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RESOLUTION OF CONJECTURES RELATED TO LIGHTS OUT! AND CARTESIAN PRODUCTS*

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Abstract. Lights Out! is a game played on a 5×5 grid of lights, or more generally on a graph. Pressing lights on the grid allows the player to turn off neighboring lights. The goal of the game is to start with a given initial configuration of lit lights and reach a state where all lights are out. Two conjectures posed in a recently published paper about Lights Out! on Cartesian products of graphs are resolved.

Key words. Matrix, Graph, Lights Out!, Sylvester equation.

AMS subject classifications. 05C50, 15A15, 15A03, 15B33.

1. Introduction. In this short note, we resolve two conjectures from [5] concerning Lights Out! on Cartesian products of graphs.

We begin by providing the setting for the Lights Out! problem for a simple graph G with vertices $1, 2, \dots, n$. Associated with each vertex of G is a light and a button. If the button at a vertex i is pressed, the lights of the neighbors of i toggle on or off. If a vertex is considered to be a neighbor of itself, this is called *closed neighborhood switching*. If not, this is called *open neighborhood switching*. Initially, some subset of vertices have their lights on and the complementary set has their lights off. This initial configuration can be represented by the $n \times 1$ vector $b = [b_i]$, where $b_i = 1$ if the light at vertex i is initially on, and $b_i = 0$ if the light at vertex i is initially off. The goal of the Lights Out! problem is to press a sequence of buttons so that at the end of the sequence all lights are off.

First consider the open neighborhood switching, and let A_G be the adjacency matrix of G, that is, $A_G = [a_{ij}]$ is the $n \times n$ matrix with $a_{ij} = 1$ if i is adjacent to j in G, and $a_{ij} = 0$ otherwise. Let $e_1, \dots, e_n$ denote the standard basis vectors. If we start with the configuration corresponding to b and press the button at vertex i, then the resulting configuration of lights corresponds to the vector $b + A_G e_i \bmod 2$. More generally, if we press the button at vertex j exactly x_j times ($j = 1, 2, \dots, n$), then the resulting configuration of lights corresponds to the vector $b + A_G x \bmod 2$ where $x = [x_i]$. Note that the ordering in which the buttons are pressed does not matter and pressing a button an even number of times is equivalent to not pressing the button at all. Thus, the Lights Out! problem for initial vector b, graph G and open neighborhoods has a solution if and only if the system $A_G x = b$ has a solution over $\mathbb{Z}_2$.

Let $r(A_G)$ and $\nu(A_G)$ denote the rank and nullity, respectively, of A_G viewed as a matrix over $\mathbb{Z}_2$. Basic facts about linear systems over $\mathbb{Z}_2$ translate into simple facts about the Lights Outs! problem on G. Namely, the number of initial conditions that can be made to have their lights off is $2^{r(A_G)}$, and for each such initial configuration there are exactly $2^{\nu(A_G)}$ sets of vertices that can be pressed to toggle all lights to the off configuration (see [7, 8]).

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Similar statements apply to the Lights Out! problem on G for closed neighborhoods; we simply replace A_G by $A_G + I$ throughout.

Let G be a graph with vertices $1, 2, \ldots, m$ and H be a graph with vertices $1, 2, \ldots, n$. The *Cartesian product* of G and H is the graph $G \square H$ with vertex set $\{(i, j) : 1 \le i \le m, 1 \le j \le n\}$ such that (i, j) and (k, ℓ) are adjacent if and only if $i = k$ and j is adjacent to ℓ in H, or i is adjacent to k in G and $j = \ell$. The recent paper [5] posed conjectures about $\nu(A_{G\square H})$ and $\nu(A_{G\square H} + I)$. We will prove these in Section 3.

2. Sylvester's equation. Let $\mathbb{F}$ be a field, and let A and B be square matrices over $\mathbb{F}$ of orders m and n respectively, and C be an $m \times n$ matrix over $\mathbb{F}$. The *Sylvester equation corresponding to A, B and C* is $AX - XB = C$ (e.g. see [2]).

Sylvester's equation arises naturally in the Lights Out! setting for Cartesian products of graphs. To see this, let G be a graph on m vertices with adjacency matrix A and let H be a graph on n vertices with adjacency matrix B. We can view the vertices of $G \square H$ as the positions in an $m \times n$ array; the entry in the (i, j) position corresponds to the vertex i of G and j of H. As before, the entry (i, j) is 1 if the light is on and 0 otherwise. Let E_{ij} be the $m \times n$ matrix with a 1 in position (i, j) and 0s elsewhere. Note that the (k, ℓ)-entry of AE_{ij} is 1 if and only if $\ell = j$ and k is adjacent to i in G. Similarly the (k, ℓ)-entry of $E_{ij}B$ equals 1 if and only if $k = i$ and ℓ is adjacent to j in H. Thus, the matrix $AE_{ij} + E_{ij}B$, which is $AE_{ij} - E_{ij}B$ in $\mathbb{Z}_2$, records the vertices of $G \square H$ that are changed due to pressing cell (i, j) using open neighborhood switching. More generally, the configuration C of lights can be turned off using open neighborhood switching if and only if the system $AX - XB = C$ has a solution over $\mathbb{Z}_2$. Equivalently, we may consider the matrix $I_n \otimes A - B^T \otimes I_m$. Over any field of characteristic 2, since B is symmetric, $I_n \otimes A - B^T \otimes I_m = I_n \otimes A + B \otimes I_m$ is the adjacency matrix of $G \square H$.

Sylvester's equation is well-studied, and in this section, we recall some of the known results that will be useful in the Lights Out! context. It is known that $I_n \otimes A - B^T \otimes I_m$ is a matrix representation of the operator on the vector space V of $m \times n$ matrices over $\mathbb{F}$ that sends $X \in V$ to $AX - XB$ (see [1, Section 57.4]). Hence the nullity of $I_n \otimes A - B^T \otimes I_m$ is the dimension of the subspace

$$W = \{X \in V : AX = XB\}.$$

Let $\widehat{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, $\widehat{V}$ be the vector space of $m \times n$ matrices over the algebraic closure $\widehat{\mathbb{F}}$ of $\mathbb{F}$, and

$$\widehat{W} = \{X \in \widehat{V} : AX = XB\}.$$

While W and $\widehat{W}$ are not necessarily equal, their dimensions are equal as these sets represent the solution space to a homogeneous system of equations with the same coefficient matrix but over the field $\mathbb{F}$ and its extension $\widehat{\mathbb{F}}$. Both [3, Chapter VIII, Section 3] and [9, Corollary to Theorem 2] prove the following formula for the nullity of $I_n \otimes A - B^T \otimes I_m$ over the complexes. Their proofs immediately carry over to any algebraically closed field.

Theorem 2.1. *Let A and B be square matrices over the field $\mathbb{F}$ with Jordan Canonical forms over $\widehat{\mathbb{F}}$*

$$\oplus_{i=1}^{k} J(\lambda_i, m_i) \quad \textit{and} \quad \oplus_{j=1}^{\ell} J(\mu_j, n_j),$$

respectively, where $J(\gamma, p)$ denotes the $p \times p$ Jordan matrix corresponding to the eigenvalue γ. Then

$$\nu(I_n \otimes A - B^T \otimes I_m) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \delta_{\lambda_i, \mu_j} \min(m_i, n_j),$$

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where $\delta_{\cdot,\cdot}$ is the Kronecker delta.

Theorem 2.1 implies that

$$\dim W = \sum_{i=1}^{k}\sum_{j=1}^{\ell} \delta_{\lambda_i,\mu_j} \min(m_i, n_j). \tag{2.1}$$

This formula requires determining the Jordan Canonical Form of both A and B over the algebraic closure of $\mathbb{F}$. In particular, this requires factoring polynomials. As we see in the next section, the formula is sufficiently strong to resolve the two conjectures in [5]. We end this section by deriving a similar formula that allows one to do all computations over $\mathbb{F}$ and avoid factorization. We believe this formulation will be more convenient for future research.

First we need to introduce some terminology and recall some classic results which can be found in Chapter 6 of [6]. The *characteristic matrix* of A is $xI - A$. It is known that there exist matrices U and V (whose entries are polynomials in x over $\mathbb{F}$ such that both $\det U$ and $\det V$ are nonzero elements of $\mathbb{F}$) such that UAV has the form

$$S = \mathrm{diag}(s_1(x), s_2(x), \dots, s_m(x)),$$

where each $s_i(x)$ is a monic polynomial and $s_\ell(x)$ divides $s_{\ell+1}(x)$ for $k = 1, \dots, m-1$. The matrix S is the *Smith Normal Form* of $xI - A$, is unique, and can be determined from $xI - A$ using only the Euclidiean Algorithm for finding gcd's of polynomials. The $s_i(x)$ are called the *invariant factors of $xI - A$*. Let $p_1(x), \dots, p_u(x)$ be the distinct irreducible factors of the characteristic polynomial of A, $c_A(x) = s_1(x)s_2(x)\cdots s_m(x)$. Then there exist nonnegative integers $e_i(j)$ such that

$$s_i(x) = p_1(x)^{e_1(i)} p_2(x)^{e_2(i)} \cdots p_u(x)^{e_u(i)} \quad \text{for } i = 1, \dots, m.$$

Now we do the same for the $n \times n$ matrix B. Take the Smith Normal Form of B to be

$$T = \mathrm{diag}(t_1(x), t_2(x), \dots, t_n(x)),$$

where each of the $t_i(x)$ is a monic polynomial and $t_\ell(x)$ divides $t_{\ell+1}(x)$ for $\ell = 1, \dots, n-1$. Let $q_1(x), \dots, q_v(x)$ be the distinct irreducible factors of $c_B(x) = t_1(x)t_2(x)\cdots t_n(x)$. Then there exist nonnegative integers $f_i(j)$ such that

$$t_i(x) = q_1(x)^{f_1(i)} q_2(x)^{f_2(i)} \cdots q_t(x)^{f_v(i)} \quad \text{for } i = 1, \dots, n.$$

If $\mathbb{F}$ is separable, which is the case when $\mathbb{F}$ is finite or of characteristic 0, and λ is an eigenvalue of A over $\widehat{F}$, then the sizes of the Jordan blocks in the Jordan Canonical Form of A corresponding to λ are the nonzero $e_j(i)$ $(i = 1, \dots, m)$, where j is the unique index such that λ is a root of $p_j(x)$. Similarly, if μ is an eigenvalue of B, then the sizes of the Jordan blocks in the Jordan Canonical Form of B corresponding to μ are the nonzero $f_j(i)$ $(i = 1, \dots, n)$ for which μ is a root of $q_j(x)$.

Corollary 2.2. *Let A and B be $m \times m$ and $n \times n$ matrices, respectively, over the separable field $\mathbb{F}$ with invariant factors $s_1(x), \dots, s_m(x)$, and $t_1(x), \dots, t_n(x)$, respectively. Then the nullity of $I \otimes A - B^T \otimes I$, is given by*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \deg(\gcd(s_i(x), t_j(x)).$$

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Proof. Let Λ_k be the set of roots of $p_k(x)$ over $\widehat{\mathbb{F}}$, Γ_ℓ be the set of roots of $q_\ell(x)$ over $\widehat{\mathbb{F}}$, $\Lambda = \cup_{k=1}^m \Lambda_k$ and $\Gamma = \cup_{\ell=1}^n \Gamma_\ell$. Then

$$\begin{aligned}
\sum_{i=1}^m \sum_{j=1}^n \deg(\gcd(s_i(x), t_j(x))) &= \sum_{i=1}^m \sum_{j=1}^n \deg(\gcd(\prod_{k=1}^u p_k(x)^{e_i(k)}, \prod_{\ell=1}^v q_\ell(x)^{f_j(\ell)}) \\
&= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^u \sum_{\ell=1}^v \deg(\gcd(p_k(x)^{e_i(k)}, q_\ell(x)^{f_j(\ell)}) \\
&= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^u \sum_{\ell=1}^v \delta_{p_k(x), q_\ell(x)} \deg(p_k(x)) \min(e_i(k), f_j(\ell)) \\
&= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^u \sum_{\ell=1}^v \sum_{\lambda \in \Lambda_k \cap \Gamma_\ell} \min(e_i(k), f_j(\ell)) \\
&= \sum_{k=1}^u \sum_{\ell=1}^v \sum_{\lambda \in \Lambda_k} \sum_{\mu \in \Gamma_\ell} \delta_{\lambda,\mu} \sum_{i=1}^m \sum_{j=1}^n \min(e_i(k), f_j(\ell)).
\end{aligned}$$

Noting that the $e_i(k)$ and $f_j(\ell)$ are the sizes of the Jordan blocks of A, respectively B, corresponding to the eigenvalues of λ and μ respectively, the result follows from Theorem 2.1. ∎

Corollary 2.3. *Let A be an $m \times m$ matrix over the separable field $\mathbb{F}$ with invariant factors $s_1(x), \ldots, s_m(x)$. Then the nullity of $I \otimes A - A^T \otimes I$ is given by*

$$\sum_{i=1}^m (2m - 2i + 1) \deg(s_i(x))$$

Proof. Since $s_i(x)$ divides $s_{i+1}(x)$ for $i = 1, \ldots, m-1$, $\gcd(s_i(x), s_j(x)) = s_i(x)$ when $i \leq j$. The result now follows from Corollary 2.2. ∎

If A is non-derogatory (that is, its minimal and characteristic polynomial are equal), then it has only one invariant factor not equal to 1, and hence, the formula in Corollary 2.2 simplifies to

$$\sum_{j=1}^n \deg(\gcd(c_A(x), t_j(x)).$$

In particular, we have the following result for Cartesian products involving a path P_m.

Corollary 2.4. *Let G be a graph and let $(s_1, \ldots, s_n)$ be the invariant factors of $xI - A_G$ over $\mathbb{Z}_2$. Then the nullity of the adjacency matrix of $P_m \square G$ equals*

$$\sum_{i=1}^n \deg(\gcd(c_{P_m}(x), s_i(x)).$$

Proof. The submatrix obtained from A_{P_m} by deleting its first row and last column has determinant 1. Hence the geometric multiplicity of each eigenvalue of A_{P_m} is 1. This implies that the Jordan Canonical Form of A_{P_m} has exactly one block for each eigenvalue, and hence, A_{P_m} is non-derogatory. The result now follows from Corollary 2.2. ∎

We conclude this section with a few simple examples.

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EXAMPLE 2.5. Let H be the Petersen graph. The Smith Normal Form of $xI - A_H$ over $\mathbb{Z}_2$ is

$$S = \operatorname{diag}(1,1,1,1,1,(x+1),(x+1)x,(x+1)x,(x+1)x,(x+1)^2x).$$

Hence, by Corollary 2.2, the nullity of $A_{H\square H}$ is

$$\sum_{i=1}^{10}\sum_{j=1}^{10}\deg(\gcd(s_i,s_j)) = 42.$$

EXAMPLE 2.6. Consider the star graph S_m (as illustrated below in Figure 1) on n vertices. The Jordan Canonical Form of the adjacency matrix of S_m has the form

$$\begin{array}{ll} \left(\oplus_{i=1}^{m-3} J(0,1)\right) \oplus J(0,3) & \text{if } m \text{ is odd, and} \\ \left(\oplus_{i=1}^{m-2} J(0,1)\right) \oplus J(1,2) & \text{if } m \text{ is even.} \end{array}$$

Using Corollary 2.2 to find the nullity of $I \otimes A_{S_m} - A_{S_n}^T \otimes I$ over $\mathbb{Z}_2$ for any n, m, we find

$$\nu(A_{S_m\square S_n}) = \begin{cases} (m-2)(n-2)+2 & \text{if } m \text{ and } n \text{ are both even or both odd, and} \\ (m-2)(n-2) & \text{otherwise.} \end{cases}$$

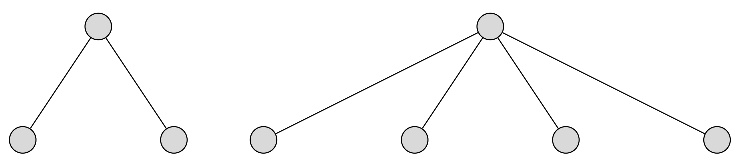

Figure 1: The star graphs S_3 and S_5.

EXAMPLE 2.7. In this example, we outline how to use our results and known results to determine the nullity of the adjacency matrix of the Cartesian product of a path and a star over a field of characteristic 2. The observations about the Jordan canonical forms of A_{S_n} in the previous example, imply the following:

1. If m is even, then the invariant factors of A_{S_m} are

$$s_1 = 1,\ s_2 = 1,\ s_3 = x,\ s_4 = x,\ \ldots,\ s_{m-1} = x, \text{ and } s_m = x(x+1)^2.$$

2. If m is odd and $m \geq 3$, then the invariant factors of A_{S_m} are

$$s_1 = 1,\ s_2 = 1,\ s_3 = x,\ s_4 = x,\ \ldots,\ s_{m-1} = x, \text{ and } s_m = x^3.$$

In particular, note that the irreducible factors of each invariant factor of A_{S_m} lie in the set $\{x, x+1\}$. Hence, in applying Corollary 2.4 to the Cartesian product $S_m\square P_n$, we need only know the algebraic multiplicities of the eigenvalues 0 and 1 of A_{P_n}.

The Fibonacci polynomials $f_n(x)$ are defined by $f_1(x) = 1$, $f_2(x) = x$ and $f_{n+1}(x) = xf_n(x) + f_{n-1}(x) \bmod 2$ for $n \geq 2$. It is can be verified that $f_{n+1}(x)$ is the characteristic polynomial of P_n. Using Lemma 2 and Theorem 7 of [4] on the factorization of Fibonacci polynomials, we can obtain the following. Here $n+1 = o2^k$, where o is an odd integer and k is a nonnegative integer.

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- If n is even, then 0 is not an eigenvalue of A_{P_n}.
- If $n \equiv 1 \pmod 4$, then $k = 1$ and the algebraic multiplicity of 0 as an eigenvalue A_{P_n} is $2^k - 1 = 1$.
- If $n \equiv 3 \pmod 4$, then the algebraic multiplicity of 0 as an eigenvalue of A_{P_n} is $2^k - 1 \geq 3$.
- if $n \equiv 2 \pmod 3$, then the algebraic multiplicity of 1 as an eigenvalue of A_{P_n} is $2^{k+1} \geq 2$.
- if $n \not\equiv 2 \pmod 3$, then 1 is not an eigenvalue of A_{P_n}.

The following chart gives the contribution to the nullity of $A_{S_m \square P_n}$ for the eigenvalue 0, assuming $m, n \geq 2$.

		n mod 4			
		0	1	2	3
m mod 2	0	0	$m-2$	0	$m-2$
	1	0	$m-2$	0	m

The following chart gives the contribution to the nullity of $A_{S_m \square P_n}$ for the eigenvalue 1, assuming $m, n \geq 2$.

		n mod 3		
		0	1	2
m mod 2	0	0	0	2
	1	0	0	0

For $m, n \geq 2$, we obtain the following formulas:

1. If m is even, then

$$\nu(A_{S_m \square P_n}) = \begin{cases} 0 & \text{if } n \equiv 0, 4 \bmod 6 \\ 2 & \text{if } n \equiv 2 \bmod 6 \\ m-2 & \text{if } n \equiv 1, 3 \bmod 6 \\ m & \text{if } n \equiv 5 \bmod 6. \end{cases}$$

2. If m is odd, then

$$\nu(A_{S_m \square P_n}) = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 2 \\ m-2 & \text{if } n \equiv 1 \bmod 4 \\ m & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

3. Proof of conjectures. Our proof depends on the following simple result about partitions, and Theorem 2.1. A *partition of the nonnegative integer* r is a tuple $\pi = (\pi_1, \ldots, \pi_k)$ of positive integers with $r = \pi_1 + \cdots + \pi_k$.

LEMMA 3.1. *Let* $\pi = (\pi_1, \ldots, \pi_k)$ *and* $\tau = (\tau_1, \ldots, \tau_\ell)$ *be partitions of* r *and* s *respectively. Then*

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} \min(\pi_i, \tau_j) \geq \min(r, s). \tag{3.2}$$

Proof. Without loss of generality we may assume that $\pi_1 \geq \tau_j$ for $j = 1, \ldots, k$. Then

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} \min(\pi_i, \tau_j) \geq \sum_{j=1}^{\ell} \min(\pi_1, \tau_j) = \sum_{j=1}^{\ell} \tau_j = s. \qquad \square$$

We note, but don't make use of, the fact that equality holds in (3.2) if and only if $k = 1$ and $s \leq r$, or $\ell = 1$ and $r \leq s$.

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Let λ be an eigenvalue of A, and let $S_\lambda = \{(j,k) : \lambda_j = \lambda = \mu_k\}$. Let $\alpha_A(\lambda)$ and $\alpha_B(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of A and B respectively. Note that the contribution to dim W in (2.1) corresponding to λ is given by

$$\sum_{(j,k)\in S_\lambda} \min(m_j, n_k).$$

By Lemma 3.1, this contribution is at least $\min(\alpha_A(\lambda), \beta_B(\lambda))$. This last quantity is the multiplicity of λ as a root of $\gcd(c_A(x), c_B(x))$. Hence we have proven the following result.

Corollary 3.2. *Let A and B be square matrices over the field $\mathbb{F}$ of order m and n respectively. Then the nullity of $I_n \otimes A - B^T \otimes I_m$ is at least the degree of the greatest common divisor of $c_A(x)$ and $c_B(x)$.*

The following corollaries prove Conjectures 4.1 and 4.2 of [5].

Corollary 3.3. *Let G and H be graphs with adjacency matrices A and B respectively, and $\mathbb{F}$ be a field of characteristic 2. Then the nullity of $A_{G\Box H}$ is at least the degree of the greatest common divisor of $c_A(x)$ and $c_B(x)$ over $\mathbb{F}$.*

Proof. Recall that the adjacency matrix of $G\Box H$ is $A \otimes I + I \otimes B$, which is $I \otimes B - A^T \otimes I$ since $\mathbb{F}$ has characteristic 2 and A is symmetric. The result now follows from Corollary 3.2. ∎

Corollary 3.4. *Let G and H be graphs with adjacency matrices A and B respectively, and $\mathbb{F}$ be a field of characteristic 2. Then the nullity of $A_{G\Box H} + I$ is at least the degree of the greatest common divisor of $c_A(x+1)$ and $c_B(x)$ over $\mathbb{F}$.*

Proof. Note that $A_{G\Box H} + I$ is $(A+I)\otimes I + I \otimes B$, which is $I\otimes B - (A+I)^T \otimes I$ since $\mathbb{F}$ has characteristic 2 and $A+I$ is symmetric. The result follows from Theorem 3.2 and the observation that $c_{A+I}(x) = c_A(x+1)$. ∎

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REFERENCES

[1] P. Benner. Control theory. In: Leslie Hogben (editor), *Handbook of Linear Algebra*, second edition, CRC Press, 2014.
[2] R. Bhatia and P. Rosenthal. How and why to solve the operator equation $AX - XB = Y$? *Bull. Lond. Math. Soc.*, 29:1–21, 1997.
[3] F.R. Gantmacher. *The Theory of Matrices*, Vol. 1. Chelsea Publishing, New York, 1959.
[4] J. Goldwasser, W. Klostermeyer, and H. Ware. Fibonacci polynomials and parity dominations in graphs. *Graphs Combin.*, 18:271–283, 2002.
[5] J. Goldwasser, T. Peters, and M. Young. Lights Out! on Cartesian products. *Electron. J. Linear Algebra*, 32:464–474, 2018.
[6] L. Hogben. Canonical forms. In: Leslie Hogben (editor), *Handbook of Linear Algebra*, second edition, CRC Press, 2014.
[7] K. Sutner. Linear cellular automata and the Garden-of-Eden. *Math. Intelligencer*, 11:49–53, 1989.
[8] R. Fleischer and J.A. Yu. Survey of the game "Lights Out!". In: A. Brodnik, A, Lopez-Ortiz, V. Raman, and A. Viola (editors), *Space-Efficient Data Structures, Streams, and Algorithms*, Lecture Notes in Comput. Sci., Springer, Berlin, 8066:176–198, 2013.
[9] V. Kučera. The matrix equation $AX + XB = C$. *SIAM J. Appl. Math.*, 26:15–25, 1974.