i} c_j a_{i,j} - a_{i,q} \right] \subset \boldsymbol{\alpha}_{i,\overline{\jmath}(i)} \tag{3.5}$$

if $c_{\overline{\jmath}(i)} \in \mathbb{I}$ (i.e., $\overline{\jmath}(i) \in \{r+1, \dots, q-1\}$).

Choice of the $\tilde{c}_j$ for $j = r+1, \dots, q-1$. If $X = \emptyset$, for any $j = r+1, \dots, q-1$, choose $\tilde{c}_j$ in the set

$$\left(\cap_{i \in \{1,\dots,p\}} V_j^i \right) \cap \mathbb{Q}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

If $X \neq \emptyset$, consider the submatrix of A given by the rows indicized by X and the columns $r+1,\ldots,q-1$ and reduce it in row echelon form by elementary row operations; let T be the set of the $j \in \{r+1,\ldots,q-1\}$ corresponding to some pivot, and let S be the set $\{r+1,\ldots,q-1\} \setminus T$. For any $j \in S$, choose $\tilde{c}_j \in \left(\cap_{i\in\{1,\ldots,p\}\setminus X} V_j^i\right) \cap \mathbb{Q}$ in such a way that, called $\tilde{c}_j$ for $j \in T$ the solutions of the linear systems given by the equations

$$\sum_{j\in R_i} c_j a_{i,j} + \sum_{j\in\tilde{R}_i} \tilde{c}_j a_{i,j} = a_{i,q}, \tag{3.6}$$

for $i \in X$, we have that $\tilde{c}_j \in \cap_{i\in\{1,\ldots,p\}\setminus X} V_j^i$ for any $j \in T$.

Choice of the $\tilde{a}_{i,j}$ for $i \in \{k+1,\ldots,p\} \setminus X$, $j \in N_i \cup \tilde{N}_i$. Now, for any $i \in \{k+1,\ldots,p\} \setminus X$, choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$ for any $j \in N_i \cup \tilde{N}_i \setminus \{\overline{\jmath}(i)\}$ and define $\tilde{a}_{i,\overline{\jmath}(i)}$ to be

$$-\frac{1}{c_{\overline{\jmath}(i)}}\left[\sum_{j\in N_i\setminus\{\overline{\jmath}(i)\}} c_j\tilde{a}_{i,j} + \sum_{j\in\tilde{N}_i}\tilde{c}_j\tilde{a}_{i,j} + \sum_{j\in\tilde{R}_i}\tilde{c}_j a_{i,j} + \sum_{j\in R_i} c_j a_{i,j} - a_{i,q}\right] \tag{3.7}$$

if $c_{\overline{\jmath}(i)} \in \mathbb{Q}$ (i.e., $\overline{\jmath}(i) \in N_i$),

$$-\frac{1}{\tilde{c}_{\overline{\jmath}(i)}}\left[\sum_{j\in N_i} c_j\tilde{a}_{i,j} + \sum_{j\in\tilde{N}_i\setminus\{\overline{\jmath}(i)\}}\tilde{c}_j\tilde{a}_{i,j} + \sum_{j\in\tilde{R}_i}\tilde{c}_j a_{i,j} + \sum_{j\in R_i} c_j a_{i,j} - a_{i,q}\right] \tag{3.8}$$

if $c_{\overline{\jmath}(i)} \in \mathbb{I}$ (i.e., $\overline{\jmath}(i) \in \tilde{N}_i$). By (3.4) and (3.5), we have that $\tilde{a}_{i,\overline{\jmath}(i)} \in \mathbb{Q} \cap \boldsymbol{\alpha}_{i,\overline{\jmath}(i)}$.

Let B be the $p \times q$ matrix such that, for every $i = 1,\ldots,p$ and $j = 1,\ldots,q-1$,

$$B_{i,j} = \begin{cases} \tilde{a}_{i,j} & \text{if } a_{i,j} \in \mathbb{I} \\ a_{i,j} & \text{if } a_{i,j} \in \mathbb{Q} \end{cases}$$

and such that

$$B^{(q)} = \sum_{j=1,\ldots,r} c_j B^{(j)} + \sum_{j=r+1,\ldots,q-1} \tilde{c}_j B^{(j)}.$$

By the choice of $\tilde{c}_j$ for $j = r+1,\ldots,q-1$ (see (3.6)) and the choice of $\tilde{a}_{i,j}$ for $i \in \{k+1,\ldots,p\} \setminus X$, $j \in N_i \cup \tilde{N}_i$ (see (3.7) and (3.8)), we have that $b_{i,q} = a_{i,q}$ for $i = k+1,\ldots,p$. By the choice of $\tilde{c}_j$ for $j = r+1,\ldots,q-1$ such that $\tilde{c}_j \in \cap_{i\in\{1,\ldots,p\}\setminus X} V_j^i$ and by (3.3), we get that $b_{i,q} \in \boldsymbol{\alpha}_{i,q}$ for $i = 1,\ldots,k$. So the matrix B is contained in $\boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$. ∎

From Theorem 1 and Theorem 8, we get immediately the following corollary.

Corollary 9. *Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times p$ rational interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i,j. Let $Y_p = \{-1,1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x.*

Then $\boldsymbol{\alpha}$ is a full-rank rational interval matrix if and only if, for each $x, y \in Y_p$,

$$\det\Big(\mathrm{mid}(\boldsymbol{\alpha})\Big)\,\det\Big(\mathrm{mid}(\boldsymbol{\alpha}) - T_x\,\mathrm{rad}(\boldsymbol{\alpha})\,T_y\Big) > 0.$$

Before stating the second theorem, we enunciate a lemma that will be useful in the proof of the theorem.

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A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

LEMMA 10. *Let $A \in M(m \times n, \mathbb{Q})$ for some $m, n \in \mathbb{N} \setminus \{0\}$. Let $c \in \mathbb{R}^n \setminus \{0\}$ be such that $Ac = 0$. Then, for every V neighbourhood of c, we can find $\tilde{c} \in V \cap \mathbb{Q}^n$ such that $A\tilde{c} = 0$. If, in addition, $c \in (\mathbb{R} \setminus \{0\})^n$, we can find $\tilde{c}$ in $V \cap (\mathbb{Q} \setminus \{0\})^n$ such that $A\tilde{c} = 0$.*

Proof. Let $\overline{A}$ be a matrix in row echelon form obtained from A by elementary row operations. We can suppose that the columns containing the pivots are the first k. Since c is nonzero and $Ac = 0$, we have that $k < n$. Write $c = \begin{pmatrix} c' \\ c'' \end{pmatrix}$ with c' given by the first k entries of c and c'' given by the last $n-k$ entries and let V' and V'' be respectively neighbourhood of c' and c'' such that $V' \times V''$ is contained in V. There exists a neighbourhood U of c'' contained in V'' such that, if $b'' \in U$ and $\begin{pmatrix} b' \\ b'' \end{pmatrix}$ is the solution of the linear system $\overline{A}x = 0$ with vector of the last $n-k$ entries equal to b'', we have that $b' \in V'$. So, if we take $\tilde{c}'' \in U \cap \mathbb{Q}^{n-k}$ and $\begin{pmatrix} \tilde{c}' \\ \tilde{c}'' \end{pmatrix}$ is the solution of the linear system $\overline{A}x = 0$ with vector of the last $n-k$ entries equal to $\tilde{c}''$, we have that $\begin{pmatrix} \tilde{c}' \\ \tilde{c}'' \end{pmatrix} \in V \cap \mathbb{Q}^n$.

Finally, the last statement is obvious, because, if $c \in (\mathbb{R} \setminus \{0\})^n$, we can find a neighbourhood W of c contained in $V \cap (\mathbb{R} \setminus \{0\})^n$ and, by applying the previous statement to W, we get $\tilde{c} \in W \cap \mathbb{Q}^n$ such that $A\tilde{c} = 0$, thus $\tilde{c} \in V \cap (\mathbb{Q} \setminus \{0\})^n$ and $A\tilde{c} = 0$. ∎

THEOREM 11. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i, j. If there exists $A \in \alpha$ with $rk(A) = 1$, then there exists $B \in \alpha \cap M(p \times q, \mathbb{Q})$ with $rk(B) = 1$.*

Proof. We can suppose that every entry of $A^{(1)}$ is nonzero and, for $j = 2, \ldots, q$, we have that $A^{(j)} = c_j A^{(1)}$ for some $c_j \in \mathbb{R} \setminus \{0\}$. For any $i \in \{1, \ldots, p\}$ such that $\boldsymbol{\alpha}_{i,1}$ is nondegenerate, let $\tilde{\boldsymbol{\alpha}}_{i,1}$ be a closed nondegenerate interval neighbourhood of $a_{i,1}$, contained either in $\boldsymbol{\alpha}_{i,1} \cap \mathbb{R}_{>0}$ or in $\boldsymbol{\alpha}_{i,1} \cap \mathbb{R}_{<0}$.

We can suppose also that $c_2, \ldots, c_k \in \mathbb{I}$ and $c_{k+1}, \ldots, c_q \in \mathbb{Q}$.

For any $j = 2, \ldots, q$, let $I_j = \{i \in \{1, \ldots, p\} |\ a_{i,j} \in \mathbb{I}\}$ and let $Q_j = \{i \in \{1, \ldots, p\} |\ a_{i,j} \in \mathbb{Q}\}$.

Let $j \in \{2, \ldots, q\}$ and $i \in I_j$; obviously, $\boldsymbol{\alpha}_{i,j}$ is nondegenerate, because it has rational endpoints and contains $a_{i,j}$ which is irrational;

if $a_{i,1} \in \mathbb{I}$, we define A_i^j to be a neighbourhood of $a_{i,1}$ contained either in $\boldsymbol{\alpha}_{i,1} \cap \mathbb{R}_{>0}$ or in $\boldsymbol{\alpha}_{i,1} \cap \mathbb{R}_{<0}$, and, if $c_j \in \mathbb{I}$ (i.e., $j \in \{2, \ldots, k\}$), we define V_i^j to be a neighbourhood of c_j, such that:

$$\begin{array}{ll} V_i^j a_{i,1} \subset \boldsymbol{\alpha}_{i,j} & \text{if } \ a_{i,1} \in \mathbb{Q} \ \text{ and } \ c_j \in \mathbb{I}, \\ c_j A_i^j \subset \boldsymbol{\alpha}_{i,j} & \text{if } \ a_{i,1} \in \mathbb{I} \ \text{ and } \ c_j \in \mathbb{Q}, \\ V_i^j A_i^j \subset \boldsymbol{\alpha}_{i,j} & \text{if } \ a_{i,1} \in \mathbb{I} \ \text{ and } \ c_j \in \mathbb{I}. \end{array}$$

For any $j = 2, \ldots, k$, let $\tilde{c}_j$ be such that

(1) $\tilde{c}_j \in \left(\cap_{i \in I_j} V_i^j\right) \cap \left(\cap_{i \in Q_j} \frac{a_{i,j}}{\tilde{\boldsymbol{\alpha}}_{i,1}}\right) \cap (\mathbb{Q} \setminus \{0\})$ (observe that, if $i \in Q_j$, then, since $a_{i,j} \in \mathbb{Q}$ and $c_j \in \mathbb{I}$, we have that $a_{i,1} \in \mathbb{I}$, so $\boldsymbol{\alpha}_{i,1}$ is nondegenerate),

(2) $\frac{a_{i,j}}{\tilde{c}_j} = \frac{a_{i,j'}}{\tilde{c}_{j'}}$ for any $i \in \{1, \ldots, p\}$ and $j, j' \in \{2, \ldots, k\}$ such that $i \in Q_j \cap Q_{j'}$,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

(3) $\tilde{c}_j \in \frac{a_{i,j}}{A_i^{j'}}$ for any $i \in \{1,\dots,p\}$, $j \in \{2,\dots,k\}$, $j' \in \{2,\dots,q\}$ such that $i \in Q_j \cap I_{j'}$.

By Lemma 10, we can find $\tilde{c}_j$ satisfying (1),(2),(3) because V_i^j for $i \in I_j$, $\frac{a_{i,j}}{\tilde{\boldsymbol{\alpha}}_{i,1}}$ for $i \in Q_j$ and $\frac{a_{i,j}}{A_i^{j'}}$ for $i \in Q_j \cap I_{j'}$ are neighbourhoods of c_j and the equations in (2) give a homogeneous linear system in the variables $\tilde{c}_j$ satisfied by the c_j.

We define B to be the matrix such that, for any $i = 1,\dots,p$,

$$b_{i,1} = \begin{cases} \frac{a_{i,j}}{\tilde{c}_j} & \text{if } a_{i,1} \in \mathbb{I} \text{ and } i \in Q_j \text{ for some } j \in \{2,\dots,k\}, \quad \text{(1st case)} \\ \text{an element of } \cap_{j=2,\dots,q} A_i^j \cap \mathbb{Q} & \text{if } a_{i,1} \in \mathbb{I} \text{ and } i \in I_j \ \forall j \in \{2,\dots,q\}, \quad \text{(2nd case)} \\ a_{i,1} & \text{if } a_{i,1} \in \mathbb{Q} \quad \text{(3rd case)} \end{cases}$$

and such that

$$B^{(j)} = \begin{cases} \tilde{c}_j B^{(1)} & \text{for } j = 2,\dots,k, \\ c_j B^{(1)} & \text{for } j = k+1,\dots,q. \end{cases}$$

Observe that in the definition of $b_{i,1}$, the 1^{st} case and the 2^{nd} case cover all the case $a_{i,1} \in \mathbb{I}$, because, if $a_{i,1} \in \mathbb{I}$ and $i \in Q_j$ for some $j \in \{2,\dots,q\}$, then $c_j \in \mathbb{I}$, so $j \in \{2,\dots,k\}$.

Observe also that the definition of $b_{i,1}$ in the 1^{st} case is good by condition (2). Moreover $b_{i,1} \in \mathbb{Q}$ for any $i \in \{1,\dots,p\}$ and, finally, $b_{i,1}$ is an element of $\boldsymbol{\alpha}_{i,1}$: in the 1^{st} case, this follows from condition (1), in the other cases it is obvious.

Now we want to prove that $\tilde{c}_j B^{(1)} \in \boldsymbol{\alpha}^{(j)}$ for $j = 2,\dots,k$ and that $c_j B^{(1)} \in \boldsymbol{\alpha}^{(j)}$ for $j = k+1,\dots,q$.

• First, let us prove that $\tilde{c}_j b_{i,1} \in \boldsymbol{\alpha}_{i,j}$ for $j = 2,\dots,k$, $i = 1,\dots,p$. Let us fix $i \in \{1,\dots,p\}$.

1st Case. In this case, $a_{i,1} \in \mathbb{I}$, $i \in Q_l$ for some $l \in \{2,\dots,k\}$ and $b_{i,1}$ is defined to be $\frac{a_{i,l}}{\tilde{c}_l}$; therefore, for $j = 2,\dots,k$,

$$\tilde{c}_j b_{i,1} = \tilde{c}_j \frac{a_{i,l}}{\tilde{c}_l} = a_{i,j} \in \boldsymbol{\alpha}_{i,j}$$

if $i \in Q_j$ (where the second equality holds by condition (2)), and

$$\tilde{c}_j b_{i,1} = \tilde{c}_j \frac{a_{i,l}}{\tilde{c}_l} \in V_i^j A_i^j \subset \boldsymbol{\alpha}_{i,j}$$

if $i \in I_j$ (where the first inclusion holds by conditions (1) and (3) and the second by the definition of V_i^j and A_i^j).

2nd Case. In this case, $a_{i,1} \in \mathbb{I}$, $i \in I_j$ for any $j \in \{2,\dots,q\}$ and $b_{i,1}$ is defined to be a rational element of $\cap_{j=2,\dots,q} A_i^j$; hence, for $j = 2,\dots,k$,

$$\tilde{c}_j b_{i,1} \in V_i^j A_i^j \subset \boldsymbol{\alpha}_{i,j}.$$

3rd Case. In this case, $a_{i,1} \in \mathbb{Q}$ and $b_{i,1}$ is defined to be $a_{i,1}$; so, for any $j \in \{2,\dots,k\}$, we get:

$$\tilde{c}_j b_{i,1} = \tilde{c}_j a_{i,1} \in V_i^j a_{i,1} \subset \boldsymbol{\alpha}_{i,j},$$

where the first inclusion holds by condition (1) (observe that, since $c_j \in \mathbb{I}$ and $a_{i,1} \in \mathbb{Q}$, we have that $a_{i,j} \in \mathbb{I}$, thus $i \in \mathbb{I}_j$) and the second by the definition of V_i^j.

• Finally, let us prove that $c_j b_{i,1} \in \boldsymbol{\alpha}_{i,j}$ for $j = k+1,\dots,q$, $i = 1,\dots,p$. Fix $i \in \{1,\dots,p\}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

1st Case. In this case, $a_{i,1} \in \mathbb{I}$, $i \in Q_l$ for some $l \in \{2, \dots, k\}$ and $b_{i,1}$ is defined to be $\frac{a_{i,l}}{\tilde{c}_l}$; so, for $j = k+1, \dots, q$,

$$c_j b_{i,1} = c_j \frac{a_{i,l}}{\tilde{c}_l} \in c_j A_i^j \subset \boldsymbol{\alpha}_{i,j},$$

where the first inclusion holds by condition (3) since $i \in Q_l \cap I_j$ and the last inclusion holds by definition of A_i^j.

2nd Case. In this case, $a_{i,1} \in \mathbb{I}$, $i \in I_j$ for any $j \in \{2, \dots, q\}$ and $b_{i,1}$ is defined to be a rational element of $\cap_{j=2,\dots,q} A_i^j$; hence, for $j = k+1, \dots, q$,

$$c_j b_{i,1} \in c_j A_i^j \subset \boldsymbol{\alpha}_{i,j},$$

where the last inclusion holds by definition of A_i^j.

3rd Case. In this case, $a_{i,1} \in \mathbb{Q}$ and $b_{i,1}$ is defined to be $a_{i,1}$; hence,

$$c_j b_{i,1} = c_j a_{i,1} = a_{i,j} \in \boldsymbol{\alpha}_{i,j}$$

for any $j = k+1, \dots, q$. ◻

Theorem 3 and Theorem 11 imply obviously the following corollary.

COROLLARY 12. *Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ reduced rational interval matrix with $p, q \geq 2$ and $0 \leq \underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. There exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = 1$ if and only if, for any $h \in \mathbb{N}$ with $2 \leq h \leq 2^{\min\{p,q\}-1}$, for any $i_1, \dots, i_h \in \{1, \dots, p\}$, for any $j_1, \dots, j_h \in \{1, \dots, q\}$ and for any $\sigma \in \Sigma_h$, we have:*

$$\underline{\alpha}_{i_1,j_1} \cdots \underline{\alpha}_{i_h,j_h} \leq \overline{\alpha}_{i_1,j_{\sigma(1)}} \cdots \overline{\alpha}_{i_h,j_{\sigma(h)}}.$$

Observe that, as for (real) interval matrices (see Remarks 8 and 9 in [18]), to study when a reduced rational interval matrix contains a rank-one matrix it is sufficient to study the problem for a reduced rational interval matrix $\boldsymbol{\alpha}$, with $\boldsymbol{\alpha}_{i,j} \subseteq \mathbb{R}_{\geq 0}$ for every i, j.

In fact, let $\boldsymbol{\alpha}$ be a $p \times q$ reduced rational interval matrix. Let $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ be such that $\underline{\alpha}_{i,j} \leq 0 \leq \overline{\alpha}_{i,j}$. Define $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ to be the rational interval matrices such that $\boldsymbol{\alpha}'_{i,j} = [\underline{\alpha}_{i,j}, 0]$, $\boldsymbol{\alpha}''_{i,j} = [0, \overline{\alpha}_{i,j}]$ and $\boldsymbol{\alpha}'_{t,s} = \boldsymbol{\alpha}''_{t,s} = \boldsymbol{\alpha}_{t,s}$ for any $(t,s) \neq (i,j)$ (observe that obviously the definition of $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ do depend on i, j we have fixed). Then

$$\{A \in \boldsymbol{\alpha}\} = \{A \in \boldsymbol{\alpha}'\} \cup \{A \in \boldsymbol{\alpha}''\};$$

hence, there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = r$, for any $r \in \mathbb{N}$, if and only if either there exists $A \in \boldsymbol{\alpha}'$ with $\mathrm{rk}(A) = r$ or there exists $A \in \boldsymbol{\alpha}''$ with $\mathrm{rk}(A) = r$. In particular, to study whether a rational interval matrix $\boldsymbol{\alpha}$ contains a rank-r matrix, it is sufficient to consider the case where, for any i, j, either $\boldsymbol{\alpha}_{i,j} \subseteq \mathbb{R}_{\geq 0}$ or $\boldsymbol{\alpha}_{i,j} \subseteq \mathbb{R}_{\leq 0}$. Observe that splitting every entry of $\boldsymbol{\alpha}$ into the nonnegative part and the nonpositive part can give 2^{pq} matrices in the worst case.

Moreover, by Remark 6, we can suppose $\boldsymbol{\alpha}_{i,j} \subseteq \mathbb{R}_{\geq 0}$ for every (i,j) such that either i or j is equal to 1. Finally for such a matrix $\boldsymbol{\alpha}$, if there exists (i,j) such that $\boldsymbol{\alpha}_{i,j} \subseteq \mathbb{R}_{<0}$, then $\boldsymbol{\alpha}$ does not contain a rank-one matrix. Otherwise, that is $\overline{\alpha}_{i,j} \geq 0$ for any i, j, define $\hat{\boldsymbol{\alpha}}$ to be the rational interval matrix such that

$$\hat{\boldsymbol{\alpha}}_{i,j} = [\max\{0, \underline{\alpha}_{i,j}\}, \overline{\alpha}_{i,j}]$$

for any i, j. Obviously, $\boldsymbol{\alpha}$ contains a rank-one matrix if and only if $\hat{\boldsymbol{\alpha}}$ contains a rank-one matrix.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

Remark 13. In [3] and [9], the authors showed that there exists a sign pattern Q such that the minimal rank $r_Q^{\mathbb{R}}$ of the real matrices with sign pattern Q is strictly smaller than the minimal rank $r_Q^{\mathbb{Q}}$ of the rational matrices with sign pattern Q. Let A be a real matrix with sign pattern Q and rank $r_Q^{\mathbb{R}}$. Let $\boldsymbol{\alpha}$ be an interval matrix containing A and such that, for any i, j, we have:

$\boldsymbol{\alpha}_{i,j} = \{0\}$ if and only if $a_{i,j} = 0$,

$\boldsymbol{\alpha}_{i,j} \subset \mathbb{R}_{>0}$ if and only if $a_{i,j} > 0$,

$\boldsymbol{\alpha}_{i,j} \subset \mathbb{R}_{<0}$ if and only if $a_{i,j} < 0$.

Obviously, $\mathrm{mrk}(\boldsymbol{\alpha}) = r_Q^{\mathbb{R}}$ and, since there does not exist a rational matrix with sign pattern Q and rank $r_Q^{\mathbb{R}}$, there does not exist a rational matrix in $\boldsymbol{\alpha}$ with rank $r_Q^{\mathbb{R}}$. So Theorem 8 and Theorem 11 are not generalizable to any rank, that is, it is not true for any r, that, if an interval matrix contains a rank-r real matrix, then it contains a rank-r rational matrix.

4. Maximal rank of matrices contained in a subset matrix over any field.

Definition 14. Given a $p \times p$ subset matrix over e a field K, $\boldsymbol{\alpha}$, a *partial generalized diagonal* (*pg-diagonal* for short) of length k of $\boldsymbol{\alpha}$ is a k-uple of the kind

$$(\boldsymbol{\alpha}_{i_1,j_1}, \dots, \boldsymbol{\alpha}_{i_k,j_k})$$

for some $\{i_1, \dots i_k\}$ and $\{j_1, \dots, j_k\}$ subsets of $\{1, \dots, p\}$.

Its *complementary matrix* is defined to be the submatrix of $\boldsymbol{\alpha}$ given by the rows and columns whose indices are respectively in $\{1, \dots, p\} \setminus \{i_1, \dots, i_k\}$ and in $\{1, \dots, p\} \setminus \{j_1, \dots, j_k\}$.

We say that a pg-diagonal is *totally nondegenerate* if and only if all its entries are not degenerate.

We define $\det^c(\boldsymbol{\alpha})$ to be

$$\sum_{\sigma \in \Sigma_p \text{ s.t. } \boldsymbol{\alpha}_{1,\sigma(1)}, \dots, \boldsymbol{\alpha}_{p,\sigma(p)} \text{ are degenerate}} \epsilon(\sigma)\, \boldsymbol{\alpha}_{1,\sigma(1)} \cdots \boldsymbol{\alpha}_{p,\sigma(p)}$$

if there exists $\sigma \in \Sigma_p$ such that $\boldsymbol{\alpha}_{1,\sigma(1)}, \dots, \boldsymbol{\alpha}_{p,\sigma(p)}$ are degenerate; we define $\det^c(\boldsymbol{\alpha})$ to be equal to 0 otherwise.

For every pg-diagonal of length p, say $\boldsymbol{\alpha}_{1,\sigma(1)}, \dots, \boldsymbol{\alpha}_{p,\sigma(p)}$ for some $\sigma \in \Sigma_p$, we call $\epsilon(\sigma)$ also the sign of the pg-diagonal.

In [7], Hladík introduced the notion of strongly singular interval matrix. We generalize it to subset matrices over any field.

Definition 15. (Hladík) Let $\boldsymbol{\alpha}$ be a $p \times p$ subset matrix over a field K. We say that it is strongly singular if $\mathrm{Mrk}(\boldsymbol{\alpha}) < p$, that is, if every $A \in \boldsymbol{\alpha}$ is singular.

Theorem 16. *Let $\boldsymbol{\alpha}$ be a $p \times p$ subset matrix over a field K. Then $\boldsymbol{\alpha}$ is* strongly singular *if and only if the following conditions hold:*

(1) in $\boldsymbol{\alpha}$ there is no totally nondegenerate pg-diagonal of length p,

(2) the complementary matrix of every totally nondegenerate pg-diagonal of length between 0 *and* $p-1$ *has* $\det^c$ *equal to* 0 *(in particular,* $\det^c(\boldsymbol{\alpha}) = 0$*).*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

The proof is quite similar to the one of Theorem 13 in [18]; for the convenience of the reader, we sketch the proof here.

Proof. ($\Longrightarrow$) We argue by induction on p. For $p = 1$ the statement is obvious. Suppose $p \geq 2$ and that the statement is true for $(p-1) \times (p-1)$ subset matrices. Let $\boldsymbol{\alpha}$ be a $p \times p$ subset matrix such that $\mathrm{Mrk}(\boldsymbol{\alpha}) < p$; so $\det(A) = 0$ for every $A \in \boldsymbol{\alpha}$.

If $\boldsymbol{\alpha}$ contained a totally nondegenerate pg-diagonal of length p, say $\boldsymbol{\alpha}_{i_1,j_1}, \ldots, \boldsymbol{\alpha}_{i_p,j_p}$, then $\boldsymbol{\alpha}_{\hat{i_1},\hat{j_1}}$ would have obviously a totally nondegenerate pg-diagonal of length $p-1$; hence, by induction assumption, there would exist $B \in \boldsymbol{\alpha}_{\hat{i_1},\hat{j_1}}$ with $\det(B) \neq 0$. Hence, for any choice of elements $x_{i_1,j} \in \boldsymbol{\alpha}_{i_1,j}$ for $j \neq j_1$ and $x_{i,j_1} \in \boldsymbol{\alpha}_{i,j_1}$ for $i \neq i_1$, we could find $x \in \boldsymbol{\alpha}_{i_1,j_1}$ such that the determinant of the matrix X defined by $X_{\hat{i_1},\hat{j_1}} = B$, $X_{i_1,j_1} = x$, $X_{i,j_1} = x_{i,j_1}$ for any $i \neq i_1$ and $X_{i_1,j} = x_{i_1,j}$ for any $j \neq j_1$ is nonzero, which is absurd. So (1) holds.

Now, by contradiction, suppose (2) does not hold. Thus in $\boldsymbol{\alpha}$ there exists a totally nondegenerate pg-diagonal of length k with $0 \leq k \leq p-1$ whose complementary matrix has $\det^c$ nonzero. If there exists such a diagonal with $k \geq 1$, say $\boldsymbol{\alpha}_{i_1,j_1}, \ldots, \boldsymbol{\alpha}_{i_k,j_k}$, then also $\boldsymbol{\alpha}_{\hat{i_1},\hat{j_1}}$ does not satisfy (2), so, by induction assumption, there exists $B \in \boldsymbol{\alpha}_{\hat{i_1},\hat{j_1}}$ with $\det(B) \neq 0$ and, as before, we can get a contradiction. On the other hand, suppose that $\det^c(\boldsymbol{\alpha}) \neq 0$ and the complementary matrix of every totally nondegenerate pg-diagonal of length k with $1 \leq k \leq p-1$ has $\det^c$ equal to zero; we call this assumption $(*)$.

Let $A \in \boldsymbol{\alpha}$. By (1), we can write $\det(A)$ as the sum of:

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such the corresponding entries of $\boldsymbol{\alpha}$ are all degenerate,

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of $\boldsymbol{\alpha}$ apart from one are degenerate,

$\vdots$

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of $\boldsymbol{\alpha}$ apart from $p-1$ are degenerate.

We call $(\star)$ this way to write $det(A)$.

The first sum coincides with $\det^c(\boldsymbol{\alpha})$, so it is nonzero by the assumption $(*)$; we can write the second sum by collecting the terms containing the same entry corresponding to the nondegenerate entry of $\boldsymbol{\alpha}$; so, by assumption $(*)$, we get that this sum is zero; we argue analogously for the other sums. So we can conclude that $\det(A)$ is nonzero, a contradiction.

($\Longleftarrow$) Let $\boldsymbol{\alpha}$ be a matrix satisfying (1) and (2) and let $A \in \boldsymbol{\alpha}$. By (1), we can write $\det(A)$ as in $(\star)$.

The first sum is zero by assumption; we can write the second sum by collecting the terms containing the same entry corresponding to the nondegenerate entry of $\boldsymbol{\alpha}$; so by assumption we get that also this sum is zero. We argue analogously for the other sums. $\square$

Corollary 17. *Let $\boldsymbol{\alpha}$ be a subset matrix over a field K. Then* $\mathrm{Mrk}(\boldsymbol{\alpha})$ *is the maximum of the natural numbers t such that there is a $t \times t$ submatrix of $\boldsymbol{\alpha}$ either with a totally nondegenerate pg-diagonal of length t or with a totally nondegenerate pg-diagonal of length between 0 and $t-1$ whose complementary matrix has* $\det^c \neq 0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 35, pp. 285-296, July 2019.

Acknowledgments. The author wishes to thank the anonymous referee for his/her comments, which helped to improve the paper.

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