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PURE PSVD APPROACH TO SYLVESTER-TYPE QUATERNION MATRIX EQUATIONS*

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Abstract. In this paper, the pure product singular value decomposition (PSVD) for four quaternion matrices is given. The system of coupled Sylvester-type quaternion matrix equations with five unknowns $X_iA_i - B_iX_{i+1} = C_i$ is considered by using the PSVD approach, where A_i, B_i, and C_i are given quaternion matrices of compatible sizes ($i = 1, 2, 3, 4$). Some necessary and sufficient conditions for the existence of a solution to this system are derived. Moreover, the general solution to this system is presented when it is solvable.

Key words. Product singular value decomposition, Quaternion, Sylvester equations, Solvability, General solution.

AMS subject classifications. 15A03, 15A21, 15A23, 15A24.

1. Introduction. Generalized singular value decomposition (GSVD) has applications in signal processing [17], genomic signal processing [20], discrete time system [18], generalized Gauss-Markov estimation problems, open and closed loop balancing [1], and so on. The product singular value decomposition (PSVD) is an important kind of the available generalized singular value decompositions. Since Heath et. al [14] first considered the PSVD of two matrices in 1986, there have been many papers to study the PSVD (e.g. [1], [3]–[6]). For instance, De Moor, Van Dooren and Zha ([16], [18], [19]) gave the PSVD of any number of complex matrices. Chu and De Moor [2] considered the nonuniqueness of the PSVD.

In this paper, we will apply PSVD for quaternion matrices to the following system of four one-sided coupled Sylvester-type quaternion matrix equations with five unknowns

$$X_iA_i - B_iX_{i+1} = C_i, \quad i = 1, 2, 3, 4, \tag{1.1}$$

where $A_i(q_i \times q_{i+1}), B_i(p_i \times p_{i+1})$, and $C_i(p_i \times q_{i+1})$ are given quaternion matrices. We make use of the PSVD that brings the system (1.1) to a canonical form. Based on this canonical form, we give necessary and sufficient conditions for the existence and the general solution to the system (1.1).

Sylvester-type matrix equations have wide applications in control theory, for instance, Sylvester-type matrix equations can be used in robust control [26], output feedback control [25], the almost noninteracting control by measurement feedback problem [27]. The study on the Sylvester-type matrix equations over the quaternion algebra has attracted more and more attention in the last ten years (e.g. [4], [7]–[13], [15], [21]–[24], [28], [29], [30]).

The remainder of this paper is organized as follows. In Section 2, we extend the PSVD for four complex matrices to the quaternion algebra. By using the similar approach, we consider the PSVD for n quaternion matrices. In Section 3, we use PSVD for four quaternion matrices to give a canonical form for the system

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(1.1). In Section 4, we give some necessary and sufficient conditions for the solvability of the system (1.1). We also present its general solution when it is solvable.

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real field and the space of all $m\times n$ matrices over the quaternion algebra

$$\mathbb{H}=\left\{a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\,\middle|\,\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\, a_0,a_1,a_2,a_3\in\mathbb{R}\right\}.$$

It is well known that the quaternion algebra is an associative and noncommutative division algebra. The symbol I means an identity matrix of appropriate dimensions. The rank of $A\in\mathbb{H}^{m\times n}$ is defined as the (quaternion) dimension of

$$\mathrm{Ran}(A):=\left\{Ax\,\middle|\,x\in\mathbb{H}^{m\times 1}\right\},$$

the range of A (Definition 3.2.3 in [21]). More properties of the rank of quaternion matrix can be found in the page 31 of [21]. If $A\in\mathbb{H}^{m\times n}$, then there exist nonsingular matrices P and Q such that

$$PAQ=\begin{pmatrix} I_{r(A)} & 0\\ 0 & 0\end{pmatrix}, \tag{1.2}$$

where the symbol $r(A)$ is the rank of A. The decomposition (1.2) is also called rank decomposition. It is easy to see that $r(A)=r(PAQ)$, where P and Q are any nonsingular quaternion matrices.

2. PSVD for four quaternion matrices. In this section, we consider the PSVD for four quaternion matrices. The PSVD for four complex matrices is presented in [16], [18] and [19]. The following result extends the PSVD for four complex matrices to the quaternion algebra.

Lemma 2.1. (PSVD for four quaternion matrices) *Consider a set of four quaternion matrices with compatible dimensions: $A_1\in\mathbb{H}^{q_1\times q_2}, A_2\in\mathbb{H}^{q_2\times q_3}, A_3\in\mathbb{H}^{q_3\times q_4}$, $A_4\in\mathbb{H}^{q_4\times q_5}$. Then there exist nonsingular matrices $T_0\in\mathbb{H}^{q_1\times q_1}, T_1\in\mathbb{H}^{q_2\times q_2}, T_2\in\mathbb{H}^{q_3\times q_3}, T_3\in\mathbb{H}^{q_4\times q_4}$, and $T_4\in\mathbb{H}^{q_5\times q_5}$ such that*

$$A_1=T_0D_1T_1^{-1},\quad A_2=T_1D_2T_2^{-1},\quad A_3=T_2D_3T_3^{-1},\quad A_4=T_3D_4T_4^{-1},$$

where

$$D_1=\begin{array}{c c} & \begin{array}{cc} r_1 & q_2-r_1\end{array}\\ \begin{array}{c} r_1\\ q_1-r_1\end{array} & \begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\end{array},\quad D_2=\begin{array}{c c} & \begin{array}{ccc} r_2^1 & r_2^2 & q_3-r_2\end{array}\\ \begin{array}{c} r_2^1\\ r_1-r_2^1\\ r_2^2\\ q_2-r_1-r_2^2\end{array} & \begin{pmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\end{pmatrix}\end{array}, \tag{2.3}$$

$$D_3=\begin{array}{c c} & \begin{array}{cccc} r_3^1 & r_3^2 & r_3^3 & q_4-r_3\end{array}\\ \begin{array}{c} r_3^1\\ r_2^1-r_3^1\\ r_3^2\\ r_2^2-r_3^2\\ r_3^3\\ q_3-r_2-r_3^3\end{array} & \begin{pmatrix} I & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\end{array}, \tag{2.4}$$

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$$D_4 = \begin{array}{c} \\ r_4^1 \\ r_3^1 - r_4^1 \\ r_4^2 \\ r_3^2 - r_4^2 \\ r_4^3 \\ r_3^3 - r_4^3 \\ r_4^4 \\ q_4 - r_3 - r_4^4 \end{array} \begin{array}{c} \begin{array}{ccccc} r_4^1 & r_4^2 & r_4^3 & r_4^4 & q_5 - r_4 \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{2.5}$$

The expressions for the block dimensions $r_1, r_2, r_3, r_4, r_2^1, r_2^2, r_3^1, r_3^2, r_3^3, r_4^1, r_2^2, r_4^3, r_4^4$ *are given by:*

$$r_1 = r(A_1), \quad r_2 = r(A_2), \quad r_3 = r(A_3), \quad r_4 = r(A_4), \tag{2.6}$$

$$r_2^1 = r(A_1A_2), \quad r_2^2 = r(A_2) - r(A_1A_2), \quad r_3^1 = r(A_1A_2A_3), \tag{2.7}$$

$$r_3^2 = r(A_2A_3) - r(A_1A_2A_3), \quad r_3^3 = r(A_3) - r(A_2A_3), \quad r_4^1 = r(A_1A_2A_3A_4), \tag{2.8}$$

$$r_4^2 = r(A_2A_3A_4) - r(A_1A_2A_3A_4), \quad r_4^3 = r(A_3A_4) - r(A_2A_3A_4), \quad r_4^4 = r(A_4) - r(A_3A_4). \tag{2.9}$$

Proof. We will use the rank decomposition (1.2) and elementary matrix operations to give the PSVD for four quaternion matrices. First, we construct the following block quaternion matrix

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} A_1 & & & \\ I & A_2 & & \\ & I & A_3 & \\ & & I & A_4 \end{pmatrix} \end{array}. \tag{2.10}$$

Then, we perform the same row operations to each row and the same column operations to each column of the block quaternion matrix (2.10). In order to find the nonsingular matrices T_0, T_1, T_2, T_3, and T_4 in the PSVD, we need to keep the identity matrices unchanged after every step. That is to say, the row operations of the second, third and fourth rows are the inverses of the column operations of the first, second and third columns, respectively. Finally, we will obtain the following block matrix

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{pmatrix} D_1 & & & \\ I & D_2 & & \\ & I & D_3 & \\ & & I & D_4 \end{pmatrix} \end{array},$$

where D_1, D_2, D_3, and D_4 are given in (2.3)–(2.5).

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Now we give the steps of this transformation. We transform the quaternion matrices A_1, A_2, A_3, A_4 to quasi-diagonal matrices in turn. Below is a sequence of displays of the matrix block that illustrates the transformations. The symbols $\star, \blacktriangle, \blacklozenge$ are nonzero block quaternion matrices.

$$\begin{array}{cc} & \begin{array}{cccc} q_2 & q_3 & q_4 & q_5 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} & \begin{pmatrix} A_1 & & & \\ I & A_2 & & \\ & I & A_3 & \\ & & I & A_4 \end{pmatrix} \end{array} \xrightarrow[Q_1^{-1}A_2:=\begin{pmatrix}\star_1\\ \star_2\end{pmatrix}]{P_1A_1Q_1=\begin{pmatrix}I&0\\0&0\end{pmatrix},\ Q_1^{-1}I_{q_2}Q_1=\begin{pmatrix}I&0\\0&I\end{pmatrix}} \left(\begin{array}{ccccc} I & 0 & & & \\ 0 & 0 & & & \\ \hline I & 0 & \star_1 & & \\ 0 & I & \star_2 & & \\ \hline & & I_{q_3} & A_3 & \\ \hline & & & I_{q_4} & A_4 \end{array}\right)$$

$$\xrightarrow[Q_2^{-1}I_{q_3}Q_2=I,\ Q_2^{-1}A_3:=\blacktriangle_1]{P_2\star_1Q_2=\begin{pmatrix}I&0\\0&0\end{pmatrix},\ \star_2Q_2:=\begin{pmatrix}\star_3 & \star_4\end{pmatrix}} \left(\begin{array}{ccccccc} I & 0 & 0 & & & & \\ 0 & I & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \hline I & 0 & 0 & I & 0 & & \\ 0 & I & 0 & 0 & 0 & & \\ 0 & 0 & I & \star_3 & \star_4 & & \\ \hline & & & I & 0 & \blacktriangle_1 & \\ & & & 0 & I & \blacktriangle_2 & \\ \hline & & & & & I_{q_4} & A_4 \end{array}\right) \xrightarrow{\text{Using elementary matrix operations}}$$

$$\left(\begin{array}{ccccccc} I & 0 & 0 & & & & \\ 0 & I & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \hline I & 0 & 0 & I & 0 & & \\ 0 & I & 0 & 0 & 0 & & \\ 0 & 0 & I & 0 & \star_4 & & \\ \hline & & & I & 0 & \blacktriangle_1 & \\ & & & 0 & I & \blacktriangle_2 & \\ \hline & & & & & I_{q_4} & A_4 \end{array}\right) \xrightarrow[Q_3^{-1}\blacktriangle_2:=\begin{pmatrix}\blacktriangle_3\\ \blacktriangle_4\end{pmatrix}]{P_3\star_4Q_3=\begin{pmatrix}I&0\\0&0\end{pmatrix}} \left(\begin{array}{ccccccccc} I & 0 & 0 & 0 & & & & & \\ 0 & I & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ \hline I & 0 & 0 & 0 & I & 0 & 0 & & \\ 0 & I & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & I & 0 & 0 & I & 0 & & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & & \\ \hline & & & & I & 0 & 0 & \blacktriangle_1 & \\ & & & & 0 & I & 0 & \blacktriangle_3 & \\ & & & & 0 & 0 & I & \blacktriangle_4 & \\ \hline & & & & & & & I_{q_4} & A_4 \end{array}\right) \xrightarrow[Q_1^{-1}A_4:=\begin{pmatrix}\blacklozenge_1\\ \blacklozenge_2\end{pmatrix}]{P_4\blacktriangle_1Q_4=\begin{pmatrix}I&0\\0&0\end{pmatrix}}$$

$$\left(\begin{array}{cccccccccccc} I & 0 & 0 & 0 & 0 & & & & & & & \\ 0 & I & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & I & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & & & \\ \hline I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & & \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & & & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & & \\ \hline & & & & & I & 0 & 0 & 0 & I & 0 & \\ & & & & & 0 & I & 0 & 0 & 0 & 0 & \\ & & & & & 0 & 0 & I & 0 & \blacktriangle_5 & \blacktriangle_6 & \\ & & & & & 0 & 0 & 0 & I & \blacktriangle_7 & \blacktriangle_8 & \\ \hline & & & & & & & & & I & 0 & \blacklozenge_1 \\ & & & & & & & & & 0 & I & \blacklozenge_2 \end{array}\right) \xrightarrow{\text{Using elementary matrix operations}} \left(\begin{array}{cccccccccccc} I & 0 & 0 & 0 & 0 & & & & & & & \\ 0 & I & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & I & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & & & \\ \hline I & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & & & \\ 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 & & & \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & & & \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & & & \\ \hline & & & & & I & 0 & 0 & 0 & I & 0 & \\ & & & & & 0 & I & 0 & 0 & 0 & 0 & \\ & & & & & 0 & 0 & I & 0 & 0 & \blacktriangle_6 & \\ & & & & & 0 & 0 & 0 & I & 0 & \blacktriangle_8 & \\ \hline & & & & & & & & & I & 0 & \blacklozenge_1 \\ & & & & & & & & & 0 & I & \blacklozenge_2 \end{array}\right)$$

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$$\xrightarrow[Q_5^{-1}\blacklozenge_2:=\begin{pmatrix}\blacklozenge_3\\ \blacklozenge_4\end{pmatrix}]{P_5\blacktriangle_6Q_5=\begin{pmatrix}I&0\\0&0\end{pmatrix},\ \blacktriangle_8Q_5:=\begin{pmatrix}\blacktriangle_9&\blacktriangle_{10}\end{pmatrix}}\left(\begin{array}{cccc}
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&0&0&0\end{matrix} & & & \\ \hline
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0&0&0\\0&I&0&0&0\\0&0&0&0&0\\0&0&I&0&0\\0&0&0&I&0\\0&0&0&0&0\end{matrix} & & \\ \hline
 & \begin{matrix}I&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&I&0\\0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0\\0&0&0\\0&I&0\\0&0&0\\0&\blacktriangle_9&\blacktriangle_{10}\end{matrix} & \\ \hline
 & & \begin{matrix}I&0&0\\0&I&0\\0&0&I\end{matrix} & \begin{matrix}\blacklozenge_1\\ \blacklozenge_3\\ \blacklozenge_4\end{matrix}
\end{array}\right)$$

$$\xrightarrow{\text{Using elementary matrix operations}}\left(\begin{array}{cccc}
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&0&0&0\end{matrix} & & & \\ \hline
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0&0&0\\0&I&0&0&0\\0&0&0&0&0\\0&0&I&0&0\\0&0&0&I&0\\0&0&0&0&0\end{matrix} & & \\ \hline
 & \begin{matrix}I&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&I&0\\0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0\\0&0&0\\0&I&0\\0&0&0\\0&0&\blacktriangle_{10}\end{matrix} & \\ \hline
 & & \begin{matrix}I&0&0\\0&I&0\\0&0&I\end{matrix} & \begin{matrix}\blacklozenge_1\\ \blacklozenge_3\\ \blacklozenge_4\end{matrix}
\end{array}\right)\xrightarrow[Q_6^{-1}\blacklozenge_4:=\begin{pmatrix}\blacklozenge_5\\ \blacklozenge_6\end{pmatrix}]{P_6\blacktriangle_{10}Q_6=\begin{pmatrix}I&0\\0&0\end{pmatrix}}$$

$$\left(\begin{array}{cccc}
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&0&0&0\end{matrix} & & & \\ \hline
\begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&0&0\end{matrix} & & \\ \hline
 & \begin{matrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&0&0&0&I\end{matrix} & \begin{matrix}I&0&0&0\\0&0&0&0\\0&I&0&0\\0&0&0&0\\0&0&I&0\\0&0&0&0\end{matrix} & \\ \hline
 & & \begin{matrix}I&0&0&0\\0&I&0&0\\0&0&I&0\\0&0&0&I\end{matrix} & \begin{matrix}\blacklozenge_1\\ \blacklozenge_3\\ \blacklozenge_5\\ \blacklozenge_6\end{matrix}
\end{array}\right).$$

We pay attention to the transformation of the structure of A_3. Note that

$$A_3\rightarrow\begin{pmatrix}\blacktriangle_1\\ \blacktriangle_3\\ \blacktriangle_4\end{pmatrix}\rightarrow\begin{pmatrix}I&0\\0&0\\ \blacktriangle_5&\blacktriangle_6\\ \blacktriangle_7&\blacktriangle_8\end{pmatrix}\rightarrow\begin{pmatrix}I&0\\0&0\\0&\blacktriangle_6\\0&\blacktriangle_8\end{pmatrix}\rightarrow\begin{pmatrix}I&0&0\\0&0&0\\0&I&0\\0&0&0\\0&\blacktriangle_9&\blacktriangle_{10}\end{pmatrix}$$

$$\rightarrow\begin{pmatrix}I&0&0\\0&0&0\\0&I&0\\0&0&0\\0&0&\blacktriangle_{10}\end{pmatrix}\rightarrow\begin{pmatrix}I&0&0&0\\0&0&0&0\\0&I&0&0\\0&0&0&0\\0&0&I&0\\0&0&0&0\end{pmatrix}=D_3.$$

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Hence, we can obtain the transformation of the structure of A_4

$$A_4 \to \begin{pmatrix} \divideontimes \\ \divideontimes \\ \divideontimes \\ \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & 0 \\ \divideontimes & \divideontimes \\ \divideontimes & \divideontimes \\ \divideontimes & \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & \divideontimes \\ 0 & \divideontimes \\ 0 & \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & \divideontimes & \divideontimes \\ 0 & \divideontimes & \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \divideontimes \\ 0 & 0 & \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \divideontimes & \divideontimes \end{pmatrix}$$

$$\to \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \divideontimes \end{pmatrix} \to \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = D_4,$$

where $\divideontimes$ stands for the nonzero block quaternion matrix. The expressions of $r_1, r_2, r_3, r_4, r_2^1, r_2^2$, $r_3^1, r_3^2, r_3^3, r_4^1$, r_2^2, r_4^3, r_4^4 can be obtained from the ranks of the products of A_i $(i = 1, 2, 3, 4)$. $\square$

The PSVD for a set of n complex matrices $A_1^{q_1\times q_2}, A_2^{q_2\times q_3}, \dots, A_n^{q_n\times q_{n+1}}$ is presented in [16], [18] and [19]. We can use the similar approach to extend the PSVD for n complex matrices to the quaternion algebra. The steps are given as follows:

1. Construct the following block quaternion matrix

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{array} \begin{array}{c} \begin{array}{ccccc} q_2 & q_3 & q_4 & \cdots & q_{n+1} \end{array} \\ \begin{pmatrix} A_1 & & & & \\ I & A_2 & & & \\ & I & A_3 & & \\ & & \ddots & \ddots & \\ & & & I & A_n \end{pmatrix} \end{array}.$$

2. Perform the same row operations to each row and the same column operations to each column of the block quaternion matrix (2.10). In order to find the PSVD, we need to keep all the identity matrices unchanged after every step.
3. Give the transformation of the structure of A_n

$$A_n \to \begin{pmatrix} \divideontimes_1 \\ \divideontimes_2 \\ \vdots \\ \divideontimes_n \end{pmatrix} \xrightarrow{P_1 \divideontimes_1 Q_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},\ \begin{pmatrix} \divideontimes_2 \\ \vdots \\ \divideontimes_n \end{pmatrix} Q_1 := \begin{pmatrix} \blacklozenge_{11} & \blacklozenge_{12} \\ \vdots & \\ \blacklozenge_{n-1,1} & \blacklozenge_{n-1,2} \end{pmatrix}} \begin{pmatrix} I & 0 \\ 0 & 0 \\ \blacklozenge_{11} & \blacklozenge_{12} \\ \vdots & \vdots \\ \blacklozenge_{n-1,1} & \blacklozenge_{n-1,2} \end{pmatrix} \to \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & \blacklozenge_{12} \\ \vdots & \vdots \\ 0 & \blacklozenge_{n-1,2} \end{pmatrix}$$

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$$\to \cdots \to \begin{pmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = D_n,$$

where $\divideontimes$ and $\blacklozenge$ stand for the nonzero block quaternion matrices.

4. Give the PSVD for n quaternion matrices

$$A_1 = T_0 D_1 T_1^{-1}, \quad A_2 = T_1 D_2 T_2^{-1}, \quad \ldots, \quad A_i = T_{i-1} D_i T_i^{-1}, \quad \ldots, \quad A_n = T_{n-1} D_n T_n^{-1},$$

where $T_0 \in \mathbb{H}^{q_1 \times q_1}, \ldots, T_n \in \mathbb{H}^{q_{n+1} \times q_{n+1}}$ are nonsingular matrices, D_i are quasi-diagonal.

3. A canonical form for the system (1.1). In this section, we use PSVD (Lemma 2.1) to present a canonical form for the system (1.1), i.e.,

$$\begin{cases} X_1 A_1 - B_1 X_2 = C_1, \\ X_2 A_2 - B_2 X_3 = C_2, \\ X_3 A_3 - B_3 X_4 = C_3, \\ X_4 A_4 - B_4 X_5 = C_4. \end{cases} \tag{3.11}$$

Note that the sizes of the matrices A_i in the system (3.11) are as follows

$$A_1 \in \mathbb{H}^{q_1 \times q_2}, \quad A_2 \in \mathbb{H}^{q_2 \times q_3}, \quad A_3 \in \mathbb{H}^{q_3 \times q_4}, \quad A_4 \in \mathbb{H}^{q_4 \times q_5}.$$

Applying Lemma 2.1 to the matrices A_i, we find nonsingular matrices $T_0 \in \mathbb{H}^{q_1 \times q_1}$, $T_1 \in \mathbb{H}^{q_2 \times q_2}$, $T_2 \in \mathbb{H}^{q_3 \times q_3}$, $T_3 \in \mathbb{H}^{q_4 \times q_4}$, and $T_4 \in \mathbb{H}^{q_5 \times q_5}$ such that

$$A_1 = T_0 D_1 T_1^{-1}, \quad A_2 = T_1 D_2 T_2^{-1}, \quad A_3 = T_2 D_3 T_3^{-1}, \quad A_4 = T_3 D_4 T_4^{-1}, \tag{3.12}$$

where D_i are quasi-diagonal and given in (2.3) and (2.5), their only nonzero blocks being identity matrices. In a similar way, for the matrices B_i, we find nonsingular matrices $Q_0 \in \mathbb{H}^{p_1 \times p_1}, Q_1 \in \mathbb{H}^{p_2 \times p_2}, Q_2 \in \mathbb{H}^{p_3 \times p_3}, Q_3 \in \mathbb{H}^{p_4 \times p_4}$, and $Q_4 \in \mathbb{H}^{p_5 \times p_5}$ such that

$$B_1 = Q_0 S_1 Q_1^{-1}, \quad B_2 = Q_1 S_2 Q_2^{-1}, \quad B_3 = Q_2 S_3 Q_3^{-1}, \quad B_4 = Q_3 S_4 Q_4^{-1}, \tag{3.13}$$

where S_i are quasi-diagonal. It follows from (3.12) and (3.13) that the system (3.11) is equivalent to the following system

$$\begin{cases} X_1 T_0 D_1 T_1^{-1} - Q_0 S_1 Q_1^{-1} X_2 = C_1, \\ X_2 T_1 D_2 T_2^{-1} - Q_1 S_2 Q_2^{-1} X_3 = C_2, \\ X_3 T_2 D_3 T_3^{-1} - Q_2 S_3 Q_3^{-1} X_4 = C_3, \\ X_4 T_3 D_4 T_4^{-1} - Q_3 S_4 Q_4^{-1} X_5 = C_4, \end{cases}$$

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i.e.,

$$\left\{\begin{array}{l}(Q_0^{-1}X_1T_0)D_1 - S_1(Q_1^{-1}X_2T_1) = Q_0^{-1}C_1T_1,\\ (Q_1^{-1}X_2T_1)D_2 - S_2(Q_2^{-1}X_3T_2) = Q_1^{-1}C_2T_2,\\ (Q_2^{-1}X_3T_2)D_3 - S_3(Q_3^{-1}X_4T_3) = Q_2^{-1}C_3T_3,\\ (Q_3^{-1}X_4T_3)D_4 - S_4(Q_4^{-1}X_5T_4) = Q_3^{-1}C_4T_4.\end{array}\right. \tag{3.14}$$

Set

$$Y_k = Q_{k-1}^{-1}X_kT_{k-1}, \quad k = 1, 2, \ldots, 5,$$

and

$$E_i = Q_{i-1}^{-1}C_iT_i, \quad i = 1, 2, \ldots, 4. \tag{3.15}$$

Then the original system (3.11) becomes

$$\left\{\begin{array}{l}Y_1D_1 - S_1Y_2 = E_1,\\ Y_2D_2 - S_2Y_3 = E_2,\\ Y_3D_3 - S_3Y_4 = E_3,\\ Y_4D_4 - S_4Y_5 = E_4.\end{array}\right. \tag{3.16}$$

Hence, we have the following theorem which plays a role in solving the system (3.11).

THEOREM 3.1. *The system (3.11) is equivalent to the system (3.16), where the matrices D_i and S_i are quasi-diagonal and given in (3.12) and (3.13), and E_i is given in (3.15).*

We will use Theorem 3.1 to consider the system (3.11) in the next section.

4. The solvability conditions and general solution to the system (3.11). In this section, we consider the solvability conditions and general solution to the system (3.11). In the following theorem, we will give some practical necessary and sufficient conditions for the existence of a solution to the system (3.11).

THEOREM 4.1. *The system (3.11) is consistent if and only if the following* 10 *rank equalities hold*

$$r\begin{pmatrix}C_i & B_i\\ A_i & 0\end{pmatrix} = r(A_i) + r(B_i), \quad i = 1, 2, 3, 4, \tag{4.17}$$

$$r\begin{pmatrix}C_1A_2 + B_1C_2 & B_1B_2\\ A_1A_2 & 0\end{pmatrix} = r(A_1A_2) + r(B_1B_2), \tag{4.18}$$

$$r\begin{pmatrix}C_2A_3 + B_2C_3 & B_2B_3\\ A_2A_3 & 0\end{pmatrix} = r(A_2A_3) + r(B_2B_3), \tag{4.19}$$

$$r\begin{pmatrix}C_3A_4 + B_3C_4 & B_3B_4\\ A_3A_4 & 0\end{pmatrix} = r(A_3A_4) + r(B_3B_4), \tag{4.20}$$

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$$r\begin{pmatrix} C_1A_2A_3+B_1C_2A_3+B_1B_2C_3 & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} = r(A_1A_2A_3)+r(B_1B_2B_3), \tag{4.21}$$

$$r\begin{pmatrix} C_2A_3A_4+B_2C_3A_4+B_2B_3C_4 & B_2B_3B_4 \\ A_2A_3A_4 & 0 \end{pmatrix} = r(A_2A_3A_4)+r(B_2B_3B_4), \tag{4.22}$$

$$\begin{aligned} & r\begin{pmatrix} C_1A_2A_3A_4+B_1C_2A_3A_4+B_1B_2C_3A_4+B_1B_2B_3C_4 & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix} \\ & = r(A_1A_2A_3A_4)+r(B_1B_2B_3B_4). \end{aligned} \tag{4.23}$$

Proof. "only if"-part. Let us assume that there exists a solution $(X_1^0, X_2^0, X_3^0, X_4^0, X_5^0)$ which satisfies the system (3.11), then clearly

$$X_i^0A_i - B_iX_{i+1}^0 = C_i, \quad i = 1,2,3,4. \tag{4.24}$$

We will use (4.24) and elementary matrix operations to prove the rank equalities (4.17)–(4.23).

1. For the rank equalities (4.17). It follows from (4.24) that
$$r\begin{pmatrix} C_i & B_i \\ A_i & 0 \end{pmatrix} = r\begin{pmatrix} X_i^0A_i - B_iX_{i+1}^0 & B_i \\ A_i & 0 \end{pmatrix} = r\begin{pmatrix} 0 & B_i \\ A_i & 0 \end{pmatrix} = r(A_i)+r(B_i) \Rightarrow (4.17).$$
2. For the rank equality (4.18). Note that
$$\begin{aligned} r\begin{pmatrix} C_1A_2+B_1C_2 & B_1B_2 \\ A_1A_2 & 0 \end{pmatrix} &= r\begin{pmatrix} (X_1^0A_1-B_1X_2^0)A_2+B_1(X_2^0A_2-B_2X_3^0) & B_1B_2 \\ A_1A_2 & 0 \end{pmatrix} \\ &= r\begin{pmatrix} X_1^0A_1A_2-B_1B_2X_3^0 & B_1B_2 \\ A_1A_2 & 0 \end{pmatrix} \\ &= r(A_1A_2)+r(B_1B_2) \Rightarrow (4.18). \end{aligned}$$
Similarly, we can show the rank equalities (4.19) and (4.20).
3. For the rank equality (4.21). Note that
$$\begin{aligned} & r\begin{pmatrix} C_1A_2A_3+B_1C_2A_3+B_1B_2C_3 & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} \\ =& r\begin{pmatrix} (X_1^0A_1-B_1X_2^0)A_2A_3+B_1(X_2^0A_2-B_2X_3^0)A_3+B_1B_2(X_3^0A_3-B_3X_4^0) & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} \\ =& r\begin{pmatrix} X_1^0A_1A_2A_3-B_1B_2B_3X_4^0 & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} \\ =& r(A_1A_2A_3)+r(B_1B_2B_3) \Rightarrow (4.21). \end{aligned}$$
Similarly, we can prove the rank equality (4.22).
4. For the rank equality (4.23). Note that
$$\begin{aligned} & r\begin{pmatrix} C_1A_2A_3A_4+B_1C_2A_3A_4+B_1B_2C_3A_4+B_1B_2B_3C_4 & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix} \\ =& r\begin{pmatrix} (X_1^0A_1-B_1X_2^0)A_2A_3A_4+B_1(X_2^0A_2-B_2X_3^0)A_3A_4+B_1B_2(X_3^0A_3-B_3X_4^0)A_4+B_1B_2B_3(X_4^0A_4-B_4X_5^0) & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix} \\ =& r\begin{pmatrix} X_1^0A_1A_2A_3A_4-B_1B_2B_3B_4X_5^0 & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix} = r(A_1A_2A_3A_4)+r(B_1B_2B_3B_4). \end{aligned}$$

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"if"-part. It follows from Theorem 3.1 that the system (3.11) is equivalent to the system (3.16). In order to solve the system (3.16), we need to partition the quasi-diagonal matrices D_i and S_i with the finest possible subdivision of matrices

$$D_1 = \begin{array}{rc} & \begin{array}{cccccccc} r_4^1 & r_3^1-r_4^1 & r_2^1-r_3^1 & r_1-r_2^1 & r_4^2 & r_3^2-r_4^2 & r_2^2-r_3^2 & q_2-r_1-r_2^2 \end{array} \\ \begin{array}{r} r_4^1 \\ r_3^1-r_4^1 \\ r_2^1-r_3^1 \\ r_1-r_2^1 \\ q_1-r_1 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.25}$$

$$D_2 = \begin{array}{rc} & \begin{array}{ccccccccc} r_4^1 & r_3^1-r_4^1 & r_2^1-r_3^1 & r_4^2 & r_3^2-r_4^2 & r_2^2-r_3^2 & r_4^3 & r_3^3-r_4^3 & q_3-r_2-r_3^3 \end{array} \\ \begin{array}{r} r_4^1 \\ r_3^1-r_4^1 \\ r_2^1-r_3^1 \\ r_1-r_2^1 \\ r_4^2 \\ r_3^2-r_4^2 \\ r_2^2-r_3^2 \\ q_2-r_1-r_2^2 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.26}$$

$$D_3 = \begin{array}{rc} & \begin{array}{cccccccc} r_4^1 & r_3^1-r_4^1 & r_4^2 & r_3^2-r_4^2 & r_4^3 & r_3^3-r_4^3 & r_4^4 & q_4-r_3-r_4^4 \end{array} \\ \begin{array}{r} r_4^1 \\ r_3^1-r_4^1 \\ r_2^1-r_3^1 \\ r_4^2 \\ r_3^2-r_4^2 \\ r_2^2-r_3^2 \\ r_4^3 \\ r_3^3-r_4^3 \\ q_3-r_2-r_3^3 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.27}$$

$$D_4 = \begin{array}{rc} & \begin{array}{ccccc} r_4^1 & r_4^2 & r_4^3 & r_4^4 & q_5-r_4 \end{array} \\ \begin{array}{r} r_4^1 \\ r_3^1-r_4^1 \\ r_4^2 \\ r_3^2-r_4^2 \\ r_4^3 \\ r_3^3-r_4^3 \\ r_4^4 \\ q_4-r_3-r_4^4 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.28}$$

$$S_1 = \begin{array}{rc} & \begin{array}{cccccccc} l_4^1 & l_3^1-l_4^1 & l_2^1-l_3^1 & l_1-l_2^1 & l_4^2 & l_3^2-l_4^2 & l_2^2-l_3^2 & p_2-l_1-l_2^2 \end{array} \\ \begin{array}{r} l_4^1 \\ l_3^1-l_4^1 \\ l_2^1-l_3^1 \\ l_1-l_2^1 \\ p_1-l_1 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.29}$$

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$$S_2 = \begin{array}{c c} & \begin{array}{ccccccccc} l_4^1 & l_3^1 - l_4^1 & l_2^1 - l_3^1 & l_4^2 & l_3^2 - l_4^2 & l_2^2 - l_3^2 & l_4^3 & l_3^3 - l_4^3 & p_3 - l_2 - l_3^3 \end{array} \\ \begin{array}{l} l_4^1 \\ l_3^1 - l_4^1 \\ l_2^1 - l_3^1 \\ l_1 - r_2^1 \\ l_4^2 \\ l_3^2 - l_4^2 \\ l_2^2 - l_3^2 \\ p_2 - l_1 - l_2^2 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.30}$$

$$S_3 = \begin{array}{c c} & \begin{array}{cccccccc} l_4^1 & l_3^1 - l_4^1 & l_4^2 & l_3^2 - l_4^2 & l_4^3 & l_3^3 - l_4^3 & l_4^4 & p_4 - l_3 - l_4^4 \end{array} \\ \begin{array}{l} l_4^1 \\ l_3^1 - l_4^1 \\ l_2^1 - l_3^1 \\ l_4^2 \\ l_3^2 - l_4^2 \\ l_2^2 - l_3^2 \\ l_4^3 \\ l_3^3 - l_4^3 \\ p_3 - l_2 - l_3^3 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.31}$$

$$S_4 = \begin{array}{c c} & \begin{array}{ccccc} l_4^1 & l_4^2 & l_4^3 & l_4^4 & p_5 - l_4 \end{array} \\ \begin{array}{l} l_4^1 \\ l_3^1 - l_4^1 \\ l_4^2 \\ l_3^2 - l_4^2 \\ l_4^3 \\ l_3^3 - l_4^3 \\ l_4^4 \\ p_4 - l_3 - l_4^4 \end{array} & \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{4.32}$$

where the expressions of the block dimension

$$r_1, r_2, r_3, r_4, r_2^1, r_2^2, r_3^1, r_3^2, r_3^3, r_4^1, r_2^2, r_4^3, r_4^4$$

and

$$l_1, l_2, l_3, l_4, r_2^1, r_2^2, r_3^1, r_3^2, r_3^3, l_4^1, l_2^2, l_4^3, l_4^4$$

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can be obtained from Lemma 2.1. Let the matrices

$$Y_k := \begin{cases} \begin{pmatrix} Y_{11}^k & \cdots & Y_{15}^k \\ \vdots & \ddots & \vdots \\ Y_{51}^k & \cdots & Y_{55}^k \end{pmatrix}, & \text{if } k = 1, 5, \\[2ex] \begin{pmatrix} Y_{11}^k & \cdots & Y_{18}^k \\ \vdots & \ddots & \vdots \\ Y_{81}^k & \cdots & Y_{88}^k \end{pmatrix}, & \text{if } k = 2, 4, \\[2ex] \begin{pmatrix} Y_{11}^k & \cdots & Y_{19}^k \\ \vdots & \ddots & \vdots \\ Y_{91}^k & \cdots & Y_{99}^k \end{pmatrix}, & \text{if } k = 3, \end{cases}$$

and

$$E_1 := \left(E_{ij}^1\right)_{5\times 8}, \quad E_2 := \left(E_{ij}^2\right)_{8\times 9},$$

$$E_3 := \left(E_{ij}^3\right)_{9\times 8}, \quad E_4 := \left(E_{ij}^4\right)_{8\times 5}$$

be partitioned in accordance with (3.16) and (4.25)–(4.32). Substituting Y_k into the matrix equations in (3.16) yields

$$\begin{pmatrix} Y_{11}^1 - Y_{11}^2 & Y_{12}^1 - Y_{12}^2 & Y_{13}^1 - Y_{13}^2 & Y_{14}^1 - Y_{14}^2 & -Y_{15}^2 & -Y_{16}^2 & -Y_{17}^2 & -Y_{18}^2 \\ Y_{21}^1 - Y_{21}^2 & Y_{22}^1 - Y_{22}^2 & Y_{23}^1 - Y_{23}^2 & Y_{24}^1 - Y_{24}^2 & -Y_{25}^2 & -Y_{26}^2 & -Y_{27}^2 & -Y_{28}^2 \\ Y_{31}^1 - Y_{31}^2 & Y_{32}^1 - Y_{32}^2 & Y_{33}^1 - Y_{33}^2 & Y_{34}^1 - Y_{34}^2 & -Y_{35}^2 & -Y_{36}^2 & -Y_{37}^2 & -Y_{38}^2 \\ Y_{41}^1 - Y_{41}^2 & Y_{42}^1 - Y_{42}^2 & Y_{43}^1 - Y_{43}^2 & Y_{44}^1 - Y_{44}^2 & -Y_{45}^2 & -Y_{46}^2 & -Y_{47}^2 & -Y_{48}^2 \\ Y_{51}^1 & Y_{52}^1 & Y_{53}^1 & Y_{54}^1 & 0 & 0 & 0 & 0 \end{pmatrix} = \left(E_{ij}^1\right)_{5\times 8}, \tag{4.33}$$

$$\begin{pmatrix} Y_{11}^2 - Y_{11}^3 & Y_{12}^2 - Y_{12}^3 & Y_{13}^2 - Y_{13}^3 & Y_{15}^2 - Y_{14}^3 & Y_{16}^2 - Y_{15}^3 & Y_{17}^2 - Y_{16}^3 & -Y_{17}^3 & -Y_{18}^3 & -Y_{19}^3 \\ Y_{21}^2 - Y_{21}^3 & Y_{22}^2 - Y_{22}^3 & Y_{23}^2 - Y_{23}^3 & Y_{25}^2 - Y_{24}^3 & Y_{26}^2 - Y_{25}^3 & Y_{27}^2 - Y_{26}^3 & -Y_{27}^3 & -Y_{28}^3 & -Y_{29}^3 \\ Y_{31}^2 - Y_{31}^3 & Y_{32}^2 - Y_{32}^3 & Y_{33}^2 - Y_{33}^3 & Y_{35}^2 - Y_{34}^3 & Y_{36}^2 - Y_{35}^3 & Y_{37}^2 - Y_{36}^3 & -Y_{37}^3 & -Y_{38}^3 & -Y_{39}^3 \\ Y_{41}^2 & Y_{42}^2 & Y_{43}^2 & Y_{45}^2 & Y_{46}^2 & Y_{47}^2 & 0 & 0 & 0 \\ Y_{51}^2 - Y_{41}^3 & Y_{52}^2 - Y_{42}^3 & Y_{53}^2 - Y_{43}^3 & Y_{55}^2 - Y_{44}^3 & Y_{56}^2 - Y_{45}^3 & Y_{57}^2 - Y_{46}^3 & -Y_{47}^3 & -Y_{48}^3 & -Y_{49}^3 \\ Y_{61}^2 - Y_{51}^3 & Y_{62}^2 - Y_{52}^3 & Y_{63}^2 - Y_{53}^3 & Y_{65}^2 - Y_{54}^3 & Y_{66}^2 - Y_{55}^3 & Y_{67}^2 - Y_{56}^3 & -Y_{57}^3 & -Y_{58}^3 & -Y_{59}^3 \\ Y_{71}^2 - Y_{61}^3 & Y_{72}^2 - Y_{62}^3 & Y_{73}^2 - Y_{63}^3 & Y_{75}^2 - Y_{64}^3 & Y_{76}^2 - Y_{65}^3 & Y_{77}^2 - Y_{66}^3 & -Y_{67}^3 & -Y_{68}^3 & -Y_{69}^3 \\ Y_{81}^2 & Y_{82}^2 & Y_{83}^2 & Y_{85}^2 & Y_{86}^2 & Y_{87}^2 & 0 & 0 & 0 \end{pmatrix}$$
$$= \left(E_{ij}^2\right)_{8\times 9}, \tag{4.34}$$

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$$\begin{pmatrix} Y^3_{11}-Y^4_{11} & Y^3_{12}-Y^4_{12} & Y^3_{14}-Y^4_{13} & Y^3_{15}-Y^4_{14} & Y^3_{17}-Y^4_{15} & Y^3_{18}-Y^4_{16} & -Y^4_{17} & -Y^4_{18} \\ Y^3_{21}-Y^4_{21} & Y^3_{22}-Y^4_{22} & Y^3_{24}-Y^4_{23} & Y^3_{25}-Y^4_{24} & Y^3_{27}-Y^4_{25} & Y^3_{28}-Y^4_{26} & -Y^4_{27} & -Y^4_{28} \\ Y^3_{31} & Y^3_{32} & Y^3_{34} & Y^3_{35} & Y^3_{37} & Y^3_{38} & 0 & 0 \\ Y^3_{41}-Y^4_{31} & Y^3_{42}-Y^4_{32} & Y^3_{44}-Y^4_{33} & Y^3_{45}-Y^4_{34} & Y^3_{47}-Y^4_{35} & Y^3_{48}-Y^4_{36} & -Y^4_{37} & -Y^4_{38} \\ Y^3_{51}-Y^4_{41} & Y^3_{52}-Y^4_{42} & Y^3_{54}-Y^4_{43} & Y^3_{55}-Y^4_{44} & Y^3_{57}-Y^4_{45} & Y^3_{58}-Y^4_{46} & -Y^4_{47} & -Y^4_{48} \\ Y^3_{61} & Y^3_{62} & Y^3_{64} & Y^3_{65} & Y^3_{67} & Y^3_{68} & 0 & 0 \\ Y^3_{71}-Y^4_{51} & Y^3_{72}-Y^4_{52} & Y^3_{74}-Y^4_{53} & Y^3_{75}-Y^4_{54} & Y^3_{77}-Y^4_{55} & Y^3_{78}-Y^4_{56} & -Y^4_{57} & -Y^4_{58} \\ Y^3_{81}-Y^4_{61} & Y^3_{82}-Y^4_{62} & Y^3_{84}-Y^4_{63} & Y^3_{85}-Y^4_{64} & Y^3_{87}-Y^4_{65} & Y^3_{88}-Y^4_{66} & -Y^4_{67} & -Y^4_{68} \\ Y^3_{91} & Y^3_{92} & Y^3_{94} & Y^3_{95} & Y^3_{97} & Y^3_{98} & 0 & 0 \end{pmatrix} = \left(E^3_{ij}\right)_{9\times 8}, \tag{4.35}$$

$$\begin{pmatrix} Y^4_{11}-Y^5_{11} & Y^4_{13}-Y^5_{12} & Y^4_{15}-Y^5_{13} & Y^4_{17}-Y^5_{14} & -Y^5_{15} \\ Y^4_{21} & Y^4_{23} & Y^4_{25} & Y^4_{27} & 0 \\ Y^4_{31}-Y^5_{21} & Y^4_{33}-Y^5_{22} & Y^4_{35}-Y^5_{23} & Y^4_{37}-Y^5_{24} & -Y^5_{25} \\ Y^4_{41} & Y^4_{43} & Y^4_{45} & Y^4_{47} & 0 \\ Y^4_{51}-Y^5_{31} & Y^4_{53}-Y^5_{32} & Y^4_{55}-Y^5_{33} & Y^4_{57}-Y^5_{34} & -Y^5_{35} \\ Y^4_{61} & Y^4_{63} & Y^4_{65} & Y^4_{67} & 0 \\ Y^4_{71}-Y^5_{41} & Y^4_{73}-Y^5_{42} & Y^4_{75}-Y^5_{43} & Y^4_{77}-Y^5_{44} & -Y^5_{45} \\ Y^4_{81} & Y^4_{83} & Y^4_{85} & Y^4_{87} & 0 \end{pmatrix} = \left(E^4_{ij}\right)_{8\times 5}. \tag{4.36}$$

Hence, the matrix equations (4.33)–(4.36) are consistent if and only if

$$E^1_{55}=0,\ E^1_{56}=0,\ E^1_{57}=0,\ E^1_{58}=0,\ E^2_{47}=0,\ E^2_{48}=0,\ E^2_{49}=0,\ E^2_{87}=0,\ E^2_{88}=0,\ E^2_{89}=0, \tag{4.37}$$

$$E^3_{37}=0,\ E^3_{38}=0,\ E^3_{67}=0,\ E^3_{68}=0,\ E^3_{97}=0,\ E^3_{98}=0,\ E^4_{25}=0,\ E^4_{45}=0,\ E^4_{65}=0,\ E^4_{85}=0, \tag{4.38}$$

$$E^1_{45}+E^2_{44}=0, \quad E^1_{46}+E^2_{45}=0, \quad E^1_{47}+E^2_{46}=0, \tag{4.39}$$

$$E^2_{37}+E^3_{35}=0, \quad E^2_{38}+E^3_{36}=0, \quad E^2_{77}+E^3_{65}=0, \quad E^2_{78}+E^3_{66}=0, \tag{4.40}$$

$$E^3_{27}+E^4_{24}=0, \quad E^3_{57}+E^4_{44}=0, \quad E^3_{87}+E^4_{64}=0, \tag{4.41}$$

$$E^1_{35}+E^2_{34}+E^3_{33}=0, \quad E^1_{36}+E^2_{35}+E^3_{34}=0, \tag{4.42}$$

$$E^2_{27}+E^3_{25}+E^4_{23}=0, \quad E^2_{67}+E^3_{55}+E^4_{43}=0, \tag{4.43}$$

$$E^1_{25}+E^2_{24}+E^3_{23}+E^4_{22}=0. \tag{4.44}$$

Now we will use (3.12), (3.13), (3.15), (4.25)–(4.32) to prove that (4.17)–(4.23) $\Longrightarrow$ (4.37)–(4.44).

1. Note that

$$\begin{aligned} r\begin{pmatrix} C_1 & B_1 \\ A_1 & 0 \end{pmatrix} = r(A_1)+r(B_1) &\Longrightarrow r\begin{pmatrix} Q_0E_1T_1^{-1} & Q_0S_1Q_1^{-1} \\ T_0D_1T_1^{-1} & 0 \end{pmatrix} = r(T_0D_1T_1^{-1})+r(Q_0S_1Q_1^{-1}) \\ &\Longrightarrow r\begin{pmatrix} E_1 & S_1 \\ D_1 & 0 \end{pmatrix} = r(D_1)+r(S_1) \\ &\Longrightarrow E^1_{55}=0, \quad E^1_{56}=0, \quad E^1_{57}=0, \quad E^1_{58}=0. \end{aligned}$$

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Similarly, we have

$$r\begin{pmatrix} C_2 & B_2 \\ A_2 & 0 \end{pmatrix} = r(A_2)+r(B_2) \Longrightarrow E^2_{47}=0,\ \ E^2_{48}=0,\ \ E^2_{49}=0,\ \ E^2_{87}=0,\ \ E^2_{88}=0,\ \ E^2_{89}=0,$$

$$r\begin{pmatrix} C_3 & B_3 \\ A_3 & 0 \end{pmatrix} = r(A_3)+r(B_3) \Longrightarrow E^3_{37}=0,\ \ E^3_{38}=0,\ \ E^3_{67}=0,\ \ E^3_{68}=0,\ \ E^3_{97}=0,\ \ E^3_{98}=0,$$

$$r\begin{pmatrix} C_4 & B_4 \\ A_4 & 0 \end{pmatrix} = r(A_4)+r(B_4) \Longrightarrow E^4_{25}=0,\ \ E^4_{45}=0,\ \ E^4_{65}=0,\ \ E^4_{85}=0.$$

Hence, we show that (4.17) $\Longrightarrow$ (4.37)–(4.38).

2. Note that

$$\begin{aligned}
& r\begin{pmatrix} C_1A_2+B_1C_2 & B_1B_2 \\ A_1A_2 & 0 \end{pmatrix} = r(A_1A_2)+r(B_1B_2) \\
\Longrightarrow\ & r\begin{pmatrix} Q_0E_1D_2T_2^{-1}+Q_0S_1E_2T_2^{-1} & Q_0S_1S_2Q_2^{-1} \\ T_0D_1D_2T_2^{-1} & 0 \end{pmatrix} = r(T_0D_1D_2T_2^{-1})+r(Q_0S_1S_2Q_2^{-1}) \\
\Longrightarrow\ & r\begin{pmatrix} E_1D_2+S_1E_2 & S_1S_2 \\ D_1D_2 & 0 \end{pmatrix} = r(D_1D_2)+r(S_1S_2) \\
\Longrightarrow\ & E^1_{45}+E^2_{44}=0,\ \ E^1_{46}+E^2_{45}=0,\ \ E^1_{47}+E^2_{46}=0 \\
\Longrightarrow\ & (4.39).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& r\begin{pmatrix} C_2A_3+B_2C_3 & B_2B_3 \\ A_2A_3 & 0 \end{pmatrix} = r(A_2A_3)+r(B_2B_3) \\
\Longrightarrow\ & E^2_{37}+E^3_{35}=0,\ \ E^2_{38}+E^3_{36}=0,\ \ E^2_{77}+E^3_{65}=0,\ \ E^2_{78}+E^3_{66}=0 \\
\Longrightarrow\ & (4.40), \\
& r\begin{pmatrix} C_3A_4+B_3C_4 & B_3B_4 \\ A_3A_4 & 0 \end{pmatrix} = r(A_3A_4)+r(B_3B_4) \\
\Longrightarrow\ & E^3_{27}+E^4_{24}=0,\ \ E^3_{57}+E^4_{44}=0,\ \ E^3_{87}+E^4_{64}=0 \\
\Longrightarrow\ & (4.41).
\end{aligned}$$

Hence, we show that (4.18)–(4.20) $\Longrightarrow$ (4.39)–(4.41).

3. Note that

$$\begin{aligned}
& r\begin{pmatrix} C_1A_2A_3+B_1C_2A_3+B_1B_2C_3 & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} = r(A_1A_2A_3)+r(B_1B_2B_3) \\
\Longrightarrow\ & r\begin{pmatrix} Q_0E_1D_2D_3T_3^{-1}+Q_0S_1E_2D_3T_3^{-1}+Q_0S_1S_2E_3T_3^{-1} & Q_0S_1S_2S_3Q_3^{-1} \\ T_0D_1D_2D_3T_3^{-1} & 0 \end{pmatrix} = \\
& r(T_0D_1D_2D_3T_3^{-1})+r(Q_0S_1S_2S_3Q_3^{-1}) \\
\Longrightarrow\ & r\begin{pmatrix} E_1D_2D_3+S_1E_2D_3+S_1S_2E_3 & S_1S_2S_3 \\ D_1D_2D_3 & 0 \end{pmatrix} = r(D_1D_2D_3)+r(S_1S_2S_3) \\
\Longrightarrow\ & E^1_{35}+E^2_{34}+E^3_{33}=0,\ \ E^1_{36}+E^2_{35}+E^3_{34}=0 \Longrightarrow (4.42).
\end{aligned}$$

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Similarly, we have

$$r\begin{pmatrix} C_2A_3A_4+B_2C_3A_4+B_2B_3C_4 & B_2B_3B_4 \\ A_2A_3A_4 & 0 \end{pmatrix} = r(A_2A_3A_4)+r(B_2B_3B_4)$$
$$\Longrightarrow E^2_{27}+E^3_{25}+E^4_{23}=0,\ \ E^2_{67}+E^3_{55}+E^4_{43}=0 \Longrightarrow (4.43).$$

Hence, we show that (4.21)–(4.22)$\Longrightarrow$ (4.42)–(4.43).

4. Finally, we prove that (4.23) $\Longrightarrow$ (4.44).

$$r\begin{pmatrix} C_1A_2A_3A_4+B_1C_2A_3A_4+B_1B_2C_3A_4+B_1B_2B_3C_4 & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix}$$
$$= r(A_1A_2A_3A_4)+r(B_1B_2B_3B_4)$$
$$\Longrightarrow r\begin{pmatrix} Q_0E_1D_2D_3D_4T_4^{-1}+Q_0S_1E_2D_3D_4T_4^{-1}+Q_0S_1S_2E_3D_4T_4^{-1}+Q_0S_1S_2S_3E_4T_4^{-1} & Q_0S_1S_2S_3S_4Q_4^{-1} \\ T_0D_1D_2D_3D_4T_4^{-1} & 0 \end{pmatrix} =$$
$$r(T_0D_1D_2D_3D_4T_4^{-1})+r(Q_0S_1S_2S_3S_4Q_4^{-1})$$
$$\Longrightarrow r\begin{pmatrix} E_1D_2D_3D_4+S_1E_2D_3D_4+S_1S_2E_3D_4+S_1S_2S_3E_4 & S_1S_2S_3S_4 \\ D_1D_2D_3D_4 & 0 \end{pmatrix} = r(D_1D_2D_3D_4)+r(S_1S_2S_3S_4)$$
$$\Longrightarrow E^1_{25}+E^2_{24}+E^3_{23}+E^4_{22}=0 \Longrightarrow (4.44).$$ ∎

The following theorem gives the general solution to the system (3.11).

THEOREM 4.2. *Assume that the system (3.11) has a solution. Then, the general solution to the system (3.11) can be expressed as*

$$X_1=Q_0Y_1T_0^{-1},\ \ X_2=Q_1Y_2T_1^{-1},\ \ X_3=Q_2Y_3T_2^{-1},\ \ X_4=Q_3Y_4T_3^{-1},\ \ X_5=Q_4Y_5T_4^{-1}, \tag{4.45}$$

where

$$Y_1=\begin{pmatrix} E^1_{11}+E^2_{11}+E^3_{11}+E^4_{11}+Y^5_{11} & E^1_{12}+E^2_{12}+E^3_{12}+Y^4_{12} & E^1_{13}+E^2_{13}+Y^3_{13} & E^1_{14}+Y^2_{14} & Y^1_{15} \\ E^1_{21}+E^2_{21}+E^3_{21}+E^4_{21} & E^1_{22}+E^2_{22}+E^3_{22}+Y^4_{22} & E^1_{23}+E^2_{23}+Y^3_{23} & E^1_{24}+Y^2_{24} & Y^1_{25} \\ E^1_{31}+E^2_{31}+E^3_{31} & E^1_{32}+E^2_{32}+E^3_{32} & E^1_{33}+E^2_{33}+Y^3_{33} & E^1_{34}+Y^2_{34} & Y^1_{35} \\ E^1_{41}+E^2_{41} & E^1_{42}+E^2_{42} & E^1_{43}+E^2_{43} & E^1_{44}+Y^2_{44} & Y^1_{45} \\ E^1_{51} & E^1_{52} & E^1_{53} & E^1_{54} & Y^1_{55} \end{pmatrix}, \tag{4.46}$$

$$Y_2=\begin{pmatrix} E^2_{11}+E^3_{11}+E^4_{11}+Y^5_{11} & E^2_{12}+E^3_{12}+Y^4_{12} & E^2_{13}+Y^3_{13} & Y^2_{14} & -E^1_{15} & -E^1_{16} & -E^1_{17} & -E^1_{18} \\ E^2_{21}+E^3_{21}+E^4_{21} & E^2_{22}+E^3_{22}+Y^4_{22} & E^2_{23}+Y^3_{23} & Y^2_{24} & -E^1_{25} & -E^1_{26} & -E^1_{27} & -E^1_{28} \\ E^2_{31}+E^3_{31} & E^2_{32}+E^3_{32} & E^2_{33}+Y^3_{33} & Y^2_{34} & -E^1_{35} & -E^1_{36} & -E^1_{37} & -E^1_{38} \\ E^2_{41} & E^2_{42} & E^2_{43} & Y^2_{44} & -E^1_{45} & -E^1_{46} & -E^1_{47} & -E^1_{48} \\ E^2_{51}+E^3_{41}+E^4_{31}+Y^5_{21} & E^2_{52}+E^3_{42}+Y^4_{32} & E^2_{53}+Y^3_{43} & Y^2_{54} & E^2_{54}+E^3_{43}+E^4_{32}+Y^5_{22} & E^2_{55}+E^3_{44}+Y^4_{34} & E^2_{56}+Y^3_{46} & Y^2_{58} \\ E^2_{61}+E^3_{51}+E^4_{41} & E^2_{62}+E^3_{52}+Y^4_{42} & E^2_{63}+Y^3_{53} & Y^2_{64} & E^2_{64}+E^3_{53}+E^4_{42} & E^2_{65}+E^3_{54}+Y^4_{44} & E^2_{66}+Y^3_{56} & Y^2_{68} \\ E^2_{71}+E^3_{61} & E^2_{72}+E^3_{62} & E^2_{73}+Y^3_{63} & Y^2_{74} & E^2_{74}+E^3_{63} & E^2_{75}+E^3_{64} & E^2_{76}+Y^3_{66} & Y^2_{78} \\ E^2_{81} & E^2_{82} & E^2_{83} & Y^2_{84} & E^2_{84} & E^2_{85} & E^2_{86} & Y^2_{88} \end{pmatrix}, \tag{4.47}$$

$$Y_3=\begin{pmatrix} E^3_{11}+E^4_{11}+Y^5_{11} & E^3_{12}+Y^4_{12} & Y^3_{13} & E^3_{13}+E^4_{12}+Y^5_{12} & E^3_{14}+Y^4_{14} & -E^1_{17}-E^2_{16} & -E^2_{17} & -E^2_{18} & -E^2_{19} \\ E^3_{21}+E^4_{21} & E^3_{22}+Y^4_{22} & Y^3_{23} & E^3_{23}+E^4_{22} & E^3_{24}+Y^4_{24} & -E^1_{27}-E^2_{26} & -E^2_{27} & -E^2_{28} & -E^2_{29} \\ E^3_{31} & E^3_{32} & Y^3_{33} & E^3_{33} & E^3_{34} & -E^1_{37}-E^2_{36} & -E^2_{37} & -E^2_{38} & -E^2_{39} \\ E^3_{41}+E^4_{31}+Y^5_{21} & E^3_{42}+Y^4_{32} & Y^3_{43} & E^3_{43}+E^4_{32}+Y^5_{22} & E^3_{44}+Y^4_{34} & Y^3_{46} & -E^2_{57} & -E^2_{58} & -E^2_{59} \\ E^3_{51}+E^4_{41} & E^3_{52}+Y^4_{42} & Y^3_{53} & E^3_{53}+E^4_{42} & E^3_{54}+Y^4_{44} & Y^3_{56} & -E^2_{67} & -E^2_{68} & -E^2_{69} \\ E^3_{61} & E^3_{62} & Y^3_{63} & E^3_{63} & E^3_{64} & Y^3_{66} & -E^2_{77} & -E^2_{78} & -E^2_{79} \\ E^3_{71}+E^4_{51}+Y^5_{31} & E^3_{72}+Y^4_{52} & Y^3_{73} & E^3_{72}+E^4_{52}+Y^5_{32} & E^3_{74}+Y^4_{54} & Y^3_{76} & E^3_{75}+E^4_{53}+Y^5_{33} & E^3_{76}+Y^4_{56} & Y^3_{79} \\ E^3_{81}+E^4_{61} & E^3_{82}+Y^4_{62} & Y^3_{83} & E^3_{83}+E^4_{62} & E^3_{84}+Y^4_{64} & Y^3_{86} & E^3_{85}+E^4_{63} & E^3_{86}+Y^4_{66} & Y^3_{89} \\ E^3_{91} & E^3_{92} & Y^3_{93} & E^3_{93} & E^3_{94} & Y^3_{96} & E^3_{95} & E^3_{96} & Y^3_{99} \end{pmatrix}, \tag{4.48}$$

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$$Y_4=\begin{pmatrix} E^4_{11}+Y^5_{11} & Y^4_{12} & E^4_{12}+Y^5_{12} & -E^1_{16}-E^2_{25}-E^3_{24} & -E^2_{17}-E^3_{15} & -E^2_{18}-E^3_{16} & -E^3_{17} & -E^3_{18}\\ E^4_{21} & Y^4_{22} & E^4_{22} & -E^1_{26}-E^2_{25}-E^3_{24} & E^4_{23} & -E^2_{28}-E^3_{26} & -E^3_{27} & -E^2_{28}\\ E^4_{31}+Y^5_{21} & Y^4_{32} & E^4_{32}+Y^5_{22} & Y^4_{34} & -E^2_{57}-E^3_{45} & -E^2_{58}-E^3_{46} & -E^3_{47} & -E^3_{48}\\ E^4_{41} & Y^4_{42} & E^4_{42} & Y^4_{44} & E^4_{43} & -E^2_{68}-E^3_{56} & -E^3_{57} & -E^3_{58}\\ E^4_{51}+Y^5_{31} & Y^4_{52} & E^4_{52}+Y^5_{32} & Y^4_{54} & E^4_{53}+Y^5_{33} & Y^3_{56} & -E^3_{77} & -E^3_{78}\\ E^4_{61} & Y^4_{62} & E^4_{62} & Y^4_{64} & E^4_{63} & Y^4_{66} & -E^3_{87} & -E^3_{88}\\ E^4_{71}+Y^5_{41} & Y^4_{72} & E^4_{72}+Y^5_{42} & Y^4_{74} & E^4_{73}+Y^5_{43} & Y^3_{76} & E^4_{74}+Y^5_{44} & Y^3_{78}\\ E^4_{81} & Y^4_{82} & E^4_{82} & Y^4_{84} & E^4_{83} & Y^4_{86} & E^3_{84} & Y^3_{88} \end{pmatrix}, \tag{4.49}$$

$$Y_5=\begin{pmatrix} Y^5_{11} & -E^1_{15}-E^2_{14}-E^3_{13}-E^4_{12} & -E^2_{17}-E^3_{15}-E^4_{13} & -E^3_{17}-E^4_{14} & -E^4_{15}\\ Y^5_{21} & Y^5_{22} & -E^2_{57}-E^3_{45}-E^4_{33} & -E^3_{47}-E^4_{34} & -E^4_{35}\\ Y^5_{31} & Y^5_{32} & Y^5_{33} & -E^3_{77}-E^4_{54} & -E^4_{55}\\ Y^5_{41} & Y^5_{42} & Y^5_{43} & Y^5_{44} & -E^4_{75}\\ Y^5_{51} & Y^5_{52} & Y^5_{53} & Y^5_{54} & Y^5_{55} \end{pmatrix}, \tag{4.50}$$

and $Q_0,\ldots,Q_4,T_0,\ldots,T_4$ are defined in (3.12) and (3.13), and the remaining $Y^1_{j_1k_1}$, $Y^2_{j_2k_2}$, $Y^3_{j_3k_3}$, $Y^4_{j_4k_4}$, and $Y^5_{j_5k_5}$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Remark 4.3. In [9], the authors use QSVD to establish the simultaneous decomposition for 15 matrices A_j, B_j, C_j, D_j, and E_j, which plays an important role in investigating the system of matrix equations $A_jX_jB_j + C_jX_{j+1}D_j = E_j$ $(j = 1, 2, 3)$. In this paper, we use PSVD for the quaternion matrices A_i and B_i to solve the system (3.11) directly.

Example 4.4. Given the quaternion matrices:

$$A_1=\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \mathbf{i}+\mathbf{j} & 2\mathbf{j} & \mathbf{j}+\mathbf{k}\end{pmatrix},\quad A_2=\begin{pmatrix}1 & \mathbf{i}+\mathbf{j} & \mathbf{j}-\mathbf{k}\\ \mathbf{i} & -1+\mathbf{k} & \mathbf{j}+\mathbf{k}\\ 1+\mathbf{i} & -1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 2\mathbf{j}\end{pmatrix},$$

$$A_3=\begin{pmatrix}\mathbf{j} & \mathbf{k} & 1+\mathbf{j} & 2\\ 0 & 1 & 2\mathbf{i} & \mathbf{k}\\ \mathbf{j} & 1+\mathbf{k} & 1+2\mathbf{i}+\mathbf{j} & 2+\mathbf{k}\end{pmatrix},\quad A_4=\begin{pmatrix}1+2\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{j}+\mathbf{k}\\ -1-2\mathbf{i} & -\mathbf{j} & -\mathbf{k} & -\mathbf{j}-\mathbf{k}\\ 2\mathbf{i} & \mathbf{j} & -\mathbf{k} & \mathbf{j}\\ 1 & 0 & 2\mathbf{k} & \mathbf{k}\end{pmatrix},$$

$$B_1=\begin{pmatrix}\mathbf{i}-\mathbf{j} & \mathbf{i}+\mathbf{k} & 1\\ -1-\mathbf{k} & -1-\mathbf{j} & \mathbf{i}\end{pmatrix},\quad B_2=\begin{pmatrix}\mathbf{j} & \mathbf{k} & 1\\ 0 & 1 & \mathbf{i}\\ \mathbf{j} & 1+\mathbf{k} & 1+\mathbf{i}\end{pmatrix},$$

$$B_3=\begin{pmatrix}\mathbf{k} & \mathbf{j} & 1 & 0\\ -\mathbf{j} & \mathbf{k} & \mathbf{i} & 0\\ -\mathbf{j}+\mathbf{k} & \mathbf{j}+\mathbf{k} & 1+\mathbf{i} & 0\end{pmatrix},\quad B_4=\begin{pmatrix}1 & 2 & 0 & \mathbf{i}\\ \mathbf{i} & 2\mathbf{i} & 0 & -1\\ \mathbf{j} & \mathbf{k} & 1 & \mathbf{i}\\ \mathbf{i}+\mathbf{j} & 2\mathbf{i}+\mathbf{k} & 1 & -1+\mathbf{i}\end{pmatrix},$$

$$C_1=\begin{pmatrix}1-2\mathbf{i}+3\mathbf{j}-\mathbf{k} & 1-2\mathbf{i}-\mathbf{k} & -2-2\mathbf{j}-\mathbf{k}\\ 3\mathbf{i}+\mathbf{j}+2\mathbf{k} & 2\mathbf{i}+4\mathbf{j}+\mathbf{k} & -\mathbf{i}+\mathbf{j}\end{pmatrix},$$

$$C_2=\begin{pmatrix}2-2\mathbf{k} & -1-\mathbf{i}-2\mathbf{k} & -3-\mathbf{i}-2\mathbf{j}\\ -2+\mathbf{i}+\mathbf{j}-2\mathbf{k} & -1-2\mathbf{i}+2\mathbf{k} & 1-2\mathbf{i}-\mathbf{j}+3\mathbf{k}\\ 2+\mathbf{i}-3\mathbf{k} & -1-2\mathbf{i}+3\mathbf{j}+\mathbf{k} & -3\mathbf{i}-\mathbf{j}+\mathbf{k}\end{pmatrix},$$

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$$C_3 = \begin{pmatrix} -1+\mathbf{j} & -1+\mathbf{i}+2\mathbf{k} & -1+2\mathbf{i}+4\mathbf{j} & 1-\mathbf{i}+2\mathbf{j}+\mathbf{k} \\ -\mathbf{i}+\mathbf{k} & -1-\mathbf{i}-2\mathbf{j} & -2-\mathbf{i}+4\mathbf{k} & 1+\mathbf{i}-\mathbf{j}+2\mathbf{k} \\ -1-\mathbf{i}+\mathbf{j}+\mathbf{k} & -2-2\mathbf{j}+2\mathbf{k} & -3+\mathbf{i}+4\mathbf{j}+4\mathbf{k} & 2+\mathbf{j}+3\mathbf{k} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -1-4\mathbf{i}+2\mathbf{j}-2\mathbf{k} & -2+\mathbf{j}+\mathbf{k} & -4+\mathbf{i}+\mathbf{j}-2\mathbf{k} & -3+\mathbf{i}-\mathbf{j}-2\mathbf{k} \\ 4-\mathbf{i}+2\mathbf{j}+2\mathbf{k} & -2\mathbf{i}-\mathbf{j}+\mathbf{k} & -1-4\mathbf{i}+2\mathbf{j}+\mathbf{k} & -1-3\mathbf{i}+2\mathbf{j}-\mathbf{k} \\ -1-3\mathbf{i}+\mathbf{j}+2\mathbf{k} & 1-2\mathbf{i}-2\mathbf{j} & -1-2\mathbf{j}-5\mathbf{k} & -3\mathbf{j}-3\mathbf{k} \\ 1+\mathbf{i}+2\mathbf{k} & 1-2\mathbf{i}-\mathbf{j}+3\mathbf{k} & -2\mathbf{i}-2\mathbf{k} & 1 \end{pmatrix}.$$

Now we consider the system (3.16). Upon computation, we have

$$r\begin{pmatrix} C_i & B_i \\ A_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 3, & \text{if } i=1 \\ 3, & \text{if } i=2 \\ 3, & \text{if } i=3 \\ 4, & \text{if } i=4 \end{cases}, \tag{4.51}$$

$$r\begin{pmatrix} C_1A_2 + B_1C_2 & B_1B_2 \\ A_1A_2 & 0 \end{pmatrix} = r(A_1A_2) + r(B_1B_2) = 2, \tag{4.52}$$

$$r\begin{pmatrix} C_2A_3 + B_2C_3 & B_2B_3 \\ A_2A_3 & 0 \end{pmatrix} = r(A_2A_3) + r(B_2B_3) = 2, \tag{4.53}$$

$$r\begin{pmatrix} C_3A_4 + B_3C_4 & B_3B_4 \\ A_3A_4 & 0 \end{pmatrix} = r(A_3A_4) + r(B_3B_4) = 3, \tag{4.54}$$

$$r\begin{pmatrix} C_1A_2A_3 + B_1C_2A_3 + B_1B_2C_3 & B_1B_2B_3 \\ A_1A_2A_3 & 0 \end{pmatrix} = r(A_1A_2A_3) + r(B_1B_2B_3) = 2, \tag{4.55}$$

$$r\begin{pmatrix} C_2A_3A_4 + B_2C_3A_4 + B_2B_3C_4 & B_2B_3B_4 \\ A_2A_3A_4 & 0 \end{pmatrix} = r(A_2A_3A_4) + r(B_2B_3B_4) = 2, \tag{4.56}$$

$$\begin{aligned} & r\begin{pmatrix} C_1A_2A_3A_4 + B_1C_2A_3A_4 + B_1B_2C_3A_4 + B_1B_2B_3C_4 & B_1B_2B_3B_4 \\ A_1A_2A_3A_4 & 0 \end{pmatrix} \\ & = r(A_1A_2A_3A_4) + r(B_1B_2B_3B_4) = 2. \end{aligned} \tag{4.57}$$

Therefore, all rank equalities in (4.17)–(4.23) hold, and thus, the system (3.16) is consistent. We can use the steps in the proofs of Lemma 2.1 and Theorem 4.1 to find the solution. Below is the algorithm for finding the solution to the system (3.16).

1. Computer PSVD for the quaternion matrices A_i and B_i $(i = 1, 2, 3, 4)$.
2. Computer $E_i = Q_{i-1}^{-1} C_i T_i$ $(i = 1, 2, \ldots, 4)$.
3. Computer $Y_1, \ldots, Y_5$ in (4.46)–(4.49).
4. Computer $X_1, \ldots, X_5$ in (4.45).

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In addition, $X_1,\ldots,X_5$ with the following structures satisfy the system (3.16):

$$X_1=\begin{pmatrix}\mathbf{j} & \mathbf{k}\\ 1+\mathbf{i} & 1\end{pmatrix},\quad X_2=\begin{pmatrix}1 & \mathbf{i} & \mathbf{j}\\ \mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{i}-\mathbf{k}\\ 0 & 1 & 0\end{pmatrix},\quad X_3=\begin{pmatrix}\mathbf{j} & \mathbf{k} & 1\\ \mathbf{k} & -\mathbf{j} & \mathbf{i}\\ \mathbf{j}+\mathbf{k} & -\mathbf{j}+\mathbf{k} & 1+\mathbf{i}\end{pmatrix},$$

$$X_4=\begin{pmatrix}\mathbf{j} & 1 & \mathbf{i} & \mathbf{k}\\ \mathbf{k} & \mathbf{i} & -1 & -\mathbf{j}\\ 0 & 2 & 1 & \mathbf{i}\\ \mathbf{i}+\mathbf{j} & \mathbf{j}+\mathbf{k} & 0 & 0\end{pmatrix},\quad X_5=\begin{pmatrix}2\mathbf{i} & 1+2\mathbf{j} & 0 & 1\\ -1 & -2\mathbf{j} & 1 & \mathbf{k}\\ -1+2\mathbf{i} & 1 & 1 & 1+\mathbf{k}\\ \mathbf{j}+\mathbf{k} & 0 & \mathbf{j} & 0\end{pmatrix}.$$

5. Conclusions. We have used the pure product singular value decomposition (PSVD) approach to consider the system of four coupled Sylvester-type quaternion matrix equations with five unknowns (1.1). We have given some necessary and sufficient conditions for the existence of a solution to the system (1.1) using the ranks of the given quaternion matrices. We have also derived the general solution to this system when it is solvable.

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