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UNIONS OF A CLIQUE AND A CO-CLIQUE AS STAR COMPLEMENTS FOR NON-MAIN GRAPH EIGENVALUES*

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Abstract. Graphs consisting of a clique and a co-clique, both of arbitrary size, are considered in the role of star complements for an arbitrary non-main eigenvalue. Among other results, the sign of such a eigenvalue is discussed, the neigbourhoods of star set vertices are described, and the parameters of all strongly regular extensions are determined. It is also proved that, apart from a specified special case, if the size of a co-clique is fixed then there is a finite number of possibilities for our star complement and the corresponding non-main eigenvalue. Numerical data on these possibilities is presented.

Key words. Adjacency matrix, Non-main part of the spectrum, Graph extension, Strongly regular graph, Block design.

AMS subject classifications. 05C50, 65F15.

1. Introduction. If μ is an eigenvalue of (the $n \times n$ adjacency matrix $A = (a_{ij})$ of a finite simple) graph G with multiplicity k, then a *star set* for μ in G is a subset X of the set of vertices $V(G)$ such that $|X| = k$ and μ is not an eigenvalue of the induced subgraph $G - X$. In this situation, the graph $G - X$ (of order $n - k$) is called a *star complement* for μ in G. The main properties of star sets, star complements and corresponding eigenvalues can be found in [5, Chapter 5]. Obviously, star sets and star complements exist for any eigenvalue of any graph.

An eigenvalue μ of G is a *main eigenvalue* if the corresponding eigenspace $\mathcal{E}(\mu)$ is not orthogonal to the all-1 vector $\mathbf{j}$ in $\mathbb{R}^n$. Otherwise, it is a non-main eigenvalue.

Star complements are studied in an extensive literature. A database with around 1500 examples is described by Cvetković et al. in [4], while surveys of results obtained before 2004 can be found in [6, 12]. For more recent results, see [1, 15, 17]. In particular, some infinite families of star complements in regular graphs are studied by Rowlinson in [10, 11, 13] or Rowlinson and Tayfeh-Rezaie [14] (the latter paper contains a survey and further references).

In this paper, we consider an infinite family of graphs (that depend on two parameters) in the role of star complements for an arbitrary non-main eigenvalue. In this context, we use $C(p, q)$ to denote the (disjoint) union of a clique (i.e., a complete graph) with p vertices and a co-clique (i.e., a totally disconnected graph) with q vertices.

Main results of this paper are announced in the Abstract. Section 2 is preparatory. In the next three sections, we consider $C(p, q)$ as a star complement for an arbitrary non-main eigenvalue. The special case $q = 0$ and certain general results are separated in Section 3. Sections 4 and 5 are devoted to the special case $q = 1$ and the general case $q \geq 2$, respectively. A few concluding remarks are given is Section 6. The Appendix is reserved for numerical data that arises from the previous theoretical considerations.

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2. Preliminaries. We use the standard graph-theoretic notation. For example, we write K_n and nK_1 for the complete and totally disconnected graph with n vertices, respectively. The degree of a regular graph is denoted r. We recall that such a graph with n vertices that is neither complete nor totally disconnected is said to be *strongly regular* with parameters

$$(n,\, r,\, a,\, b) \tag{2.1}$$

if there exist non-negative integers a and b such that every two adjacent vertices have exactly a common neighbours and every two non-adjacent vertices have exactly b common neighbours. It is well known that every connected strongly regular graph has exactly three (distinct) eigenvalues and that exactly one of them is negative (cf. [16, Theorem 3.4.7]).

We now mention certain results on star complements that are relevant to this paper. If X is a set of k vertices of a graph G, then the adjacency matrix of G can be written in the form

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}, \tag{2.2}$$

where A_X is the adjacency matrix of the graph induced by X. According to the Reconstruction Theorem [5, Theorem 5.1.7], X is a star set for the eigenvalue μ of G (and consequently, the graph H determined by the adjacency matrix C is the corresponding star complement) if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{2.3}$$

Following an extensive literature, for a graph H with t vertices and the adjacency matrix C we define a bilinear form

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T(\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

For a subset U of the vertex set $V(H)$ and a vertex u not in $V(H)$, we denote by $H(U)$ the graph obtained from H by joining u to all the vertices of U. We say that $H(U)$ is a *good extension* of H for an eigenvalue μ if μ is not an eigenvalue of H but is an eigenvalue of $H(U)$. By equating entries in (2.3), we conclude that $H(U)$ is a good extension if and only if

$$\langle \mathbf{b}_u, \mathbf{b}_u\rangle = \mu, \tag{2.4}$$

where $\mathbf{b}_u$ indicates the neighbourhood of u in $V(H)$.

Clearly, we may combine a pair of good extensions $H(U_1)$ and $H(U_2)$ to obtain two different graphs: in the first the vertices u_1 and u_2 are adjacent (we write $u_1 \sim u_2$), while in the second they are not ($u_1 \nsim u_2$). We say that the vertices u_1 and u_2 as well as the corresponding subsets U_1 and U_2 are *compatible* if μ is an eigenvalue of multiplicity two in some of these graphs. Again, the identity (2.3) leads to the conclusion that u_1 and u_2 are compatible if and only if

$$\langle \mathbf{b}_{u_1}, \mathbf{b}_{u_2}\rangle \in \{-1, 0\}. \tag{2.5}$$

The condition (2.5) is referred to as the *compatibility condition*. In this situation, u_1 and u_2 are adjacent (resp., non-adjacent) when $\langle \mathbf{b}_{u_1}, \mathbf{b}_{u_2}\rangle = -1$ (resp., $\langle \mathbf{b}_{u_1}, \mathbf{b}_{u_2}\rangle = 0$). It follows from the identity (2.3) that, for $\mu \notin \{-1, 0\}$, a subset of $V(H)$ cannot be compatible with itself. Consequently, there is a finite number of

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extensions that can be obtained from any star complement for any eigenvalue, unless that eigenvalue belongs to $\{-1, 0\}$.

In addition, it follows (again by the Reconstruction Theorem) that a collection of k subsets of $V(H)$ that correspond to good extensions and are compatible in pairs give rise to the extension G (of H) with the adjacency matrix (2.2) having the eigenvalue μ of multiplicity k. Thus, G is determined by μ, a star complement H for μ and the H-neighbourhoods of the vertices $u \in X$. The last fact reveals an effective tool for constructing large graphs from their smaller parts – star complements. If μ and H are given, then an essential part in determining G is a description of H-neighbourhoods of vertices in X, and this is one of the main points in the forthcoming sections.

A graph G is a *maximal extension* of a star complement H for an eigenvalue μ if it arises from a maximal collection of compatible subsets of $V(H)$. Maximal extensions are of special interest since any other extension is an induced subgraph of some maximal extension. We mention in passing that, according to [5, Theorem 5.3.1], for $\mu \notin \{-1, 0\}$, unless $\mu = 1$ and either $G \cong K_2$ or $G \cong 2K_2$, if $|V(H)| = t$ then $|V(G)| \leq \binom{t+1}{2}$.

It follows from the identity (2.3) that the eigenspace $\mathcal{E}(\mu)$ consists of vectors

$$\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1} B\mathbf{x} \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{R}^k$. Using this argument, we conclude that, with the notation from the above, μ is a non-main eigenvalue of G if and only if

$$\langle \mathbf{b}_u, \mathbf{j} \rangle = -1 \quad \text{for all } u \in X. \tag{2.6}$$

We are now ready to present our results.

3. $C(p,q)$ **as a star complement for a non-main eigenvalue.** We first single out the case $q = 0$. The first result is not new. For example, it is mentioned in passing in [14].

Theorem 3.1. *If a graph G contains the complete graph K_n as a star complement for an eigenvalue μ, then μ is a main eigenvalue in G, unless $\mu \in \{-1, 0\}$.*

Proof. If C is the adjacency matrix of K_n, then we have $(\mu+1-n)(\mu+1)(\mu I-C)^{-1} = (\mu+2-n)I+C$. Consider the extension $K_n(U)$ where U is a set of s vertices of K_n. Using the equalities (2.4) and (2.6), we get

$$s(\mu - n + s + 1) = \mu(\mu - n + 1)(\mu + 1) \quad \text{and} \quad s(\mu + 1) = -(\mu - n + 1)(\mu + 1).$$

Solving this system, we get $(\mu, s) = (-1, n)$ or $(\mu, s) = (0, n - 1)$.

A similar conclusion can be derived for the totally disconnected graph nK_1 in the role of a star complement.

Both solutions from the previous proof give infinite families of graphs. In fact, since μ is not an eigenvalue of the star complement, in the first solution we have $n = 1$, and then the corresponding family consists of complete graphs. The second family, among other graphs, contains the family of complete multipartite strongly regular graphs with spectrum $[k(n-1), 0^{(k-1)n}, (-k)^{n-1}]$ (the multiplicities of eigenvalues are given in the exponents). It is worth mentioning that a strongly regular graph has the eigenvalue 0 if and only if it is a complete multipartite regular graph (cf. [16, Theorem 3.4.9]), and therefore complete multipartite strongly

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regular graphs are the only strongly regular graphs that can be obtained from complete star complements. Observe that here we have a special case of so-called *uniform* star set introduced by Rowlinson [13]: The star set is called uniform if all vertices in the corresponding star complement have equal number of neighbours in X. Observe also that the regularity of G, in conjunction with the uniformity of X, yields the regularity of the star complement $G-X$. We shall meet more uniform star sets in the forthcoming sections.

We now proceed with the union of a clique and a co-clique in the role of a star complement. Resuming the notation from the introductory section, we denote such a graph by $C(p,q)$. In addition, since complete graphs and totally disconnected ones are discussed in the above, we always assume (somewhere without noting) that $p \geq 2$ and $q \geq 1$. If C stands for the adjacency matrix of $C(p,q)$, then the inverse matrix $(\mu I - C)^{-1}$ satisfies the identity

$$m(\mu)(\mu I - C)^{-1} = C^2 + (\mu + 2 - p)C + (\mu(\mu + 2 - p) + 1 - p)\, I, \tag{3.7}$$

where $m(x) = (x + 1 - p)x(x + 1)$ is the minimal polynomial of C. If u is a vertex adjacent to s vertices of the clique and t ($t \geq 1$) vertices of the co-clique of $C(p,q)$, then using the conditions (2.4) and (2.6), we arrive at the system which, after simplifying, reads as follows:

$$\mu s(\mu - p + s + 1) + (\mu + 1)t(\mu - p + 1) = \mu^2(\mu - p + 1)(\mu + 1)$$

and

$$(\mu + 1)\left(t(\mu - p + 1) + \mu s\right) = -\mu(\mu - p + 1)(\mu + 1).$$

Under the assumption that $s \notin \{\mu + 1, (\mu + 1)^2\}$, this system gives

$$p = \frac{(\mu + 1)^3 - s^2}{(\mu + 1)^2 - s} \quad \text{and} \quad t = \frac{\mu^2(\mu + 1)}{\mu - s + 1}. \tag{3.8}$$

The excluded possibilities for s are easily eliminated by the imposed restrictions $p \geq 2$, $q \geq 1$ (and the fact that μ is not an eigenvalue of the star complement).

By the system (3.8), μ may take both positive or negative values, but according to our numerical computations, the only positive integer triple (p, s, t) and a positive μ satisfying (3.8) consists of $(2, 1, 1)$ and $\frac{\sqrt{5}-1}{2}$ (producing the path with 4 vertices).

For any fixed star complement, we obviously have a finite number of possibilities for μ. On the contrary, if μ is fixed, according to the expressions (3.8) we have finite numbers of possibilities for p and t (since they and s are non-negative integers). In the next lemma and the subsequent text, we establish a less obvious result.

LEMMA 3.2. *For any fixed $t \geq 2$, there is a finite number of pairs (μ, p) such that $C(p,t)$ is a star complement for a non-main eigenvalue μ in a graph in which the corresponding star set contains a vertex adjacent to all vertices of the co-clique tK_1.*

Proof. Expressing p of (3.8) in terms of μ and t, we get

$$p = -\frac{(\mu + 1)(\mu^3 - 2\mu t - t^2)}{t(\mu + t)}$$

(a simple analysis of (3.8) leads to the conclusion that the equality $t = -\mu$ is impossible under our assumption $t \geq 2$), which means that $(\mu + 1)(\mu^3 - 2\mu t - t^2) \equiv 0 \mod (\mu + t)$. The last gives

$$\mu^2(\mu + 1)^2 - (\mu + 1)(\mu + t)^2 \equiv 0 \mod (\mu + t).$$

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Equivalently,

$$\mu^2(\mu+1)^2 \equiv 0 \mod (\mu+t).$$

Now, for any fixed $t \geq 2$ and $|\mu|$ sufficiently large, $\mu+t$ cannot divide $\mu^2(\mu+1)^2$, which gives the assertion. ∎

Consequently, if the size of a co-clique is fixed then, unless every vertex of a star set is adjacent to exactly one vertex of a co-clique, there is a finite number of possibilities for the size of a clique and a non-main eigenvalue. In the forthcoming Section 5, we impose more restrictions for the neighbourhoods in $C(p,q)$, and then establish (in the Appendix) certain possibilities that correspond to the previous lemma.

4. Special case $q = 1$. In the following theorem, as well as in the forthcoming Theorem 5.1, we avoid the presentation of a part containing a straightforward algebraic calculus (i.e., solving the systems of non-linear equations). The system is formed on the basis of the conditions (2.4)–(2.6) and the identity (3.7), while the calculus can be performed either by hand or by any advanced mathematical software.

Theorem 4.1. *If U_1 and U_2 are compatible subsets of $V(C(p,1))$, $p \geq 2$, for a non-main eigenvalue $\mu \notin \{-1,0\}$, then the isolated vertex of $C(p,1)$ belongs to $U_1 \cap U_2$ and we also have $p = -\mu^3+2\mu+1$, $|U_1| = |U_2| = -(\mu+1)^2(\mu-1)+1$ and*

$$|U_1 \cap U_2| = \begin{cases} -\mu(\mu+1)^2+1, & \text{if } u_1 \sim u_2, \\ -(\mu+1)(\mu^2+\mu-1)+1, & \text{if } u_1 \not\sim u_2. \end{cases}$$

Proof. First, if at least one of the sets U_1 or U_2 does not contain the isolated vertex, then μ cannot be a non-main eigenvalue, by Theorem 3.1. Using the conditions (2.4)–(2.6), we arrive at the two systems (one for $u_1 \sim u_2$, the other for $u_1 \not\sim u_2$) in five variables ($\mu, p, |U_1|, |U_2|$ and $|U_1 \cap U_2|$). Solving them under the restriction $\mu \notin \{-1,0\}$, we arrive at the desired result. ∎

The largest eigenvalue of any connected graph is main (since the coordinates of an associated eigenvector are non-zero and of the same sign). Recall (say from [16, Corollary 2.1.5]) that this is the only main eigenvalue if and only if a graph under consideration is regular. Motivated by this fact, we consider regular extensions of $C(p,1)$.

Theorem 4.2. *If G is a regular extension of $C(p,1)$, $p \geq 2$, then its order n and degree r are equal to $(\mu+1)^2(\mu-2)^2$ and $(\mu^2-2)(\mu^2-\mu-1)$, respectively. If G is, in addition, strongly regular then its parameters are given by*

$$\left((\mu+1)^2(\mu-2)^2,\ (\mu^2-2)(\mu^2-\mu-1),\ -\mu(\mu-1)(\mu^2+\mu-1),\ (\mu^2-1)(\mu^2-2)\right). \tag{4.9}$$

Proof. The vertices of G can be partitioned into the two sets: the star set X and the vertex set $V(C(p,1))$ of the corresponding star complement. Since, by the previous theorem, the isolated vertex of $C(p,1)$ is adjacent to all vertices of X, we have $|X| = r$. By the same theorem, every vertex of X has additional $-(\mu+1)^2(\mu-1)$ neighbours in the clique K_p of the star complement. In other words, there are $-r(\mu+1)^2(\mu-1)$ edges between X and K_p. This means that every vertex of K_p has $\frac{-r(\mu+1)^2(\mu-1)}{p}$ neigbours in X, giving

$$r = \frac{-r(\mu+1)^2(\mu-1)}{p} + p - 1.$$

After replacing p by its value obtained in Theorem 4.1, this leads to the desired value of r. Since G has $p+1+r$ vertices, by using the value of p obtained in the previous theorem we get $n = (\mu+1)^2(\mu-2)^2$.

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TABLE 1
The parameters (4.9) for small $-\mu$'s.

μ	p	n	r	a	b
-2	5	16	10	6	6
-3	22	100	77	60	56
-4	57	324	266	220	210
-5	116	784	667	570	552
-6	205	1600	1394	1218	1190

If G is strongly regular, then the number of common neighbours of two non-adjacent vertices is equal to b (we resume the notation of (2.1)). In particular, the number of common neighbours of the isolated vertex of $C(p,1)$ and any of the vertices of K_p is equal to the number of neighbours of the latter vertex in X, so we have $b = \frac{-r(\mu+1)^2(\mu-1)}{p}$. Replacing p and r, we arrive at $b = (\mu^2-1)(\mu^2-2)$. The third parameter a can be computed in various ways, for example it is expressible in terms of the remaining three as $a = r-1-\frac{b(n-r-1)}{r}$, from the well-known equality $r(r-a-1) = (n-r-1)b$ valid for strongly regular graphs with parameters (2.1) – see, for instance, [8], and we are done. $\square$

Observing the parameters (4.9), we conclude that the only strongly regular graphs that correspond to a positive μ are $2K_2$ and the pentagon. Here is a partial converse of the previous result.

THEOREM 4.3. *Every strongly regular graph with parameters (4.9) (where μ is its negative eigenvalue) contains the graph $C(-\mu^3+2\mu+1,1)$ as a star complement for μ.*

Proof. Under the notation from the previous proof, the degree of the complementary strongly regular graph $\overline{G}$ is $p = -\mu^3+2\mu+1$, while its third parameter is equal to $n-2(r+1)+b$. In fact, the last expression is equal to zero, which means that $\overline{G}$ contains the star $K_{1,p}$ as an induced subgraph, which again means that G contains an induced subgraph isomorphic to $C(p,1)$. In addition, μ is not an eigenvalue of the latter induced subgraph, while its multiplicity in G is computed in terms of parameters as $(\mu^2-2)(\mu^2-\mu-1)$ (cf. [16, p. 74]). The last expression is equal to $n - |V(C(p,1))|$, and the result follows. $\square$

Some sets of parameters (4.9) obtained for small $-\mu$'s are given in Table 1.

For $\mu=-2$, we have that two subsets of $V(C(5,1))$ with four vertices each (the isolated vertex and three other from K_5) are compatible if they have two or three common vertices. In the former case the corresponding vertices of X are non-adjacent, while in the latter they are adjacent. Observing that any pair of such distinct subsets has two or three common vertices, we conclude that all of them are compatible and give rise to a unique maximal extension. This is the strongly regular graph commonly known as the complement of the Clebsch graph. According to [2], there is a unique strongly regular graph with parameters as in the second row of Table 1 and there is no strongly regular graph with parameters as in the third row (see also [7, 9]). At present, it is not known whether strongly regular graphs with data as in the remaining rows exist or not.

Suppose for a moment that there exists a strongly regular graph, say G, with parameters $(324, 266, 220, 210)$ (as in the third row). By Theorem 4.3, G contains $C(57,1)$ as a star complement. If so, then the corresponding star set X induces a strongly regular graph, say G_X, with parameters $(266, 220, 183, 176)$. Now, does the non-existence of G implies the non-existence of G_X? Unfortunately, the answer is negative.

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Namely, by identifying the vertices of G_X with points and the vertices of K_{57} with blocks, we arrive at a partially balanced incomplete block design with two associate classes (for more details on such designs, see [16, Subsection 3.8.2]), say P, containing 266 points arranged into 57 blocks in such a way that every point appears in 45 blocks and every block has constant size equal to 210. In addition, considering the edges between G_X and K_{57}, we conclude that every two points determined by the adjacent vertices (such points are called the first associates) occur together in 36 blocks, while every two points determined by the non-adjacent vertices (the second associates) occur together in 33 blocks. Now, the non-existence of G implies that at least one of G_X or P does not exist. In addition, if P does exist then its 266×57 point-block incidence matrix N satisfies $NN^T = 45I + 36A + 33(J - I - A)$, where A is the adjacency matrix that determines G_X. In other words, the existence of P implies the existence of G_X. Altogether, only what we can conclude on the basis of the non-existence of G is summarized in the following corollary.

Corollary 4.4. *There is no partial incomplete block design described in the above discussion.*

And so, the (non-)existence of a strongly regular graph with parameters $(266, 220, 183, 176)$ remains an open problem.

5. Case $q \geq 2$. The neighbourhoods of star set vertices are described in the following theorem.

Theorem 5.1. *Let the vertices u_1 and u_2 be compatible for a star complement $C(p,q)$ and a negative non-main eigenvalue μ ($\mu \neq -1$). If, for $i \in \{1,2\}$, u_i is adjacent to s_i vertices of K_p and t_i vertices of qK_1, and the number of common neighbours in K_p (resp., qK_1) is c_p (resp., c_q), then the following equalities hold:*

$$p = -\frac{(\mu+1)(\mu^3 - 2\mu t_1 - t_1^2)}{t_1(\mu+t_1)}, \quad s_1 = s_2 = \mu + 1 - \frac{\mu^2(\mu+1)}{t_1}, \quad t_2 = t_1,$$

$$c_p = -\frac{(\mu+1)\left(\mu^2(\mu+t_1) - t_1(t_1 - c_q)\right)}{\mu t_1} \quad \textit{for} \ \ u_1 \sim u_2,$$

$$c_p = -\frac{(\mu+1)\left(\mu(\mu^2+\mu t_1 - t_1) - t_1(t_1 - c_q)\right)}{\mu t_1} \quad \textit{for} \ \ u_1 \not\sim u_2.$$

Proof. Solving the system that arise from the conditions (2.4)–(2.6), we arrive at the solution given in the theorem and also at another solution giving the unchanged expressions for p and s_1, different expressions for s_2 and c_p (which are not important for this proof) and also

$$t_2 = -\frac{\mu^2(\mu+t_1)}{\mu^2 - t_1}. \tag{5.10}$$

By Theorem 3.1, both t_1 and t_2 are non-zero. In what follows, we prove that t_2 of (5.10) cannot be positive. Namely, considering s_1 (given in the theorem) we get $t_1 < \mu^2$. According to this, if t_2 is positive then there also must be $t_1 < -\mu$. Rewriting the equality (5.10) as

$$-\mu^2(\mu+t_1) = t_2(\mu^2 - t_1),$$

we conclude that $t_2 < -\mu$ must hold (otherwise, the right-hand side is larger than the left). The equality (5.10) also gives

$$\mu^2(\mu + t_1 + t_2) = t_1 t_2.$$

Since the left-hand side must be positive, we have $t_1 + t_2 > -\mu$, and then this side is larger than μ^2. On the contrary, since $\max\{t_1, t_2\} < -\mu$, the right-hand side is less than μ^2, and we are done. ∎

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Under the notation from the last result, strongly regular extensions that may appear in the case $t_1 = t_2 = c_q = 1$ are considered in the previous section. In addition, it is not difficult to see that the case $t_1 = t_2 = 1, c_q = 0$ cannot occur (c_p cannot be an integer). In what follows, we show that there is no strongly regular extension for $t_1 \geq 2$. We start with a lemma.

LEMMA 5.2. *The parameter c_p of Theorem 5.1 cannot be an integer, unless $t_1 = c_q$.*

Proof. Assuming that c_p (for $u_1 \sim u_2$) is an integer, we get

$$-(\mu+1)\left(\mu^2(\mu+t_1) - t_1(t_1-c_q)\right) \equiv 0 \mod \mu t_1.$$

Since it holds $-(\mu+1)\mu^3 \equiv 0 \mod \mu t_1$ (this is because s_1 is an integer) and also $-(\mu+1)\mu^2 t_1 \equiv 0 \mod \mu t_1$, we get that there must be $-(\mu+1)t_1(t_1-c_q) \equiv 0 \mod \mu t_1$, i.e., $(t_1-c_q) \equiv 0 \mod \mu$. Since $c_q \leq t_1$ and $t_1 < -\mu$ (the last is because p is also an integer), we conclude that the last congruence is possible only if $t_1 = c_q$. The other possibility for c_p (for $u_1 \not\sim u_2$) is considered in a very similar way. $\square$

The announced result reads as follows.

THEOREM 5.3. *There is no strongly regular extension of $C(p,q)$, for $p, q \geq 2$.*

Proof. If G is a strongly regular extension of $C(p,q)$, $p, q \geq 2$, then by Theorem 5.1 and the previous lemma, we have $q = t_1 = t_2 = c_q$. This, in particular, means that every vertex of qK_1 is adjacent to all vertices of the star set X, which implies that the number of common neighbours of every two non-adjacent vertices of G is equal to its degree. Bearing in mind the existence of K_p, we conclude that the last is impossible, and the result follows. $\square$

Regular (but not strongly regular) extensions of the above star complements may exist for specified values of p, q and μ. Some of these values are given in the Appendix.

6. Concluding remarks. Observe first that every regular graph G containing $C(p,q)$ as a star complement for some eigenvalue admits an equitable partition of its vertices into the sets $V(K_p), V(qK_1)$ and X. In particular, the same holds for $q = 0$ where we deal with equitable partitions defined in [3]; Equivalently $V(K_p)$ and X are regular sets in sense introduced in the same paper. Equitable partitions were introduced by Sachs 1960's and then extensively studied in frameworks of graph spectra, automorphism groups of graphs, walk partitions, coloration or distance-regular graphs (for details and further references, see [8]).

Inspecting the structure of strongly regular graphs (containing various cliques and co-cliques), at the first sight one may expect many families of those with star complements described in the title of this paper. The fact is that they occur only in special cases considered in Sections 3 and 4.

The main purpose of this paper was to describe considered star complements and adjacencies between them and the corresponding star sets on a basis of theoretical reasoning. The next natural step is a determination of possible (maximal) extensions. This step requires a computer search which extends beyond our considerations.

7. Appendix. If $C(p,q)$ is a star complement for a negative non-main eigenvalue μ ($\mu \neq -1$) and a related star set contains at least two compatible vertices, then the corresponding parameters are expressed in Theorem 5.1. By Lemma 5.2, all compatible vertices share the same neighbourhood in the co-clique qK_1, and therefore we may ignore the remaining isolated vertices (by saying that $q = t_1 = t_2 = c_q$). Under this assumption, by Lemma 3.2, there is a finite number of pairs (μ, p) such that $C(p,q)$ is a star complement for μ,

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Table 2
The parameters of Theorem 5.1 (for a negative integer $\mu \geq -10$ and $q = t_1$).

μ	p	s_1, s_2	q, t_1, t_2, c_q	$c_p \sim$	$c_p \nsim$
-3	19	7	2	3	1
-4	39	21	2	12	9
-4	49	13	3	4	1
-5	101	21	4	5	1
-6	105	55	3	30	25
-6	115	40	4	15	10
-6	181	31	5	6	1
-7	155	92	3	56	50
-7	295	43	6	7	1
-8	203	105	4	56	49
-8	449	57	7	8	1
-9	304	208	3	144	136
-9	292	100	6	36	28
-9	649	73	8	9	1
-10	351	216	4	135	126
-10	333	171	5	90	81
-10	901	91	9	10	1

Table 3
The parameters of Theorem 5.1 which correspond to regular graphs (for a negative integer $\mu \geq -1000$).

μ	p	s_1, s_2	q, t_1, t_2, c_q	$c_p \sim$	$c_p \nsim$	r
-10	351	216	4	135	126	910
-33	5248	3840	9	2816	2784	19 557
-55	19 845	16 281	10	13 365	13 311	110 495
-65	17 408	10 752	25	6656	6592	45 526
-76	34 125	27 000	16	21 375	21 300	163 436
-99	55 909	43 561	22	33 957	33 859	253 139
-145	146 016	120 960	25	100 224	100 080	850 915
-246	481 915	411 600	36	351 575	351 330	3 302 874
-260	273 911	156 066	112	88 985	88 726	636 658
-385	1 330 176	1 161 216	49	1 013 760	1 013 376	10 472 105
-442	824 229	509 355	169	314 874	314 433	2 157 538
-445	1 408 960	1 171 864	75	974 728	974 284	8 372 840
-451	1 032 750	756 000	121	553 500	553 050	3 853 917
-568	3 219 993	2 857 680	64	2 536 191	2 535 624	28 617 112
-589	3 960 964	3 578 176	57	3 232 432	3 231 844	40 986 739
-801	7 048 000	6 336 000	81	5 696 000	5 695 200	69 767 271

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whenever $q \geq 2$. For such a q and a negative integer $\mu \geq -10$, we compute the parameters from Theorem 5.1. Since all the remaining parameters are expressed in terms of μ and t_1 and we have $q = t_1 < -\mu$ (see this detail in the proof of Lemma 5.2), the computation is realized easily, and the data is summarized in Table 2.

If we consider a regular extension then its degree satisfies $r = |X|$, and then we also have $\frac{s_1 r}{p} + p - 1 = p - 1$, giving $r = \frac{p-1}{p-s_1}$ which must be an integer. Computing r and all the previous parameters for a negative integer $\mu \geq -1000$, we arrive at the data presented in Table 3. The total number of vertices in a regular graph with data as in this table is $p + q + r$.

REFERENCES

[1] F.K. Bell and S.K. Simić. On graphs whose star complement for -2 is a path or a cycle. *Linear Algebra Appl.*, 347:249–265, 2004.

[2] A.E. Brouwer. Strongly regular graphs, https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html.

[3] D.M. Cardoso and P. Rama. Equitable bipartitions of graphs and related results. *J. Math. Sci.*, 120:869–880, 2004.

[4] D. Cvetković, M. Lepović, P. Rowlinson, and S. Simić. A database of star complements of graphs. *Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat.*, 9:103–112, 1998.

[5] D. Cvetković, P. Rowlinson, and S.K. Simić. *An Introduction to the Theory of Graph Spectra.* Cambridge University Press, Cambridge, 2010.

[6] D. Cvetković, P. Rowlinson, and S.K. Simić. *Spectral Generalizations of Line Graphs: On Graphs with Least Eigenvalue* -2. Cambridge University Press, Cambridge, 2004.

[7] A.L. Gavrilyuk and A.A. Makhnev. On Krein graphs without triangles. *Doklady Math.*, 72:591–594, 2005.

[8] C.D. Godsil. *Algebraic Combinatorics.* Chapman & Hall, New York, 1993.

[9] P. Kaski and P.R.J. Östergard. There are exactly five biplanes with $k = 11$. *J. Combin. Designs*, 16:117–127, 2007.

[10] P. Rowlinson. On induced matchings as star complements in regular graphs. *J. Math. Sci.*, 182:159–163, 2012.

[11] P. Rowlinson. Regular star complements in strongly regular graphs. *Linear Algebra Appl.*, 436:1482–1488, 2012.

[12] P. Rowlinson. Star complements in finite graphs: A survey. *Rendiconti Sem. Mat. Messina*, 8:145–162, 2002.

[13] P. Rowlinson. Star partitions and regularity in graphs. *Linear Algebra Appl.*, 226-228:247–265, 1995.

[14] P. Rowlinson and B. Tayfeh-Rezaie. Star complements in regular graphs: Old and new results. *Linear Algebra Appl.*, 432:2230–2242, 2010.

[15] Z. Stanić. On graphs whose second largest eigenvalue equals 1 – the star complement technique. *Linear Algebra Appl.*, 420:700–710, 2007.

[16] Z. Stanić. *Regular Graphs. A Spectral Approach.* De Gruyter, Berlin, 2017.

[17] Z. Stanić. Some star complements for the second largest eigenvalue of a graph. *Ars Math. Contemp.*, 1:126–136, 2008.