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IN-SPHERE PROPERTY AND REVERSE INEQUALITIES FOR MATRIX MEANS*

TRUNG-HOA DINH†, TIN-YAU TAM‡, AND BICH-KHUE T. VO§

Abstract. The in-sphere property for matrix means is studied. It is proved that the matrix power mean satisfies in-sphere property with respect to the Hilbert-Schmidt norm. A new characterization of the matrix arithmetic mean is provided. Some reverse AGM inequalities involving unitarily invariant norms and operator monotone functions are also obtained.

Key words. In-sphere property of matrix means, Matrix Heinz mean, Matrix power mean, Unitarily invariant norms.

AMS subject classifications. 46L30, 15A45.

1. Introduction. A *mean* M of non-negative numbers is a map from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ such that (see, for example, [1]):

1) $M(x, x) = x$ for every $x \in \mathbb{R}^+$;
2) $M(x, y) = M(y, x)$ for every $x, y \in \mathbb{R}^+$;
3) If $x < y$, then $x < M(x, y) < y$;
4) If $x < x_0$ and $y < y_0$, then $M(x, y) < M(x_0, y_0)$;
5) $M(x, y)$ is continuous;
6) $M(tx, ty) = tM(x, y)$ for $t, x, y \in \mathbb{R}^+$.

Some well-known examples are the arithmetic mean $\dfrac{a+b}{2}$, the geometric mean $\sqrt{ab}$, and the harmonic mean $\left(\dfrac{a^{-1}+b^{-1}}{2}\right)^{-1}$. Property 3) says that for $0 \le a \le b$,

$$\frac{a+b}{2} - M(a, b) \le \frac{b-a}{2}. \tag{1.1}$$

In other words, $M(a, b)$ lies inside the interval $[a, b]$ which is contained in the circle with the center at the arithmetic mean $\dfrac{a+b}{2}$ and the radius equal a half of the distance between a and b. We call this *the in-sphere property* of scalar means with respect to the Euclidian distance on $\mathbb{R}$. In particular, for $t \in [0, 1]$ and $p > 0$, the t-weighted geometric mean $M(a, b) = a^{1-t}b^t$ and the t-power mean (or binomial mean) $\mu_p(a, b, t) = ((1-t)a^p + tb^p)^{1/p}$ satisfy the in-sphere property (1.1).

Now, let us denote by $\mathbb{M}_n$ the algebra of all complex matrices of order n and by I_n the identity matrix in $\mathbb{M}_n$. Let $\mathbb{P}_n$ and $\mathbb{H}_n^+$ denote the sets of positive definite and positive semi-definite matrices of order n,

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†Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Viet Nam; Faculty of Civil Engineering, Ton Duc Thang University, Ho Chi Minh City, Viet Nam (dinhtrunghoa@tdtu.edu.vn).

‡Department of Mathematics and Statistics, University of Nevada, Reno, USA (ttam@unr.edu).

§Department of Fundamental Sciences, University of Finance and Marketing, Ho Chi Minh City, Viet Nam (bksphcm@gmail.com).

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respectively. For Hermitian matrices A and B, the notation $A \leq B$ means $B - A \geq 0$. This is the well-known Loewner order on Hermitian matrices.

One of the most important matrix generalizations of (1.1) is the famous Powers-Størmer inequality [2] which states that for any $A, B \in \mathbb{H}_n^+$ and for any $s \in [0,1]$,

$$\mathrm{Tr}\left(\frac{A+B}{2} - \frac{1}{2}|A-B|\right) \leq \mathrm{Tr}\,(A^s B^{1-s}),$$

where $|A| = (A^*A)^{1/2}$. The value $\mathrm{Tr}\,(A^s B^{1-s})$ is called *the non-logarithmic quantum Chernoff bound* in quantum hypothesis testing theory.

Another matrix generalization of (1.1) was studied by Dinh, Vo and Osaka [3]. They proved that for any $A, B \in \mathbb{P}_n$ such that $AB + BA \geq 0$ and for any operator Kubo-Ando mean σ [4],

$$\frac{A+B}{2} - \frac{1}{2}|A-B| \leq A\sigma B. \tag{1.2}$$

Then, Dinh showed in [5, Theorem 2.1] that for any $A, B \in \mathbb{P}_n$ (without the condition $AB + BA \geq 0$) and for any operator mean σ,

$$\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} \leq A\sigma B. \tag{1.3}$$

Notice that both (1.2) and (1.3) are matrix generalizations of (1.1).

The matrix power mean which was first studied by Bhagwat and Subramanian [6] is

$$\mu_p(A,B,t) = \left(tA^p + (1-t)B^p\right)^{1/p}, \quad A, B \in \mathbb{H}_n^+,\ p \in \mathbb{R}.$$

It is worth mentioning that $\mu_p(A,B,t)$ is a mean in the sense of Kubo-Ando if and only if $p = \pm 1$. The power means with $p > 1$ have many important applications in mathematical physics and in the theory of operator spaces, where they form the basis of certain generalizations of l_p norms to non-commutative vector-valued L_p spaces [7].

In this paper, we consider some matrix generalizations of (1.1) involving unitarily invariant norms. More precisely, we prove in Section 2 that the matrix power mean $\mu_p(A,B,t)$ satisfies the in-sphere property with respect to the Hilbert-Schmidt norm. We also obtain a new characterization of the arithmetic mean. In Section 3 we establish some reverse inequalities for the matrix Heinz mean with unitarily invariant norms.

2. In-sphere property for matrix means. Using the fact that for $p \in [1,2]$ the function $x^{1/p}$ is operator concave and the function $x^{2/p}$ is operator convex, we will prove that the matrix power mean $\mu_p(A,B,t)$ satisfies the in-sphere property with respect to the Hilbert-Schmidt norm $||\cdot||_2$.

Theorem 2.1. *Let $p \in [1,2]$ and $A, B \in \mathbb{H}_n^+$. Then for $t \in [0,1]$,*

$$\left\|\frac{A+B}{2} - \mu_p(A,B,t)\right\|_2 \leq \frac{1}{2}\|A-B\|_2. \tag{2.4}$$

Proof. Since $||A||_2 = (\mathrm{Tr}\,(A^2))^{1/2}$, (2.4) is equivalent to the following:

$$\mathrm{Tr}\,(\mu_p(A,B,t)^2) - \mathrm{Tr}\,((A+B)\mu_p(A,B,t)) \leq -\mathrm{Tr}\,(AB). \tag{2.5}$$

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It is obvious that (2.5) holds for $t = 0$ and $t = 1$. If we can show that the set of t satisfying (2.5) is a connected subset in $[0, 1]$, then it coincides with $[0, 1]$. Indeed, let (2.5) hold for $s, t \in (0, 1)$ and it suffices to show that (2.5) is also true for $(t + s)/2$. Notice that

$$\begin{aligned}\mu_p(A, B, (t+s)/2) &= \left(\frac{t+s}{2}A^p + (1 - \frac{t+s}{2})B^p\right)^{1/p} \\ &= \left(\frac{1}{2}(tA^p + (1-t)B^p) + \frac{1}{2}(sA^p + (1-s)B^p)\right)^{1/p} \\ &= \left(\frac{1}{2}\mu_p^p(A, B, t) + \frac{1}{2}\mu_p^p(A, B, s)\right)^{1/p}.\end{aligned}$$

For $p \in (1, 2)$, the function $x^{1/p}$ is operator concave, hence we have

$$\begin{aligned}\mu_p(A, B, (t+s)/2) &= \left(\frac{1}{2}\mu_p^p(A, B, t) + \frac{1}{2}\mu_p^p(A, B, s)\right)^{1/p} \\ &\geq \frac{1}{2}\mu_p(A, B, t) + \frac{1}{2}\mu_p(A, B, s).\end{aligned}$$

Consequently,

$$\operatorname{Tr}((A+B)\mu_p(A, B, (t+s)/2)) \geq \frac{1}{2}\operatorname{Tr}((A+B)\mu_p(A, B, t) + (A+B)\mu_p(A, B, s)). \tag{2.6}$$

On the other hand, for $p \in [1, 2]$ the function $x^{2/p}$ is operator convex. Then we have

$$\begin{aligned}\mu_p(A, B, (t+s)/2)^2 &= \left(\frac{1}{2}\mu_p^p(A, B, t) + \frac{1}{2}\mu_p^p(A, B, s)\right)^{2/p} \\ &\leq \frac{1}{2}\mu_p^2(A, B, t) + \frac{1}{2}\mu_p^2(A, B, s). \qquad (2.7)\end{aligned}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned}&\operatorname{Tr}(\mu_p(A, B, (t+s)/2)^2) - \operatorname{Tr}((A+B)\mu_p(A, B, (t+s)/2)) \\ &\leq \frac{1}{2}\operatorname{Tr}(\mu_p^2(A, B, t)) + \frac{1}{2}\operatorname{Tr}(\mu_p^2(A, B, s)) - \frac{1}{2}\operatorname{Tr}((A+B)\mu_p(A, B, t)) - \frac{1}{2}\operatorname{Tr}((A+B)\mu_p(A, B, s)) \\ &\leq -\operatorname{Tr}(AB).\end{aligned}$$

Therefore, (2.5) holds for $(s + t)/2$. ☐

Recall that a norm $|||\cdot|||$ on $\mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any unitary matrices U, V and any $A \in \mathbb{M}_n$. Ky Fan Dominance Theorem [9] asserts that given $A, B \in \mathbb{M}_n$, $s(A) \prec_w s(B)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$, where $s(A)$ denotes the vector of singular values of A.

In the following theorem, we establish a new characterization of the arithmetic mean. The proof is adapted from the proof of [5, Theorem 2.3]. For the convenience of readers, we provide a full proof.

THEOREM 2.2. *Let σ be any symmetric mean and $|||\cdot|||$ an arbitrary unitarily invariant norm on $\mathbb{M}_n$. If*

$$|||\frac{A+B}{2} - A\sigma B||| \leq \frac{1}{2}|||A - B||| \tag{2.8}$$

holds whenever $A, B \in \mathbb{P}_n$, then σ is the arithmetic mean.

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Proof. By [4, Theorem 4.4], the symmetric operator mean σ has the representation:

$$A\sigma B = \frac{\alpha}{2}(A+B) + \int_{(0,\infty)} \frac{\lambda+1}{\lambda}\{((\lambda A):B) + (A:(\lambda B))\}d\mu(\lambda), \quad A, B \in \mathbb{P}_n, \tag{2.9}$$

where $\lambda \geq 0$ and μ is a positive measure on $(0,\infty)$ with $\alpha + \mu((0,\infty)) = 1$ and $A : B = (A^{-1} + B^{-1})^{-1}$ is the parallel sum of A and B. Given two orthogonal projections P, Q acting on a Hilbert space H denote by $P \wedge Q$ their infimum which is the orthogonal projection on the subspace $P(H) \cap Q(H)$. If $P \wedge Q = 0$, then by [4, Theorem 3.7],

$$(\lambda P) : Q = P : (\lambda Q) = \frac{\lambda}{\lambda+1} P \wedge Q.$$

Consequently, from (2.9), we get

$$P\sigma Q = \frac{\alpha}{2}(P+Q). \tag{2.10}$$

For $\theta > 0$, let us consider the following orthogonal projections

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

It is easy to see that $P \wedge Q = 0$. By (2.10) and (2.8) we have

$$(1-\alpha)|||P+Q||| \leq |||P-Q|||,$$

or

$$(1-\alpha)|||P+Q||| \leq |\sin\theta| \cdot |||H|||, \tag{2.11}$$

where $H = \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$. Since it is true for all $\theta > 0$, as θ in (2.11) tends 0^+, we obtain $1-\alpha \leq 0$. Thus, $\alpha \geq 1$. This shows that $\mu = 0$ and σ is the arithmetic mean. ∎

Remark 2.3. Firstly, note that the matrix power mean is not symmetric. So, Theorem 2.1 is not covered by Theorem 2.2.

Secondly, notice that for any operator mean σ and for any $A, B \in \mathbb{H}_n^+$ with $AB + BA \geq 0$ (2.8) follows from (1.2). Therefore, (2.8) geometrically says that for any operator mean σ, the point $A\sigma B$ lies inside the sphere centered at $\frac{A+B}{2}$ and the radius equal to $\frac{1}{2}|||A-B|||$. However, if we fix some symmetric operator mean σ that is different from the arithmetic mean, then we can find matrices A, B not satisfying the condition $AB + BA \geq 0$ and $A\sigma B$ lies outside of the sphere with the center at $\frac{A+B}{2}$ and the radius $|||A-B|||/2$.

3. Reverse inequalities.

It is well-known that the Heinz mean $\frac{a^s b^{1-s} + a^{1-s}b^s}{2}$, $s \in [0,1]$, interpolates between the geometric mean $a^{1/2}b^{1/2}$ and the arithmetic mean $\frac{a+b}{2}$, and that [9] for any unitarily invariant norm $|||\cdot|||$, for any $A, B \in \mathbb{H}_n^+$, and for $s \in [0,1]$,

$$|||A^{1/2}B^{1/2}||| \leq |||\frac{A^sB^{1-s} + A^{1-s}B^s}{2}||| \leq |||\frac{A+B}{2}|||. \tag{3.12}$$

In this section, we will prove some inequalities reverse to (3.12).

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Observe that the following matrix generalization of (1.1)

$$\frac{A+B}{2} \le A^{s/2}B^{1-s}A^{s/2} + \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} \tag{3.13}$$

is false in general. Indeed, for $s = 1/2$, let us consider the following positive definite matrices

$$A = \begin{pmatrix} 0.699 & 1.1455 \\ 1.1455 & 4.9308 \end{pmatrix}, \quad B = \begin{pmatrix} 0.9249 & 0.7064 \\ 0.7064 & 0.5928 \end{pmatrix}.$$

Using Mathlab, one can see that the matrix

$$A^{1/4}B^{1/2}A^{1/4} + \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} - \frac{A+B}{2}$$

has eigenvalues 1.2956 and -0.0234. Therefore, (3.13) is false. However, the eigenvalues of $A^{1/4}B^{1/2}A^{1/4}$ are 0.1531 and 2.1184 and the eigenvalues of $\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}$ are 0.9665 and 0.0327. That means,

$$\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} \prec_w A^{1/4}B^{1/2}A^{1/4},$$

or, equivalently, for any unitarily invariant norm $|||\cdot|||$,

$$|||\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}||| \le |||A^{1/4}B^{1/2}A^{1/4}|||. \tag{3.14}$$

In the following theorem, we establish (3.14) for general $A, B \in \mathbb{P}_n$ in the context of operator monotone functions. The proof is adapted from [3, Proposition 3.1].

THEOREM 3.1. *Let f be an operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(0) = 0$. Let $g(t) = \frac{t}{f(t)}$ $(t \in (0,\infty))$ and $g(0) = 0$. Then for any $A, B \in \mathbb{P}_n$,*

$$\begin{aligned} |||\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}||| &\le |||f(A)^{1/2}g(B)f(A)^{1/2}||| \\ &\le |||f(A)g(B)|||, \end{aligned}$$

where $|||\cdot|||$ is an arbitrary unitarily invariant norm on $\mathbb{M}_n$.

Proof. Let us prove the first inequality. Suppose that $A \le B$. We have $A^{-1/2}BA^{-1/2} \ge I_n$. Therefore,

$$A + B - A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} = 2A.$$

Since g is operator monotone, we have $g(A) \le g(B)$. Then

$$f(A)^{1/2}g(A)f(A)^{1/2} \le f(A)^{1/2}g(B)f(A)^{1/2},$$

or

$$A \le f(A)^{1/2}g(B)f(A)^{1/2}.$$

Therefore,

$$|||A||| \le |||f(A)^{1/2}g(B)f(A)^{1/2}|||.$$

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Next, we consider the general case. For the matrix $I_n - A^{-1/2}BA^{-1/2}$, let $P = (I_n - A^{-1/2}BA^{-1/2})^+$ and $Q = (I_n - A^{-1/2}BA^{-1/2})^-$ be its positive and negative parts according to its spectral decomposition, respectively. Then we have

$$I_n - A^{-1/2}BA^{-1/2} = P - Q \quad \text{and} \quad |I_n - A^{-1/2}BA^{-1/2}| = P + Q.$$

Consequently,

$$A - B = A^{1/2}PA^{1/2} - A^{1/2}QA^{1/2} \quad \text{and} \quad A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2} = A^{1/2}PA^{1/2} + A^{1/2}QA^{1/2}.$$

It is obvious that $A - A^{1/2}PA^{1/2} \in \mathbb{H}_n^+$. Since $A - A^{1/2}PA^{1/2} = B - A^{1/2}QA^{1/2} \leq B$ from the above argument we have

$$A - A^{1/2}PA^{1/2} \leq f(A - A^{1/2}PA^{1/2})^{1/2}g(B)f(A - A^{1/2}PA^{1/2})^{1/2}.$$

Consequently,

$$|||A - A^{1/2}PA^{1/2}||| \leq |||f(A - A^{1/2}PA^{1/2})^{1/2}g(B)f(A - A^{1/2}PA^{1/2})^{1/2}|||.$$

On the other hand,

$$\begin{aligned} &|||f(A - A^{1/2}PA^{1/2})^{1/2}g(B)f(A - A^{1/2}PA^{1/2})^{1/2}||| \\ &\qquad = |||f(A - A^{1/2}PA^{1/2})^{1/2}g(B)^{1/2}g(B)^{1/2}f(A - A^{1/2}PA^{1/2})^{1/2}||| \\ &\qquad \leq |||g(B)^{1/2}f(A - A^{1/2}PA^{1/2})g(B)^{1/2}||| \\ &\qquad \leq |||g(B)^{1/2}f(A)g(B)^{1/2}||| \\ &\qquad \leq |||f(A)^{1/2}g(B)f(A)^{1/2}|||. \end{aligned}$$

Therefore,

$$\begin{aligned} |||A + B - A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}||| &= 2|||A - A^{1/2}PA^{1/2}||| \\ &\leq 2|||f(A)^{1/2}g(B)f(A)^{1/2}|||. \end{aligned}$$

The second inequality in Theorem 3.1 follows immediately from the Hiai-Ando log-majorization theorem [8]. ∎

COROLLARY 3.2. *Let $A, B \in \mathbb{P}_n$ and $s \in [0,1]$. Then we have*

$$|||\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}||| \leq |||A^{1/2}B^{1/2}|||.$$

We now use Corollary 3.2 to obtain a reverse inequality for the matrix Heinz mean.

THEOREM 3.3. *Let $A, B \in \mathbb{P}_n$ and $s \in [0,1]$. Then we have*

$$|||\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}||| \leq |||\frac{A^sB^{1-s} + A^{1-s}B^s}{2}|||. \tag{3.15}$$

Proof. Since $A, B \in \mathbb{P}_n$, the function $f(s) = |||A^sB^{1-s} + A^{1-s}B^s|||$ is continuous and convex on $[0,1]$, and twice differentiable on $(0,1)$ and $f'(1/2) = 0$ (see [9, p. 265]). Hence, $f(s)$ attains the global minimum on $[0,1]$ at $s = 1/2$. That means,

$$|||A^sB^{1-s} + A^{1-s}B^s||| \geq 2|||A^{1/2}B^{1/2}|||, \quad s \in [0,1].$$

By Corollary 3.2, we get (3.15). ∎

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Remark 3.4. By using similar arguments one can prove another reverse inequality for the matrix Heinz mean as follows: for any $A, B \in \mathbb{H}_n^+$ such that $AB + BA \geq 0$ and $s \in [0, 1]$,

$$|||\frac{A+B}{2} - \frac{1}{2}|A-B|||| \leq |||\frac{A^sB^{1-s} + A^{1-s}B^s}{2}|||.$$

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