Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

TWO LINEAR PRESERVER PROBLEMS ON GRAPHS*

YANAN HU† AND ZHENHUA LYU†

Abstract. Let n, t, k be integers such that $3 \leq t, k \leq n$. Denote by $\mathcal{G}_n$ the set of graphs with vertex set $\{1, 2, \dots, n\}$. In this paper, the complete linear transformations on $\mathcal{G}_n$ mapping K_t-free graphs to K_t-free graphs are characterized. The complete linear transformations on $\mathcal{G}_n$ mapping C_k-free graphs to C_k-free graphs are also characterized when $n \geq 6$.

Key words. Cycle, Complete graph, Linear map, Vertex permutation.

AMS subject classifications. 05C69, 05C50.

1. Introduction and main results. Graphs in this paper are simple. We denote by $\mathcal{G}_n$ the set of graphs with vertex set $\langle n \rangle \equiv \{1, 2, \dots, n\}$. For a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. For graphs $G_1, G_2 \in \mathcal{G}_n$, $G_1 \cup G_2$ is the graph with vertex set $\langle n \rangle$ and edge set $E(G_1) \cup E(G_2)$. A *k-cycle*, written C_k, is a cycle of length k, which is also called a *triangle* when $k = 3$. A *complete graph* with t vertices, written K_t, is a graph whose vertices are pairwise adjacent. Note that K_3 coincides with C_3. Given a graph H, a graph G is called *H-free* if G does not contain H as a subgraph.

A map $\phi : \mathcal{G}_n \to \mathcal{G}_n$ is said to be *linear* if

$$\phi(G_1 \cup G_2) = \phi(G_1) \cup \phi(G_2) \quad \text{for all } G_1, G_2 \in \mathcal{G}_n.$$

Moreover, if $\phi(K_n) = K_n$, we call ϕ a *complete linear map.* A (complete) linear map is also called a *(complete) linear transformation.*

Linear preserver problems concern the characterization of linear maps on matrices or operators preserving special properties, which were initiated by Frobenius [5]. There are many directions and active research on preserver problems motivated by theory and applications; see [1, 2, 3, 4, 8, 9, 10, 11, 13] and their references. Hershkowitz [6] introduced linear maps on graphs and characterized the complete linear maps that map the set of all graphs containing no cycle of length greater than or equal to k into or onto itself. In [7], the authors determined the complete linear maps on $\mathcal{G}_n$ which preserve a given independence number. In this paper, we study the following problems.

Problem 1.1. *Given positive integers n and t, determine the complete linear maps $\phi : \mathcal{G}_n \to \mathcal{G}_n$ mapping all K_t-free graphs to K_t-free graphs.*

Problem 1.2. *Given positive integers n and t, determine the complete linear maps $\phi : \mathcal{G}_n \to \mathcal{G}_n$ mapping all C_t-free graphs to C_t-free graphs.*

If $t \in \{1, 2\}$ or $t > n$, both problems above are trivial. So we only discuss the cases $n \geq t \geq 3$. Note that Problem 1.1 is equivalent to Problem 1.2 when $t = 3$.

*Received by the editors on May 18, 2018. Accepted for publication on September 26, 2018. Handling Editor: Raphael Loewy. Corresponding Author: Zhenhua Lyu.

†College of Mathematics and Econometrics, Hunan University, Changsha 410082, P.R. China (huyanan530@163.com, lyuzhh@outlook.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

Given distinct $i, j \in \langle n \rangle$, denote by G_{ij} the graph in $\mathcal{G}_n$ with edge set $\{(i,j)\}$. A map $\phi : \mathcal{G}_n \to \mathcal{G}_n$ is said to be a *vertex permutation* if there is a permutation σ on $\langle n \rangle$ such that

$$\phi(G_{ij}) = G_{\sigma(i)\sigma(j)} \quad \text{for all } i, j \in \langle n \rangle \text{ with } i \neq j.$$

A complete linear map $\phi : \mathcal{G}_n \to \mathcal{G}_n$ is said to be an *edge permutation* if each $\phi(G_{ij})$ contains exactly one edge for all distinct $i, j \in \langle n \rangle$. It is easily seen that if ϕ is a vertex permutation, then ϕ is an edge permutation, but not vice versa.

We state our main results as follows.

THEOREM 1.3. *Let n, t be integers such that $3 \leq t \leq n$. Then a complete linear transformation $\phi : \mathcal{G}_n \to \mathcal{G}_n$ maps K_t-free graphs to K_t-free graphs if and only if one of the following holds.*

(i) $n > t$ with $(n,t) \neq (4,3)$ and ϕ is a vertex permutation.
(ii) $n = t$ and ϕ is an edge permutation.

THEOREM 1.4. *Let n, k be integers such that $n \geq 6$ and $3 \leq k \leq n$. Then a complete linear transformation $\phi : \mathcal{G}_n \to \mathcal{G}_n$ maps C_k-free graphs to C_k-free graphs if and only if ϕ is a vertex permutation.*

We denote by $(i_1, i_2, \ldots, i_k)$ the k-cycle $i_1 i_2 \cdots i_k i_1$ and denote by G_0 the graph in $\mathcal{G}_n$ whose edge set is empty. The following examples show that Theorem 1.3 and Theorem 1.4 do not hold for $(n,t) = (4,3)$ and $(n,k) = (5,4)$, respectively.

EXAMPLE 1.5. Let $f : \mathcal{G}_4 \to \mathcal{G}_4$ be the linear map such that

$$\begin{aligned} &f(G_{12}) = G_{12} \cup G_{34}, \quad f(G_{13}) = G_{13} \cup G_{24}, \quad f(G_{23}) = G_{23} \cup G_{14}, \\ &f(G_{14}) = f(G_{24}) = f(G_{34}) = G_0. \end{aligned}$$

Obviously, f is not a vertex permutation and it maps triangle-free graphs to triangle-free graphs.

EXAMPLE 1.6. Let $f : \mathcal{G}_5 \to \mathcal{G}_5$ be the linear map such that

$$\begin{aligned} &f(G_{12}) = G_{12}, \quad f(G_{13}) = G_{15}, \quad f(G_{14}) = G_{34}, \quad f(G_{15}) = G_{25}, \quad f(G_{23}) = G_{14}, \\ &f(G_{24}) = G_{35}, \quad f(G_{25}) = G_{24}, \quad f(G_{34}) = G_{23}, \quad f(G_{35}) = G_{45}, \quad f(G_{45}) = G_{13}. \end{aligned}$$

Since it maps $G_{12} \cup G_{14}$ to $G_{12} \cup G_{34}$, f is not a vertex permutation.

Note that there are 15 distinct 4-cycles on vertices from $\langle 5 \rangle$, which are

$$\begin{aligned} &H_1 = (1,2,3,4), \quad &&H_2 = (1,2,4,3), \quad &&H_3 = (1,3,2,4), \\ &H_4 = (1,2,3,5), &&H_5 = (1,2,5,3), &&H_6 = (1,3,2,5), \\ &H_7 = (1,2,4,5), &&H_8 = (1,2,5,4), &&H_9 = (1,4,2,5), \\ &H_{10} = (1,3,4,5), &&H_{11} = (1,3,5,4), &&H_{12} = (1,4,3,5), \\ &H_{13} = (2,3,4,5), &&H_{14} = (2,3,5,4), &&H_{15} = (2,5,3,4). \end{aligned}$$

Let $\widetilde{H}_i \in \mathcal{G}_5$ with $E(\widetilde{H}_i) = E(H_i)$ for $i \in \{1, \ldots, 15\}$. By the definition of f, we have

$$\begin{aligned} &f(\widetilde{H}_1) = \widetilde{H}_1, \quad &&f(\widetilde{H}_2) = \widetilde{H}_4, \quad &&f(\widetilde{H}_3) = \widetilde{H}_{12}, \quad &&f(\widetilde{H}_4) = \widetilde{H}_8, \quad &&f(\widetilde{H}_5) = \widetilde{H}_7, \\ &f(\widetilde{H}_6) = \widetilde{H}_9, &&f(\widetilde{H}_7) = \widetilde{H}_5, &&f(\widetilde{H}_8) = \widetilde{H}_2, &&f(\widetilde{H}_9) = \widetilde{H}_{15}, &&f(\widetilde{H}_{10}) = \widetilde{H}_6, \\ &f(\widetilde{H}_{11}) = \widetilde{H}_{10}, &&f(\widetilde{H}_{12}) = \widetilde{H}_{13}, &&f(\widetilde{H}_{13}) = \widetilde{H}_3, &&f(\widetilde{H}_{14}) = \widetilde{H}_{11}, &&f(\widetilde{H}_{15}) = \widetilde{H}_{14}. \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

One can see that f maps graphs containing 4-cycles to graphs containing 4-cycles. Since f is bijective, it maps C_4-free graphs to C_4-free graphs.

REMARK 1.7. The condition "complete" is natural here. Note that the linear transformations on $\mathcal{G}_n$ which map C_k-free graphs to C_k-free graphs do not have a uniform structure. For example, let $H \in \mathcal{G}_n$ be an arbitrary C_k-free graph and let $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be the map such that

$$\phi(G) = H \quad \text{for all } G \in \mathcal{G}_n.$$

Then ϕ is linear and it maps all C_k-free graphs to C_k-free graphs.

2. Proof of Theorem 1.3 . In this section, we present the proof of Theorem 1.3. In what follows, we always assume n and t are positive integers such that $3 \le t \le n$ and $(n, t) \neq (4, 3)$. Two distinct edges are said to be *adjacent* if they share a common vertex. Otherwise, they are said to be *nonadjacent*. We need the following lemmas.

LEMMA 2.1. *Let $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping K_t-free graphs to K_t-free graphs. Then ϕ is bijective and*

$$\mid E(\phi(G_{ij})) \mid = 1 \quad \textit{for all } i, j \in \langle n \rangle \textit{ with } i \neq j. \tag{2.1}$$

Proof. We distinguish two cases.

Case 1. $k > 3$. Firstly, we claim that $\phi(G_{ij})$ does not contain two adjacent edges for any distinct $i, j \in \langle n \rangle$. Otherwise, suppose there exist $i, j \in \langle n \rangle$ such that $\phi(G_{ij})$ contains two adjacent edges (s, w) and (s, u). Given $u_1, u_2, \ldots, u_{k-3} \in \langle n \rangle \backslash \{s, w, u\}$, let K be the complete graph on $\{s, w, u, u_1, \ldots, u_{k-3}\}$ and let

$$E(K) \setminus \{(s, w), (s, u)\} = \{e_1, \ldots, e_{t(t-1)/2-2}\}.$$

Since ϕ is complete, there exist $i_1, \ldots, i_{t(t-1)/2-2}, j_1, \ldots, j_{t(t-1)/2-2} \in \langle n \rangle$ such that

$$e_r \in E(\phi(G_{i_r j_r})) \quad \text{for all } r \in \langle t(t-1)/2 - 2 \rangle.$$

Let

$$G = G_{ij} \cup \left(\bigcup_{m=1}^{t(t-1)/2-2} G_{i_m j_m} \right).$$

Then G has at most $t(t-1)/2 - 1$ edges and it is K_t-free, but $\phi(G)$ contains a complete graph of order t, a contradiction. Therefore, $\phi(G_{ij})$ does not contain adjacent edges for all $i, j \in \langle n \rangle$ with $i \neq j$.

Applying similar arguments as above we can show $\phi(G_{ij})$ does not contain nonadjacent edges for all $i, j \in \langle n \rangle$ with $i \neq j$.

Therefore, we have

$$\mid E(\phi(G_{ij})) \mid \le 1 \quad \text{for all } i, j \in \langle n \rangle \text{ with } i \neq j. \tag{2.2}$$

Since ϕ is complete, we get (2.1) and ϕ is bijective.

Case 2. $k = 3$. Suppose $n = 3$. From the arguments in Case 1, we see that $\phi(G_{ij})$ does not contain two adjacent edges for any distinct $i, j \in \langle 3 \rangle$. Since any two edges are adjacent, (2.2) holds. Note that ϕ is complete. We get (2.1) and ϕ is bijective.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

Now assume $n \geq 5$. We prove the following claim.

Claim 1. $\phi(G_{ij}) \cup \phi(G_{sw})$ does not contain adjacent edges for any distinct $i, j, s, w \in \langle n \rangle$.

Otherwise, suppose there exist distinct $i, j, s, w \in \langle n \rangle$ such that $\phi(G_{ij}) \cup \phi(G_{sw})$ contains two adjacent edges (i_1, j_1) and (i_1, j_2). Since ϕ is complete, there exist $u, v \in \langle n \rangle$ such that $(j_1, j_2) \in E(\phi(G_{uv}))$. Now let $G = G_{ij} \cup G_{sw} \cup G_{uv}$. Then G is triangle-free and $\phi(G)$ contains a triangle, a contradiction. Therefore, we get Claim 1.

Suppose there exist distinct $i, j \in \langle n \rangle$ such that $\phi(G_{ij})$ contains two edges $(s, w), (u, v)$. Claim 1 implies that (s, w) and (u, v) are nonadjacent. Given any $p \in \langle n \rangle \backslash \{s, w, u, v\}$, let $G_1 \in \mathcal{G}_n$ with edge set $E(G_1) = \{(s, w), (w, p), (s, p)\}$. Since ϕ is complete, applying Claim 1, there exists $q \in \langle n \rangle$ such that

$$(w, p) \in E(\phi(G_{qj})) \quad \text{or} \quad (w, p) \in E(\phi(G_{qi})).$$

Without loss of generality, we assume

$$(w, p) \in E(\phi(G_{qj})). \tag{2.3}$$

Let $G_2 \in \mathcal{G}_n$ with $E(G_2) = \{(s, w), (w, u), (s, u)\}$.

We assert that there is a graph $G_3 \in \mathcal{G}_n$ with $E(G_3) = \{(i, j), (j, x), (i, x)\}$ such that G_2 is a subgraph of $\phi(G_3)$. From the definition of ϕ, there exists $H \in \mathcal{G}_n$ with 3 edges such that $\phi(H)$ contains G_2. Moreover, these edges is a triangle T. We assume $V(T) = \{a, b, c\}$ and $E(T) = \{(a, b), (a, c), (b, c)\}$. It is clear that each of $\{\phi(G_{ab}), \phi(G_{ac}), \phi(G_{bc})\}$ contains exactly one distinct edge of G_2. Without loss of generality, we let $(s, w) \in E(\phi(G_{ab}))$. If $(a, b) \neq (i, j)$, we have $G_4 = G_{bc} \cup G_{ac} \cup G_{ij}$. Obviously, $\phi(G_4)$ contains a triangle, which contradicts the definition of ϕ. Hence, we get the assertion.

Without loss of generality, we let $(s, u) \in E(\phi(G_{ix}))$ and

$$(u, w) \in E(\phi(G_{jx})). \tag{2.4}$$

Now consider $G_5 \in \mathcal{G}_n$ with $E(G_5) = \{(w, u)(u, p), (w, p)\}$. We assert that

$$(u, p) \in E(\phi(G_{qx})).$$

Otherwise, since ϕ is complete, $\phi(G_{ab})$ contains (u, p), where $(a, b) \neq (q, x)$. Let

$$G_6 = G_{ab} \cup G_{jx} \cup G_{qj},$$

which contains no triangle. By (2.3) and (2.4), $\phi(G_6)$ contains a triangle, a contradiction. It follows that

$$(u, p), (u, v) \in E(\phi(G_{ij} \cup G_{qx})),$$

which contradicts Claim 1.

Therefore, we have (2.2). Since ϕ is complete, we get (2.1) and ϕ is bijective. ∎

COROLLARY 2.2. *Let $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping K_t-free graphs to K_t-free graphs. Then $\phi(G)$ contains a copy of K_t whenever G contains a copy of K_t.*

Proof. Applying Lemma 2.1, ϕ is bijective. It follows that the numbers of K_t-free graphs in $\mathcal{G}_n$ and in $\phi(\mathcal{G}_n)$ are equal. Therefore, $\phi(G)$ contains a complete graph with k vertices whenever G contains a complete graph with k vertices. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

LEMMA 2.3. *Let $n > t$ and $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping K_t-free graphs to K_t-free graphs. Then $E(\phi(G_{ip}) \cup (\phi(G_{iq}))$ consists of two adjacent edges for any distinct $i, p, q \in \langle n\rangle$.*

Proof. As in the proof of [6, Proposition 3.14], we count the number of distinct complete graphs with t vertices. By Lemma 2.1, we have

$$\mid E(\phi(G_{ip}) \cup (\phi(G_{iq})) \mid= 2 \quad \text{for any distinct } i, p, q \in \langle n\rangle.$$

Suppose there exist distinct $i, p, q \in \langle n\rangle$ such that $E(\phi(G_{ip}) \cup (\phi(G_{iq}))$ consists of two nonadjacent edges (s, w) and (u, v). Then the number of all possible complete graphs with t vertices containing (i, p) and (i, q) is

$$f_1 \equiv \binom{n-3}{t-3} = \frac{(n-3)!}{(n-t)!(t-3)!}.$$

And the number of all possible complete graphs with t vertices containing (s, w) and (u, v) is

$$f_2 \equiv \begin{cases} \frac{(n-4)!}{(n-t)!(t-4)!}, & t > 3; \\ 0, & t = 3. \end{cases}$$

It is easily seen that $f_1 > f_2$. Applying Lemma 2.1 and Corollary 2.2, we have $f_1 = f_2$, a contradiction. Hence, $E(\phi(G_{ip}) \cup (\phi(G_{iq}))$ consists of two adjacent edges for any distinct $i, p, q \in \langle n\rangle$. ∎

LEMMA 2.4. *Let $n > k$ and $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping K_t-free graphs to K_t-free graphs. Then for every $i \in \langle n\rangle$, there exists $i' \in \langle n\rangle$ such that*

$$\phi(\bigcup_{j=1, j\neq i}^{n} G_{ij}) = \bigcup_{j=1, j\neq i'}^{n} G_{i'j}. \tag{2.5}$$

Proof. Given any $i \in \langle n\rangle$, choose $j_1, j_2 \in \langle n\rangle \setminus \{i\}$. By Lemma 2.3, there exist $i', p, q \in \langle n\rangle$ such that

$$E(\phi(G_{ij_1})) = G_{i'p} \quad \text{and} \quad E(\phi(G_{ij_2})) = G_{i'q}.$$

Suppose there exists $j \in \langle n\rangle \setminus \{i, j_1, j_2\}$ such that the only edge of $\phi(G_{ij})$ is not incident with i'. Then applying Lemma 2.3 again, we have $\phi(G_{ij}) = G_{pq}$. For any $x \in \langle n\rangle \setminus \{i, j, j_1, j_2\}$, the edge in $\phi(G_{ix})$ is not adjacent with any one of $(i', p), (i', q)$ and (p, q), which contradicts Lemma 2.3. Therefore, for every $i \in \langle n\rangle$, there exists $i' \in \langle n\rangle$ such that (2.5) holds. ∎

Proof of Theorem 1.3. The sufficiency part of this theorem is obvious. We prove the necessity part by distinguishing two cases.

Case 1. $n > t$ and $(n, t) \neq (4, 3)$. Applying Lemma 2.4, for any $i \in \langle n\rangle$ there exists $i' \in \langle n\rangle$ such that (2.5) holds. Denote by $\sigma : \langle n\rangle \to \langle n\rangle$ the map such that $\sigma(i) = i'$. Since ϕ is bijective, $i' \neq j'$ whenever $i \neq j$. Hence, σ is a permutation. Now by (2.5) we have

$$\phi(G_{ij}) = G_{i'j'} = G_{\sigma(i)\sigma(j)} \quad \text{for all distinct } i, j \in \langle n\rangle,$$

which means ϕ is a vertex permutation.

Case 2. $n = t$. Applying Lemma 2.1 we see that ϕ is an edge permutation.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

3. Proof of Theorem 1.4 . In this section, we always assume $4 \leq k \leq n$. The proof of Theorem 1.4 follows the same scheme as the proof of Theorem 1.3. We can obtain the following lemma and corollary by using similar arguments as in the proofs of Lemma 2.1 and Corollary 2.2, respectively.

LEMMA 3.1. *Let $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping C_k-free graphs to C_k-free graphs. Then ϕ is bijective and*

$$\mid E(\phi(G_{ij})) \mid = 1 \quad \textit{for all } i, j \in \langle n \rangle \textit{ with } i \neq j. \tag{3.6}$$

COROLLARY 3.2. *Let $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping C_k-free graphs to C_k-free graphs. Then $\phi(G)$ contains a k-cycle whenever G contains a k-cycle.*

LEMMA 3.3. *Let $n \geq 6$ and $\phi : \mathcal{G}_n \to \mathcal{G}_n$ be a complete linear transformation mapping C_k-free graphs to C_k-free graphs. Then $E(\phi(G_{ip}) \cup (\phi(G_{iq}))$ consists of two adjacent edges for all distinct $i, p, q \in \langle n \rangle$.*

Proof. As in the proof of Lemma 2.3 or [6, Proposition 3.14], we count the number of distinct k-cycles. By Lemma 3.1,

$$\mid E(\phi(G_{ip}) \cup (\phi(G_{iq})) \mid = 2 \quad \text{for any distinct } i, p, q \in \langle n \rangle.$$

Suppose there are distinct $i, p, q \in \langle n \rangle$ such that $E(\phi(G_{ip}) \cup (\phi(G_{iq}))$ consists of two nonadjacent edges (s, t) and (u, v). Then the number of all possible k-cycles through (i, p) and (i, q) is

$$f_3 \equiv \binom{n-3}{k-3}(k-3)! = \frac{(n-3)!}{(n-k)!}.$$

And the number of all possible k-cycles through (s, t) and (u, v) is

$$f_4 \equiv 2\binom{n-4}{k-4}(k-3)! = \frac{2(k-3)(n-4)!}{(n-k)!}.$$

Since ϕ is bijective, by Corollary 3.2, we have $f_3 = f_4$. Then $k = (n+3)/2$, which implies that n is odd and $n \geq 7$.

Now let $C = (i, p, j_1, \ldots, j_{k-3}, q)$ be a k-cycle and let $\widetilde{C} \in \mathcal{G}_n$ with $E(\widetilde{C}) = E(C)$. Then by Corollary 3.2, $\phi(\widetilde{C})$ contains a k-cycle with

$$\{(s, t), (u, v)\} \subset E(\phi(\widetilde{C})).$$

Since $n \geq 7$, we have

$$\mid \langle n \rangle \setminus \{i, p, q, j_1, \ldots, j_{k-3}\} \mid = (n-3)/2 \geq 2$$

and there are distinct $i_1, i_2 \in \langle n \rangle \setminus \{i, p, q, j_1, \ldots, j_{k-3}\}$. Replacing the role of i with i_1 and i_2 in $\widetilde{C}$, we get two graphs G_1 and G_2. Note that there are only two k-cycles containing the edges $E(\phi(\widetilde{C})) \setminus \{(s, t), (u, v)\}$. Hence, either $\phi(G_1)$ or $\phi(G_2)$ contains no C_k, which contradicts Corollary 3.2.

Therefore, $E(\phi(G_{ip} \cup G_{iq}))$ consists of two adjacent edges for any distinct $i, p, q \in \langle n \rangle$. $\square$

Applying the same arguments as in the proof of Lemma 2.4, we can prove that the statement of Lemma 2.4 holds when $n \geq 6$. Then using the same arguments as in the proof of Theorem 1.3, we obtain Theorem 1.4.

Acknowledgments. The authors thank Dr. Zejun Huang, Professor Raphael Loewy and the referee for helpful suggestions.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 602-608, November 2018.

REFERENCES

[1] A. Fošner, Z. Huang, C.-K. Li, Y.-T. Poon, and N.-S. Sze. Linear maps preserving the higher numerical ranges of tensor products of matrices. *Linear Multilinear Algebra*, 62:776–791, 2014.

[2] A. Fošner, Z. Huang, C.-K. Li, and N.-S. Sze. Linear preservers and quantum information science. *Linear Multilinear Algebra*, 61:1377–1390, 2013.

[3] A. Fošner, Z. Huang, C.-K. Li, and N.-S. Sze. Linear maps preserving Ky Fan norms and Schatten norms of tensor products of matrices. *SIAM J. Matrix Anal. Appl.*, 34:673–685, 2013.

[4] S. Friedland, C.-K. Li, Y.-T. Poon, and N.-S. Sze. The automorphism group of separable states in quantum information theory. *J. Math. Phys.*, 52:042203, doi:10.1063/1.3578015, 2011.

[5] G. Frobenius. Uber die Darstellung der endlichen Gruppen durch Linear Substitutionen. *Sitzungsber. Deutsch. Akad. Wiss. Berlin*, 994–1015, 1897.

[6] D. Hershkowitz. Linear mappings which preserve acyclicity properties of graphs and digraphs and applications to matrices. *Discrete Math.*, 64:157–190, 1987.

[7] Y. Hu and Z. Huang. Linear maps on graphs preserving a given independence number. Preprint, arXiv.org/abs/1804.11345, 2018.

[8] Z. Huang. Weakening the conditions in some classical theorems on linear preserver problems. *Linear Multilinear Algebra*, 64:923–928, 2016.

[9] Z. Huang, S. Shi, and N.-S. Sze. Linear rank preservers of tensor products of rank one matrices. *Linear Algebra Appl.*, 508:255–271, 2016.

[10] C.-K. Li and S. Pierce. Linear preserver problems. *Amer. Math. Monthly*, 108:591–605, 2001.

[11] L. Molnár. *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*. Lecture Notes in Mathematics, Vol. 1895, Springer-Verlag, Berlin, 2007.

[12] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[13] Z.X. Wan. *Geometry of Matrices*, In memory of Professor L. K. Hua (1910–1985). World Scientific Publishing Co., River Edge, NJ, 1996.