Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

DISCONTINUITY PROPAGATION IN DELAY DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract. The propagation of primary discontinuities in initial value problems for linear delay differential-algebraic equations (DDAEs) is discussed. Based on the (quasi-) Weierstraß form for regular matrix pencils, a complete characterization of the different propagation types is given and algebraic criteria in terms of the matrices are developed. The analysis, which is based on the method of steps, takes into account all possible inhomogeneities and history functions and thus serves as a worst-case scenario. Moreover, it reveals possible hidden delays in the DDAE and allows to study exponential stability of the DDAE based on the spectral abscissa. The new classification for DDAEs is compared to existing approaches in the literature and the impact of splicing conditions on the classification is studied.

Key words. Delay differential-algebraic equations, Differential-algebraic equations, Classification of DDAEs, Primary discontinuities, Splicing conditions, Exponential stability.

AMS subject classifications. 34A09, 34A12, 34K06, 65H10.

1. Introduction.

In this paper, we study *delay differential-algebraic equations* (DDAEs) of the form

$$E\dot{x}(t) = Ax(t) + Dx(t-\tau) + f(t) \tag{1.1a}$$

in the time interval $\mathbb{I} := [0, t_{\mathrm{f}}]$, where $E, A, D \in \mathbb{F}^{n,n}$ are matrices, $f : \mathbb{I} \to \mathbb{F}^n$ is the inhomogeneity, $\dot{x}$ denotes the time derivative $\frac{\mathrm{d}^+}{\mathrm{d}t}x$ of x from the right [7,18], and the field $\mathbb{F}$ is either the complex or the real numbers, i.e., $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Often, (1.1a) is formulated as an initial value problem (IVP), i.e., we equip (1.1a) with the initial condition

$$x(t) = \phi(t) \quad \text{for } t \in [-\tau, 0] \tag{1.1b}$$

with *history function* $\phi : [-\tau, 0] \to \mathbb{F}^n$. DDAEs of the form (1.1a) arise as linearization of the nonlinear implicit equation

$$F(t, x(t), \dot{x}(t), x(t-\tau)) = 0$$

around a nominal stationary solution. Typical applications are nonlinear optics, chemical reactor systems and delayed feedback control (see [11] and the references therein). Moreover, the DDAE (1.1a) may result from a realization of a transport-dominated phenomenon [21,22].

It is well-known, that the history function ϕ may not be linked smoothly to the solution x at $t = 0$. More precisely, we have

$$\lim_{t \nearrow 0} \dot{\phi}(t) \neq \lim_{t \searrow 0} \dot{x}(t) = \dot{x}(0) \tag{1.2}$$

in general (recall that $\dot{x} = \mathrm{d}^+/\mathrm{d}t$ denotes the derivative from the right). Due to the delay, this so-called *primary discontinuity* [2] is propagated to integer multiples of the delay τ. Thus, a rigorous analysis of the

*Received by the editors on March 12, 2018. Accepted for publication on October 16. Handling Editor: Michael Tsatsomeros.
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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

regularity of the solution is important for any kind of numerical integrator that is based on a Taylor series expansion of the solution. If $E = I_n$ is the $n \times n$ identity matrix, the DDAE (1.1a) is called *retarded delay differential equation* (rDDE) and it is well-known that in this case, the primary discontinuities are smoothed out, i.e., if

$$\lim_{t \nearrow k\tau} x^{(j-1)}(t) = x^{(j-1)}(k\tau) \quad \text{and} \quad \lim_{t \nearrow k\tau} x^{(j)}(t) \neq x^{(j)}(k\tau)$$

holds for some $j, k \in \mathbb{N}$, then we have

$$\lim_{t \nearrow (k+1)\tau} x^{(j)}(t) = x^{(j)}((k+1)\tau)$$

provided that f is smooth enough. This situation is specific to the case that the matrix E is nonsingular. If, in contrast, the matrix E is singular, the situation is completely different (see also [6]), as the following two examples suggest.

Example 1.1. Let $\mathbb{F} = \mathbb{R}$, $n = 1$, $E = 0$, $A = 1$, $D = 1$, $f \equiv 1$, $\tau = 1$, and $\phi(t) = t$. Then

$$x(t) = \begin{cases} k - 1 - t, & \text{if } k - 1 \leq t \leq k \text{ and } k \in \mathbb{N} \text{ odd}, \\ t + k, & \text{if } k - 1 \leq t \leq k \text{ and } k \in \mathbb{N} \text{ even} \end{cases}$$

solves the initial values problem (1.1). In particular, the solution x is continuous but $\dot{x}$ is discontinuous at every $t = k$, and thus, no smoothing occurs.

Example 1.2. Let $\mathbb{F} = \mathbb{R}$, $n = 2$, $f \equiv 0$, $\tau = 1$, and

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \phi(t) = \begin{bmatrix} \frac{1}{3}(t-1)^3 + (t-1)^2 - 1, \\ \frac{1}{3}t^3 + t^2 - 1 \end{bmatrix}.$$

Denoting the second component of x with x_2, the DDAE (1.1a) implies

$$x_2(t) = \begin{cases} t^2 - 1, & t \in [0, 1], \\ 2t - 2, & t \in [1, 2], \\ 2, & t \in [2, 3), \\ 0, & t \geq 3. \end{cases}$$

In particular, the solution becomes less smooth at multiples of the time delay and even discontinuous at $t = 3$.

The study of primary discontinuities of the scalar delay differential equation (DDE)

$$a_0\dot{x}(t) + a_1\dot{x}(t - \tau) + b_0 x(t) + b_1 x(t - \tau) = f(t) \tag{1.3}$$

is based on the the classification proposed in [3]: The DDE (1.3) is said to be of *retarded type* if $a_0 \neq 0$ and $a_1 = 0$, of *neutral type* if $a_0 \neq 0$ and $a_1 \neq 0$, and of *advanced type* if $a_0 = 0$ and $a_1 \neq 0$. Following this classification, we observe that the DDAE in Example 1.1 is of neutral type (if we differentiate the equation), while the second component in Example 1.2 satisfies a DDE of advanced type. The reason for this behavior is the so-called *index* of the differential-algebraic equation (DAE) that is encoded in the matrix pencil (E, A). The index is, roughly speaking, a measure for the smoothness requirements for the inhomogeneity f for a solution to exist. For a detailed analysis of the different index concepts, we refer to [19, 20].

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A publication of the International Linear Algebra Society
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The different classification approaches for DDAEs present in the literature, are either restricted to DDAEs in Hessenberg from with index less or equal three [1] or are based on the so-called *underlying DDE* [16]. In particular, neither of the approaches reflects the propagation of primary discontinuities and the effect of so-called *splicing conditions* [2] on the regularity of the solution. The main contributions of this work are the following:

1. We introduce a new classification for DDAE based on the propagation of primary discontinuities (Definition 3.3) and give a complete characterization of the propagation of discontinuities in terms of the matrices E, A, and D in (1.1a), cf. Theorem 3.9.
2. In Corollary 3.5, we show that multiple delays might be hidden in (1.1a) and provide a reformulation that is suitable for the stability analysis. Moreover, we show that the new classification provides a sufficient condition to analyze the stability of the DDAE in terms of the spectral abscissa (Corollary 3.13).
3. Example 4.2 illustrates that splicing conditions can have an impact on the solvability of DDAEs. Moreover, we characterize sufficient conditions for DDAEs up to index 3 to have a unique solution (cf. Theorem 4.3).
4. We show (Corollary 5.3) that in some sense the classification introduced [16] is an upper bound for the classification introduced in this paper.

Nomenclature.

$\mathbb{N}$		the set of natural numbers
$\mathbb{N}_0$	$:=$	$\{0\} \cup \mathbb{N}$
I_n		identity matrix of size $n \times n$
$\mathbb{F}$		either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$
$\mathbb{F}^{n,m}$		matrices of size $n \times m$ over the field $\mathbb{F}$
$\mathrm{GL}_n(\mathbb{F})$	$:=$	$\{A \in \mathbb{F}^{n,n} \mid A \text{ nonsingular}\}$
σ_τ		shift (backward) operator: $(\sigma_\tau x)(t) := x(t-\tau)$
$\dot{x}$	$:=$	$\frac{\mathrm{d}^+}{\mathrm{d}t} x$, the derivative of x from the right
$x^{(j)}$	$:=$	$\left(\frac{\mathrm{d}^+}{\mathrm{d}t}\right)^j x$
$x_i(t)$	$:=$	$x(t + (i-1)\tau)$ for $t \in [0, \tau]$
$\mathcal{C}(\mathbb{I}, \mathbb{F}^n)$		the vector space of all continuous functions from the real interval $\mathbb{I}$ into $\mathbb{F}^n$
$\mathcal{C}^k(\mathbb{I}, \mathbb{F}^n)$		the vector space of all k-times continuously differentiable functions from the real interval $\mathbb{I}$ into $\mathbb{F}^n$
$x^{(j)}(t^-)$	$:=$	$\lim_{s \nearrow t} x^{(j)}(s)$

2. Preliminaries and DAE theory. In this section, we review basic facts about DAE theory for linear time-invariant systems. For convenience, we omit the time argument whenever possible and use the shift (backward) operator σ_τ defined via

$$(\sigma_\tau x)(t) = x(t-\tau)$$

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A publication of the International Linear Algebra Society
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instead, such that such that (1.1a) is given by

$$E\dot{x} = Ax + D\sigma_\tau x + f. \tag{2.4}$$

Note that the formulation of the DDAE (2.4) is not restricted to one single delay, since multiple commensurate delays [13] may be rewritten as a single delay by introducing new variables [14]. A standard approach to solve the initial value problem (1.1) is via successive integration of (2.4) on the time intervals $[(i-1)\tau, i\tau)$, which is sometimes referred to as *method of steps* [16], see also [2,5]. More precisely, assume that M is the smallest integer such that $t_\mathrm{f} < M\tau$ and introduce for $i \in \mathcal{I} := \{1,\dots,M\}$ the functions

$$\begin{aligned} x_i &: [0,\tau] \to \mathbb{F}^n, & t &\mapsto (\sigma_{(1-i)\tau}x)(t) = x(t+(i-1)\tau),\\ f_i &: [0,\tau] \to \mathbb{F}^n, & t &\mapsto (\sigma_{(1-i)\tau}f)(t) = f(t+(i-1)\tau),\\ x_0 &: [0,\tau] \to \mathbb{F}^n, & t &\mapsto \phi(t-\tau). \end{aligned} \tag{2.5}$$

Then, we have to solve for each $i \in \{1,\dots,M\}$ the DAE

$$E\dot{x}_i = Ax_i + \tilde{f}_i, \quad t \in [0,\tau), \tag{2.6a}$$

$$x_i(0) = x_{i-1}(\tau^-), \tag{2.6b}$$

with $\tilde{f}_i := Dx_{i-1} + f_i$ and right continuation

$$x_{i-1}(\tau^-) := \lim_{t \nearrow \tau} x_{i-1}(t). \tag{2.7}$$

For the analysis of (2.6) we employ the following solution concept from [19]. A function $x_i \in \mathcal{C}^1([0,\tau],\mathbb{F}^n)$ is called a *solution* of (2.6a), if it satisfies (2.6a) pointwise. The function $x_i \in \mathcal{C}^1([0,\tau],\mathbb{F}^n)$ is called a *solution of the initial value problem* (2.6) if it is a solution of (2.6a) and satisfies (2.6b). An initial value $x_{i-1}(\tau^-)$ is called *consistent*, if the initial value problem (2.6) hast at least one solution.

The solvability of (2.6a) is closely connected to the matrix pencil (E,A) and the smoothness of the inhomogeneity $\tilde{f}_i$. If the inhomogeneity or some of its derivatives are discontinuous at certain points, we call this a *secondary discontinuity* [2]. For a numerical integrator, the secondary discontinuities need to be included in the time grid. However, to simplify our discussion, we assume that $\tilde{f}_i$ is arbitrarily smooth on $(0,\tau)$. A sufficient assumption to guarantee this is to assume the following.

Assumption 2.1. *The history function $\phi : [-\tau,0] \to \mathbb{F}^n$ and the inhomogeneity $f : \mathbb{I} \to \mathbb{F}^n$ are infinitely many times continuously differentiable.*

Another critical assumption that we make throughout the text is the following.

Assumption 2.2. *The matrix pencil (E,A) is* regular*, i.e., there exists $\lambda \in \mathbb{F}$ such that* $\det(\lambda E - A) \neq 0$.

Invoking Assumptions 2.1 and 2.2, the IVP (2.6) has a classical solution (cf. [19] and the discussion below) if the initial condition (2.6b) satisfies some algebraic equation. Hereby, x_i is called a *(classical) solution*, if x_i is continuously differentiable and satisfies (2.6a) pointwise. If (2.6) has a unique solution x_i for every $i \in \mathcal{I}$, we can construct a solution x of the IVP (1.1) by setting $x(t) = x_i(t-(i-1)\tau)$ for $t \in [(i-1)\tau, i\tau)$.

Remark 2.3. If (E,A) is not regular, it is still possible that the IVP (1.1) has a unique solution (in the sense of [16]). In this case, the DDAE is called *noncausal* and under some technical assumptions [16] provides algorithms to transform (2.4) such that the transformed pencil $(\tilde{E},\tilde{A})$ is regular. However, such a process adds additional restrictions on the history function [6,15].

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If (E, A) is regular, then we can characterize the smoothness requirements for the inhomogeneity $\tilde{f}_i$ in (2.6a) for a classical solution to exist. This characterization is based on the Weierstraß canonical form (cf. [12]). A more general form that is also valid for $\mathbb{F} = \mathbb{R}$ is the quasi-Weierstraß from, introduced in [4].

Theorem 2.4 (Quasi-Weierstraß form). *The matrix pencil (E, A) is regular if and only if there exist matrices $S, T \in \mathrm{GL}_n(\mathbb{F})$ such that*

$$SET = \begin{bmatrix} I_{n_\mathrm{d}} & 0 \\ 0 & N \end{bmatrix} \quad \textit{and} \quad SAT = \begin{bmatrix} J & 0 \\ 0 & I_{n_\mathrm{a}} \end{bmatrix}, \tag{2.8}$$

where $N \in \mathbb{F}^{n_\mathrm{a}, n_\mathrm{a}}$ is a nilpotent matrix with index of nilpotency ν and $J \in \mathbb{F}^{n_\mathrm{d}, n_\mathrm{d}}$. If $n_\mathrm{a} > 0$, we call ν the index of the pencil (E, A) and write $\mathrm{ind}(E, A) := \nu$. Otherwise we set $\mathrm{ind}(E, A) := 0$.

Applying the matrices S and T to the DAE (2.6a) implies a one-to-one correspondence between solutions of (2.6a) and solutions of

$$\dot{v}_i = J v_i + \tilde{g}_i, \tag{2.9a}$$

$$N \dot{w}_i = w_i + \tilde{h}_i, \tag{2.9b}$$

with

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} := T^{-1} x_i \quad \text{and} \quad \begin{bmatrix} \tilde{g}_i \\ \tilde{h}_i \end{bmatrix} := S \tilde{f}_i.$$

While (2.9a) is a standard ordinary differential equation (ODE) in v_i that can be solved with the Duhamel integral, the so called *fast subsystem* (2.9b) has the solution

$$w_i = -\sum_{k=0}^{\nu-1} N^k \tilde{h}_i^{(k)}, \tag{2.10}$$

and hence, the function $\tilde{h}_i$ must be ν times continuously differentiable for a classical solution to exist (cf. [19]), i.e., the right continuation (2.7) exists. In addition, a consistent initial value $w_i(0)$ must satisfy equation (2.10). Note that if $\tilde{g}_i$ and $\tilde{h}_i$ and the respective derivates exists at $t = \tau$, then the solution v_i and w_i, and thus, x_i can be extended to $t = \tau$, i.e., the right continuation (2.7) exists. Similar to [24], we define the matrices

$$\begin{aligned} A^{\mathrm{diff}} &:= T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, & A^{\mathrm{con}} &:= T \begin{bmatrix} I_{n_\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \\ C_0 &:= T \begin{bmatrix} I_{n_\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix} S, & C_k &:= -T \begin{bmatrix} 0 & 0 \\ 0 & N^{k-1} \end{bmatrix} S \end{aligned} \tag{2.11}$$

for $k = 1, \ldots, \mathrm{ind}(E, A)$. Note that the matrices in (2.11) do not depend on the specific choices of S and T [25].

Proposition 2.5. *Assume that the DAE (2.6a) satisfies Assumptions 2.1 and 2.2. Then any classical solution x_i of (2.6a) fullfills the so called* underlying ODE

$$\dot{x}_i = A^{\mathrm{diff}} x_i + \sum_{k=0}^{\mathrm{ind}(E,A)} C_k \tilde{f}_i^{(k)}. \tag{2.12}$$

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Conversely, let x_i be a classical solution of (2.12). *Then x_i is a solution of* (2.6a) *if and only if there exists $s \in [0, \tau]$ such that $x_i(s)$ satisfies*

$$x_i(s) = A^{\mathrm{con}} x_i(s) + \sum_{k=1}^{\mathrm{ind}(E,A)} C_k \tilde{f}_i^{(k-1)}(s). \tag{2.13}$$

Proof. Let x_i be a classical solution of (2.6a) and $S, T \in \mathrm{GL}_n(\mathbb{F})$ be matrices that satisfy (2.8) of the quasi-Weierstraß form and set $\nu := \mathrm{ind}(E, A)$. Differentiation of (2.10) yields

$$\begin{aligned}
\dot{x}_i &= T \begin{bmatrix} \dot{v}_i \\ \dot{w}_i \end{bmatrix} = T \begin{bmatrix} J v_i + \tilde{g}_i \\ -\sum_{k=0}^{\nu-1} N^k \tilde{h}_i^{(k+1)} \end{bmatrix} \\
&= T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} + T \begin{bmatrix} I_{n_{\mathrm{d}}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{g}_i \\ \tilde{h}_i \end{bmatrix} - \sum_{k=1}^{\nu} T \begin{bmatrix} 0 & 0 \\ 0 & N^{k-1} \end{bmatrix} \begin{bmatrix} \tilde{g}_i^{(k)} \\ \tilde{h}_i^{(k)} \end{bmatrix} \\
&= A^{\mathrm{diff}} x_i + \sum_{k=0}^{\nu} C_k \tilde{f}_i^{(k)}.
\end{aligned}$$

Conversely, let x_i be a classical solution of (2.12). Then there exists $x_i(0) \in \mathbb{F}^n$ such that

$$x_i(t) = \mathrm{e}^{A^{\mathrm{diff}} t} x_i(0) + \int_0^t \mathrm{e}^{A^{\mathrm{diff}}(t-s)} \sum_{k=0}^{\nu} C_k \tilde{f}_i^{(k)}(s) \mathrm{d}s. \tag{2.14}$$

Scaling (2.14) from the left by T^{-1}, we obtain

$$\begin{aligned}
v_i(t) &= \mathrm{e}^{Jt} v_i(0) + \int_0^t \mathrm{e}^{J(t-s)} \tilde{g}_i(s) \mathrm{d}s \quad \text{and} \\
w_i(t) &= w_i(0) - \sum_{k=1}^{\nu} N^{k-1} \int_0^t \tilde{h}_i^{(k)}(s) \mathrm{d}s = w_i(0) - \sum_{k=0}^{\nu-1} N^k \tilde{h}_i^{(k)}(t) + \sum_{k=0}^{\nu-1} N^k \tilde{h}_i^{(k)}(0).
\end{aligned}$$

The condition (2.13) implies the existence of $s \in [0, \tau]$ such that

$$\begin{bmatrix} v_i(s) \\ w_i(s) \end{bmatrix} = \begin{bmatrix} v_i(s) \\ -\sum_{k=0}^{\nu-1} N^k \tilde{h}_i^{(k)}(s). \end{bmatrix}$$

Together with (2.10) this implies that x_i is a solution of (2.6a). ∎

Setting $s = 0$ in the previous proposition yields the following requirement for an initial condition to be consistent.

Corollary 2.6. *Assume that the DAE* (2.6a) *satisfies Assumptions 2.1 and 2.2. Then $x_i(0)$ is consistent if and only if it satisfies the* consistency condition

$$x_i(0) = A^{\mathrm{con}} x_i(0) + \sum_{k=1}^{\mathrm{ind}(E,A)} C_k \tilde{f}_i^{(k-1)}(0). \tag{2.15}$$

In this case, the IVP (2.6) *has a unique solution $x_i \in \mathcal{C}^\infty([0, \tau], \mathbb{F}^n)$.*

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In order to reformulate (2.12) in terms of the delayed argument, i.e., by replacing $\tilde{f}_i = Dx_{i-1} + f_i$, we have to specify the solution concept for DDAEs. In contrast to DAEs, a classical solution concept is not reasonable for the DDAE (2.4), because the identity

$$\lim_{t \searrow 0} \dot{x}(t) = \dot{x}(0) = \dot{\phi}(0^-) := \lim_{t \nearrow 0} \dot{\phi}(t)$$

is in general not satisfied and this discontinuity in the first derivative at $t = 0$ may propagate over time (cf. [2] and Examples 1.1 and 1.2). Instead, we use the following solution concept.

Definition 2.7 (Solution concept). Assume that the DDAE (1.1a) satisfies Assumptions 2.1 and 2.2. We call $x \in \mathcal{C}(\mathbb{I}, \mathbb{F}^n)$ a *solution* of (1.1) if for all $i \in \mathcal{I}$ the restriction x_i of x as in (2.5) is a solution of (2.6). We call the history function ϕ *consistent* if the initial value problem (1.1) has at least one solution.

The definition and the previous discussion immediately yields the following relation between the DDAE IVP (1.1) and the sequence of DAE IVPs (2.6).

Proposition 2.8. *Let Assumptions 2.1 and 2.2 hold. If x is a solution of the IVP (1.1), then the restriction $x_i(t) = x(t + (i-1)\tau)$ is a solution of (2.6). Conversely, if the sequence (x_i) is a solution of (2.6), then*

$$x(t) = \begin{cases} x_i(t - (i-1)\tau), & \text{if } (i-1)\tau \leq t < i\tau \text{ for some } i \in \mathbb{N}, \\ \phi(t), & \text{otherwise} \end{cases}$$

is a solution of (1.1).

In order to reformulate (2.12) in terms of the delayed argument, we introduce the matrices $D_k := C_k D$ for $k = 0, \ldots, \operatorname{ind}(E, A)$. This yields the DDE

$$\dot{x} = A^{\mathrm{diff}} x + \sum_{k=0}^{\operatorname{ind}(E,A)} \left(D_k \sigma_\tau x^{(k)} + C_k f^{(k)} \right), \tag{2.16}$$

which we call the the *underlying DDE* for the DDAE (2.4).

From Corollary 2.6 and the discussion thereafter we immediately observe that a necessary condition for a history function ϕ to be consistent is that it satisfies the equation

$$\phi(0) = A^{\mathrm{con}} \phi(0) + \sum_{k=1}^{\operatorname{ind}(E,A)} \left(D_k \phi^{(k-1)}(-\tau) + C_k f^{(k-1)}(0) \right). \tag{2.17}$$

Unfortunately, as Example 1.2 suggests, this condition is not sufficient for consistency, which gives raise to the following definition.

Definition 2.9. Assume that the IVP (1.1) with history function $\phi : [-\tau, 0] \to \mathbb{F}^n$ satisfies Assumptions 2.1 and 2.2. Then ϕ is called *admissible* for the IVP (1.1) if $x_1(0) = \phi(0)$ is consistent for the DAE

$$E\dot{x}_1(t) = Ax_1(t) + D\phi(t - \tau) + f_1(t),$$

i.e., ϕ satisfies (2.17). Similarly, $x_0 : [0, \tau] \to \mathbb{F}^n$ is called *admissible* for the sequence of DAEs (2.6) if the DAE

$$\begin{aligned} E\dot{x}_1(t) &= Ax_1(t) + Dx_0(t) + f(t), \\ x_1(0) &= x_0(\tau) \end{aligned}$$

has a solution on $[0, \tau)$.

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For the analysis in the upcoming section, we introduce

$$\begin{bmatrix} D_{\mathrm{d}} \\ D_{\mathrm{a}} \end{bmatrix} := SD, \quad \begin{bmatrix} D_{\mathrm{d},1} & D_{\mathrm{d},2} \\ D_{\mathrm{a},1} & D_{\mathrm{a},2} \end{bmatrix} := SDT, \quad \begin{bmatrix} g \\ h \end{bmatrix} := Sf, \quad \text{and} \quad \begin{bmatrix} \psi \\ \eta \end{bmatrix} := T^{-1}\phi, \tag{2.18}$$

where $S, T \in \mathrm{GL}_n(\mathbb{F})$ are matrices that satisfy (2.8) from the quasi-Weierstraß form (Theorem 2.4) and we use the same block dimensions as in (2.8). Applying the matrices S, T to (2.4) yields

$$\dot{v} = Jv + D_{\mathrm{d},1}\sigma_\tau v + D_{\mathrm{d},2}\sigma_\tau w + g, \tag{2.19a}$$

$$N\dot{w} = w + D_{\mathrm{a},1}\sigma_\tau v + D_{\mathrm{a},2}\sigma_\tau w + h. \tag{2.19b}$$

3. Discontinuity propagation. In this section, we derive a classification for the DDAE (1.1a) in terms of the propagation of primary discontinuities of solutions of the IVP (1.1). Recall that for an admissible history function $\phi : [-\tau, 0] \to \mathbb{F}^n$, Assumptions 2.1 and 2.2 guarantee that there exists a number $M \in \mathbb{N}$ and a unique sequence $(x_i)_{i\in\{0,\ldots,M\}}$ that satisfies (2.6) (cf. Corollary 2.6). Hence, for any $i \in \{1, \ldots, M\}$, we can define the level ℓ_i of the primary discontinuity as

$$\ell_i := \min_{f\in\mathcal{C}^\infty(\mathbb{I},\mathbb{F}^n)} \; \min_{\substack{x_0\in\mathcal{C}^\infty([0,\tau],\mathbb{F}^n)\\ x_0 \text{ admissible}}} \; \max\left\{ \ell \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} x_j \text{ solves (2.6) for } j = 1, \ldots, i \text{ and} \\ x_i^{(\ell)}(0) = x_{i-1}^{(\ell)}(\tau^-) \end{array} \right\}. \tag{3.20}$$

If for some $j \in \mathbb{N}$ the initial condition $x_j(0) = x_{j-1}(\tau)$ is not consistent, and thus, no solution of (2.6) exists, we formally set $\ell_i := -\infty$ for all $i \geq j$. Note that this definition is independent of the specific choice of the inhomogeneity f and the history ϕ and thus serves as the worst-case scenario. To simplify the computation of the numbers ℓ_i we observe the following, which is a generalization of [18, Theorem 7.1]

Proposition 3.1. *Assume that the IVP (1.1) with admissible history function $\phi : [-\tau, 0] \to \mathbb{F}^n$ satisfies Assumptions 2.1 and 2.2. Then the solution x of (1.1) is continuously differentiable on $[-\tau, \tau)$ if and only if ϕ satisfies*

$$\dot{\phi}(0^-) = A^{\mathrm{diff}}\phi(0) + \sum_{k=0}^{\mathrm{ind}(E,A)} \left(D_k\phi^{(k)}(-\tau) + C_k f^{(k)}(0) \right). \tag{3.21}$$

The solution x of (1.1) is κ times continuously differentiable on $[-\tau, \tau)$ if and only if ϕ satisfies

$$\phi^{(p+1)}(0^-) = A^{\mathrm{diff}}\phi^{(p)}(0^-) + \sum_{k=0}^{\mathrm{ind}(E,A)} \left(D_k\phi^{(k+p)}(-\tau) + C_k f^{(k+p)}(0) \right) \tag{3.22}$$

for $p = 0, 1, \ldots, \kappa - 1$.

Proof. Since ϕ is admissible, the initial condition $x_1(0) = \phi(0)$ is consistent and following Corollary 2.6 the solution x exists on $[-\tau, \tau)$. Thus, it is sufficient to check the point $t = 0$. Using Proposition 2.5 we can consider (2.12) and thus obtain

$$\begin{aligned} \dot{x}_1(0) &= A^{\mathrm{diff}}x_1(0) + \sum_{k=0}^{\mathrm{ind}(E,A)} \left(D_k x_0^{(k)}(0) + C_k f_1^{(k)}(0) \right) \\ &= A^{\mathrm{diff}}\phi(0) + \sum_{k=0}^{\mathrm{ind}(E,A)} \left(D_k\phi^{(k)}(-\tau) + C_k f_1^{(k)}(0) \right), \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

and hence, x is continuously differentiable on $[-\tau,\tau)$ if and only if ϕ satisfies (3.21). For arbitrary $\kappa \in \mathbb{N}$ we invoke Proposition 2.5, which guarantees that the solution x exists on the interval $[0,\tau)$ and allows us to consider the underlying DDE (2.16) instead of the DDAE. Since the assumptions guarantee that x is sufficiently smooth on $[0,\tau)$ we can differentiate (2.16) $p \in \mathbb{N}$ times to obtain

$$\begin{aligned} x_1^{(p+1)}(0) &= A^{\mathrm{diff}} x_1^{(p)}(0) + \sum_{k=0}^{\mathrm{ind}(E,A)} \Big(D_k x_0^{(k+p)}(0) + C_k f_1^{(k+p)}(0) \Big) \\ &= A^{\mathrm{diff}} \phi^{(p)}(0^-) + \sum_{k=0}^{\mathrm{ind}(E,A)} \Big(D_k \phi^{(k+p)}(-\tau) + C_k f_1^{(k+p)}(0) \Big), \end{aligned}$$

which implies the result. □

Since we require $\phi \in \mathcal{C}^\infty([-\tau,0],\mathbb{F}^n)$ to be admissible we immediately obtain $\ell_1 \geq 0$. On the other hand assume that we have given the values $\phi(0)$ and $\phi^{(k)}(-\tau)$ for $k = 0,\ldots,\nu$ such that ϕ is admissible. Then we can always construct (via Hermite interpolation) ϕ in such a way that (3.21) is not satisfied, and hence, $\ell_1 \leq 0$, which yields $\ell_1 = 0$. Thus, the questions about propagation of discontinuities can be rephrased as whether there exists $k \in \mathbb{N}$ with $\ell_k > 0$ (i.e., the solution becomes smoother), or there exists $k \in \mathbb{N}$ with $\ell_k = -\infty$ (i.e., the solution becomes less smooth), or if $\ell_i = \ell_1$ for all $i \in \mathbb{N}$. Note that the smoothing may not start immediately (i.e., we cannot ask for $\ell_1 = 1$), as the following example suggests.

Example 3.2. Consider the DDAE given by $\mathbb{F} = \mathbb{R}$, $n = 2$, $f \equiv 0$, $\tau = 1$, and

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \phi(t) = \begin{bmatrix} t, \\ -1 \end{bmatrix}.$$

Since (E,A) is already in Weierstraß form, it is easy to see that the DDAE corresponds to the DDE

$$\dot{v}(t) = v(t-2\tau) \tag{3.23}$$

with coupled equation $w(t) = v(t-\tau)$. Straight forward calculations show that $\ell_1 \leq 0$ (using the specified history function ϕ) and $\ell_1 \geq 0$ implying $\ell_1 = 0$. On the other, (3.23) is a scalar delay equation and it is well-known, that the solution is continuously differentiable at $t = 2\tau$, thus we have $\ell_2 \geq 1$.

Definition 3.3 (Classification). Consider the DDAE (1.1a) on the time interval $\mathbb{I} = [0, M\tau]$, set $\mathcal{I} := \{1,\ldots,M\}$, and suppose that (1.1) satisfies Assumptions 2.1 and 2.2. We say that (1.1a) is of

- smoothing type if there exists $j \in \mathcal{I}$, $j > 1$ such that $\ell_j = 1$ and $\ell_i = 0$ for $i < j$,
- discontinuity invariant type if $\ell_i = 0$ for all $i \in \mathcal{I}$, and
- de-smoothing type if there exists $j \in \mathcal{I}$, $j > 1$ such that $\ell_j = -\infty$ and $\ell_i = 0$ for $i < j$.

In the following, we analyze in detail the DDAE (1.1a) and derive conditions for the matrices E, A, and D, from which the type can be determined. Before we analyze the general DDAE case we focus on the case of $\mathrm{ind}(E,A) \leq 1$, i.e., the system is a pure DDE or $N = 0$ in (2.9b). Note that this case includes DDEs of the form

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{D}\hat{x}(t-\tau) + \hat{B}\dot{\hat{x}}(t-\tau) + \hat{f}(t), \tag{3.24}$$

with arbitrary matrices $\hat{A}, \hat{D}, \hat{B} \in \mathbb{F}^{n,n}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

If $\operatorname{ind}(E, A) = 0$, then the matrix E is nonsingular and the DDAE is of the form

$$\dot{x}(t) = E^{-1}Ax(t) + E^{-1}Dx(t-\tau) + E^{-1}f(t) \tag{3.25}$$

and the ODE solution formula together with Proposition 3.1 directly implies $\ell_1 = 1$, i.e., (3.25) is of smoothing type.

Theorem 3.4. *Consider the DDAE (1.1a) on the interval $\mathbb{I} = [0, M\tau]$ and suppose that Assumptions 2.1 and 2.2 hold. If $\operatorname{ind}(E, A) = 1$, then (1.1a) is of smoothing type if and only if $D_{\mathrm{a},2}$ in (2.18) is nilpotent with index of nilpotency ν_D and furthermore we have $\nu_D \leq M - 1$.*

Proof. Let $S, T \in \mathrm{GL}_n(\mathbb{F})$ be matrices that transform (1.1a) into quasi-Weierstraß form (2.19). Applying the method of steps yields

$$\dot{v}_{i+1} = Jv_{i+1} + D_{\mathrm{d},1}v_i + D_{\mathrm{d},2}v_i + g_{i+1} \quad \text{and} \quad w_{i+1} = -D_{\mathrm{a},1}v_i - D_{\mathrm{a},2}w_i - h_{i+1}.$$

Since $\ell_1 = 0$ we have

$$\begin{aligned} w_1(\tau) &= -D_{\mathrm{a},1}v_0(\tau) - D_{\mathrm{a},2}w_0(\tau) - h_1(\tau) \\ &= -D_{\mathrm{a},1}v_1(0) - D_{\mathrm{a},2}w_1(0) - h_2(0) = w_2(0), \end{aligned}$$

and thus, $\ell_2 \geq 0$. By induction, we conclude $\ell_i \geq 0$ for $i \in \mathcal{I}$. Moreover, we have

$$\begin{aligned} \dot{w}_{i+1} &= -D_{\mathrm{a},1}\dot{v}_i - D_{\mathrm{a},2}\dot{w}_i - \dot{h}_{i+1} \\ &= -D_{\mathrm{a},1}\left(Jv_i + D_{\mathrm{d},1}v_{i-1} + D_{\mathrm{d},2}w_{i-1} + g_i\right) - D_{\mathrm{a},2}\dot{w}_i - \dot{h}_{i+1} \end{aligned}$$

which implies $\dot{w}_{i+1}(0^+) - \dot{w}_i(\tau^-) = D_{\mathrm{a},2}\left(\dot{w}_{i-1}(\tau^-) - \dot{w}_i(0^+)\right)$ holds. By induction, we have

$$\dot{w}_{i+1}(0^+) - \dot{w}_i(\tau^-) = (-1)^i D_{\mathrm{a},2}^i \left(\dot{w}_1(0^+) - \dot{\eta}(0^-)\right) \quad \text{for } i = 1, \ldots, M-1.$$

Thus, $\ell_{i+1} \geq 1$ holds if and only if $D_{\mathrm{a},2}^i = 0$. ∎

Applying Theorem 3.4 to the DDAE in Example 3.2 shows that this DDAE is of smoothing type, since it is already in quasi-Weierstraß form with $D_{\mathrm{a},2} = 0$. Conversely, if the DDAE (1.1a) with $\operatorname{ind}(E, A) = 1$ is of smoothing type, then the index of nilpotency indicates the number of delays present in the system. More precisely, we have the following result.

Corollary 3.5. *Suppose that the DDAE (1.1a) satisfies Assumptions 2.1 and 2.2 and is of smoothing type with $\operatorname{ind}(E, A) \leq 1$. Furthermore let ν_D denote the index of nilpotency of $D_{\mathrm{a},2}$ if $n_{\mathrm{a}} > 0$ and $\nu_D = 0$ otherwise. Then there exists matrices $B_k \in \mathbb{F}^{n_{\mathrm{d}}, n_{\mathrm{d}}}$ $(k = 0, \ldots, \nu_D)$ and an inhomogeneity ϑ such that the solution v of (2.19a) is a solution of the inital value problem*

$$\dot{z}(t) = Jz + \sum_{k=0}^{\nu_D} B_k z(t - (k+1)\tau) + \vartheta(t) \qquad \text{for } t \in [\nu_D\tau, t_{\mathrm{f}}), \tag{3.26a}$$

$$z(t) = v(t) \qquad \text{for } t \in [-\tau, \nu_D\tau]. \tag{3.26b}$$

Proof. The result is trivial for $\operatorname{ind}(E, A) = 0$, i.e., assume $\operatorname{ind}(E, A) = 1$, which implies that $N = 0$ in (2.19). Let $\Delta_{[t_0,t_1)}$ denote the characteristic function for the interval $[t_0, t_1)$, i.e.,

$$\Delta_{(t_0,t_1]}(t) = \begin{cases} 1, & \text{if } t \in [t_0, t_1), \\ 0, & \text{otherwise.} \end{cases}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

Combination of the fast subsystem (2.19b) and the initial condition yields

$$(I_{n_\mathrm{a}} + D_{\mathrm{a},2}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau)w = -D_\mathrm{a}\Delta_{[0,\tau)}\sigma_\tau\phi - D_{\mathrm{a},1}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau v - h. \tag{3.27}$$

By induction, we obtain $(\Delta_{[\tau,t_\mathrm{f})}(t)\sigma_\tau)^k = \Delta_{[k\tau,t_\mathrm{f})}(t)\sigma_{k\tau}$ and from $D_{\mathrm{a},2}^{\nu_D} = 0$ we deduce

$$\left(\sum_{k=0}^{\nu_D-1}(-1)^k\left(D_{\mathrm{a},2}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau\right)^k\right)\left(I_{n_\mathrm{a}} + D_{\mathrm{a},2}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau\right) = I_{n_\mathrm{a}}$$

such that w in (3.27) is given by

$$\begin{aligned} w &= \sum_{k=0}^{\nu_D-1}(-1)^{k+1}\left(D_{\mathrm{a},2}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau\right)^k\left(D_\mathrm{a}\Delta_{[0,\tau)}\sigma_\tau\phi + D_{\mathrm{a},1}\Delta_{[\tau,t_\mathrm{f})}\sigma_\tau v + h\right) \\ &= \sum_{k=0}^{\nu_D-1}(-1)^{k+1}D_{\mathrm{a},2}^k\left(D_\mathrm{a}\Delta_{[k\tau,(k+1)\tau)}\sigma_{(k+1)\tau}\phi + D_{\mathrm{a},1}\Delta_{[k\tau,t_\mathrm{f})}\sigma_{(k+1)\tau}v + \Delta_{[k\tau,t_\mathrm{f})}\sigma_{k\tau}h\right). \end{aligned}$$

Inserting this identity in (2.19a) and introducing for $k = 1,\ldots,\nu_D$ the matrices

$$B_0 := D_{\mathrm{d},1}, \quad B_k := (-1)^k D_{\mathrm{d},2}D_{\mathrm{a},2}^{k-1}D_{\mathrm{a},1}$$

implies that the solution v of (2.19a) is a solution of the IVP (3.26), where ϑ is given by

$$\vartheta(t) := g(t) + \sum_{k=0}^{\nu_D-1}(-1)^{k+1}D_{\mathrm{d},2}D_{\mathrm{a},2}^k h(t-(k+1)\tau). \square$$

□

Example 3.6. Consider the DDE (3.24). Introducing the new variable $y(t) = \hat{x}(t-\tau)$ yields the DDAE

$$\begin{bmatrix}-I_n & -B\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & I_n\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ y(t)\end{bmatrix} + \begin{bmatrix}D & 0\\ -I_n & 0\end{bmatrix}\begin{bmatrix}\hat{x}(t-\tau)\\ y(t-\tau)\end{bmatrix} + \begin{bmatrix}f\\ 0\end{bmatrix}.$$

The matrices $S := \left[\begin{smallmatrix}I_n & -AB\\ 0 & I_n\end{smallmatrix}\right]$ and $T := \left[\begin{smallmatrix}I_n & B\\ 0 & I_n\end{smallmatrix}\right]$ transform the DDAE to quasi-Weierstraß form given by

$$\begin{bmatrix}I_n & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{v}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & I_n\end{bmatrix}\begin{bmatrix}v(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}D+AB & (D+AB)B\\ -I_n & -B\end{bmatrix}\begin{bmatrix}v(t-\tau)\\ w(t-\tau)\end{bmatrix} + \begin{bmatrix}f(t)\\ 0\end{bmatrix}.$$

Hence, the DDE (3.24) is of smoothing type if and only if B is nilpotent. In this case, the corresponding retarded equation (3.26a) is given by

$$\dot{z}(t) = Az(t) + (D+AB)z(t-\tau) + \sum_{k=1}^{\nu_B-1}(-1)^k(D+AB)B^k z(t-(k+1)\tau) + \vartheta(t),$$

where ν_B is the index of nilpotency of B.

Remark 3.7. The delay equation (3.26) of Corollary 3.5 may be used to determine whether the DDAE (3.24) is stable (which can be done for example via DDE-biftool [10, 23]). Note that this provides an alternative way to the theory outlined in [8, 9].

For the analysis of the general DDAE case with arbitrary index, we use the following preliminary result.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

PROPOSITION 3.8. *Suppose that the IVP (1.1) satisfies Assumptions 2.1 and 2.2 and let $S, T \in \mathrm{GL}_n(\mathbb{F})$ be matrices that transform (E, A) to quasi-Weierstraß form (2.8), such that (1.1) is transformed to (2.19) with $x = T\begin{bmatrix} v^T & w^T \end{bmatrix}^T$. Then for any $m \in \mathbb{N}$ and any $\tilde{v} \in \mathbb{F}^{n_\mathrm{d}}$, $\tilde{w} \in \mathbb{F}^{n_\mathrm{a}}$ there exists an admissible history function $\phi = T^{-1}\begin{bmatrix} \psi^T & \eta^T \end{bmatrix}^T$ that is analytic and satisfies*

$$\psi^{(p)}(0^-) = v^{(p)}(0) \qquad \text{for } p = 0, 1, \ldots, m-1, \tag{3.28a}$$

$$\eta^{(p)}(0^-) = w^{(p)}(0) \qquad \text{for } p = 0, 1, \ldots, m-1, \tag{3.28b}$$

$$\tilde{v} = \psi^{(m)}(0^-) - v^{(m)}(0), \tag{3.28c}$$

$$\tilde{w} = \eta^{(m)}(0^-) - w^{(m)}(0). \tag{3.28d}$$

Proof. Let $m \in \mathbb{N}$. Proposition 3.1 implies that the solution x of the IVP (1.1) is m times continuously differentiable on $[-\tau, \tau)$ if and only if ϕ satisfies (3.22) for $p = 0, 1, \ldots, m-1$. Multiply (3.22) from the left by T^{-1} to obtain

$$\psi^{(p+1)}(0^-) = J\psi^{(p)}(0^-) + D_{\mathrm{d},1}\psi^{(p)}(-\tau) + D_{\mathrm{d},2}\eta^{(p)}(-\tau) + g^{(p)}(0), \tag{3.29a}$$

$$\eta^{(p+1)}(0^-) = -\sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k \left(D_{\mathrm{a},1}\psi^{(k+p+1)}(-\tau) + D_{\mathrm{a},2}\eta^{(k+p+1)}(-\tau) + h^{(k+p+1)}(0)\right) \tag{3.29b}$$

for $p = 0, \ldots, m-1$. We then can proceed as follows to construct ψ and η that satisfy the conditions (3.28). Choose any value for $\psi^{(p)}(-\tau)$ and $\eta^{(p)}(-\tau)$ for $p = 0, \ldots, \mathrm{ind}(E, A) + m$, and compute $\eta^{(p+1)}(0^-)$ for $p = 0, \ldots, m-2$ according to (3.29b). For an arbitrary $\psi(0)$, set $\psi^{(p+1)}(0^-)$ according to (3.29a) for $p = 0, \ldots, m-2$. Finally, set

$$\psi^{(m)}(0^-) = \tilde{v} + \left(J\psi^{(m-1)}(0^-) + D_{\mathrm{d},1}\eta^{(m-1)}(-\tau) + D_{\mathrm{d},2}\eta^{(m-1)}(-\tau) + g^{(m)}(0)\right) \quad \text{and}$$

$$\eta^{(m)}(0^-) = \tilde{w} - \sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k \left(D_{\mathrm{a},1}\psi^{(k+p+1)}(-\tau) + D_{\mathrm{a},2}\eta^{(k+p+1)}(-\tau) + h^{(k+p+1)}(0)\right)$$

The desired history functions are then given via Hermite interpolation. ∎

Applying the method of steps and the solution formula (2.10) for the fast subsystem yields

$$w_{i+1} = -\sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \left(D_{\mathrm{a},1}v_i + D_{\mathrm{a},2}w_i + h_{i+1}\right). \tag{3.30}$$

Since Assumption 2.1 implies that all functions are sufficiently smooth we obtain

$$\begin{aligned}
w_2(0) - w_1(\tau^-) &= \sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k \left(D_{\mathrm{a},1}\left(\psi^{(k)}(0^-) - v_1^{(k)}(0)\right) + D_{\mathrm{a},2}\left(\eta^{(k)}(0^-) - w_1^{(k)}(0)\right)\right) \\
&= \sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k D_\mathrm{a} T \begin{bmatrix} \psi^{(k)}(0^-) - v_1^{(k)}(0) \\ \eta^{(k)}(0^-) - w_1^{(k)}(0) \end{bmatrix} \\
&= \sum_{k=1}^{\mathrm{ind}(E,A)-1} N^k D_\mathrm{a} T \begin{bmatrix} \psi^{(k)}(0^-) - v_1^{(k)}(0) \\ \eta^{(k)}(0^-) - w_1^{(k)}(0) \end{bmatrix},
\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

where the last identity follows from the fact the ϕ is assumed to be admissible. Proposition 3.8 implies that (1.1a) is of de-smoothing type if there exists $k \in \{1, \dots, \operatorname{ind}(E, A) - 1\}$ such that $N^k D_{\mathrm{a}} \neq 0$. Assume conversely that $ND_{\mathrm{a}} = 0$. In this case, (3.30) is given by

$$w_{i+1} = -D_{\mathrm{a},1} v_i - D_{\mathrm{a},2} w_i - \sum_{k=0}^{\operatorname{ind}(E,A)-1} N^k h_{i+1}^{(k)},$$

which implies $\ell_i \geq 0$. Together with Theorem 3.4, this proofs the following theorem.

THEOREM 3.9. *Consider the DDAE* (1.1a) *on the interval* $\mathbb{I} = [0, M\tau]$ *and suppose that* Assumptions 2.1 *and* 2.2 *hold. Let* N, D_{a} *and* $D_{\mathrm{a},2}$ *be the matrices that are associated with the quasi-Weierstraß form* (2.19). *Then* (1.1a) *is of*

- *smoothing type if* $ND_{\mathrm{a}} = 0$ *and* $D_{\mathrm{a},2}$ *is nilpotent with nilpotency index* $\nu_D < M$,
- *de-smoothing type if there exists* $k \in \mathbb{N}$ *such that* $N^k D_{\mathrm{a}} \neq 0$, *and*
- *discontinuity invariant type otherwise.*

EXAMPLE 3.10. Introducing the new variable $y(t) = x(t - \tau)$ shows (similarly as in Example 3.6) that the DDAE associated with

$$x(t) = Dx(t - \tau) + B\dot{x}(t - \tau) + f(t) \tag{3.31}$$

is of de-smoothing type if and only if $B \neq 0$.

REMARK 3.11. Checking the proof of Corollary 3.5, we immediately infer from Theorem 3.9 that Corollary 3.5 is also true for arbitrary index $\operatorname{ind}(E, A)$. As a consequence, if the DDAE (1.1a) is of smoothing type, then there exists a sequence $j_k \in \mathbb{N}$ such that $\ell_{j_k} = k$, and hence, the solution becomes arbitrary smooth over time, which justifies the name smoothing type.

Note that $N^k D_{\mathrm{a}} \neq 0$ for some $k \in \mathbb{N}$ implies

$$D_{k+1} = C_{k+1} D = -T \begin{bmatrix} 0 & 0 \\ 0 & N^k \end{bmatrix} SD = -T \begin{bmatrix} 0 \\ N^k D_{\mathrm{d}} \end{bmatrix} \neq 0,$$

i.e., the DDAE (1.1a) is of de-smoothing type if $D_k \neq 0$ for some $k \geq 2$. Using

$$\begin{aligned} D_k(I_n - A^{\mathrm{con}}) &= -T \begin{bmatrix} 0 & 0 \\ N^{k-1} D_{\mathrm{a},1} & N^{k-1} D_{\mathrm{a},2} \end{bmatrix} T^{-1} T \left(\begin{bmatrix} I_{n_{\mathrm{d}}} & 0 \\ 0 & I_{n_{\mathrm{a}}} \end{bmatrix} - \begin{bmatrix} I_{n_{\mathrm{d}}} & 0 \\ 0 & 0 \end{bmatrix} \right) T^{-1} \\ &= -T \begin{bmatrix} 0 & 0 \\ 0 & N^{k-1} D_{\mathrm{a},2} \end{bmatrix} T^{-1} \end{aligned} \tag{3.32}$$

we immediately see that $D_{\mathrm{a},2}$ is nilpotent if, and only if $D_1(I_n - A^{\mathrm{con}})$ is nilpotent, which shows that the results of Theorem 3.9 can be formulated in terms of the underlying DDE (2.16).

COROLLARY 3.12. *Consider the DDAE* (1.1a) *with associated underlying DDE* (2.16) *on the interval* $\mathbb{I} = [0, M\tau]$ *and suppose that* Assumptions 2.1 *and* 2.2 *hold. Then* (1.1a) *is of*

- *smoothing type if* $D_2 = 0$ *and* $D_1(I_n - A^{\mathrm{con}})$ *is nilpotent with nilpotency index* $\nu_{D_1} \leq M$,
- *de-smoothing type if* $D_k \neq 0$ *for some* $k \geq 2$, *and*
- *discontinuity invariant type otherwise.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

A common approach to analyze the (exponential) stability of the DDAE (1.1a) is to compute the spectral abscissa, which is defined as

$$\alpha(E,A,B) := \sup\{\mathrm{Re}(\lambda) \mid \det(\lambda E - A - \exp(-\lambda\tau)B) = 0\}.$$

Surprisingly, the condition $\alpha(E,A,B) < 0$ is not sufficient for a DDAE to be exponentially stable [9]. However, based on the new classification we have the following result.

Corollary 3.13. *Suppose that the DDAE* (1.1a) *is not of de-smoothing type. Then the DDAE* (1.1a) *is exponentially stable if and only if* $\alpha(E,A,B) < 0$.

Proof. Since the DDAE (1.1a) is not of de-smoothing type, we have $ND_{\mathrm{a}} = 0$. The result follows directly from [9, Proposition 3.4 and Theorem 3.4]. ∎

Note that we refrain from using the terminology retarded, neutral, and advanced in Definition 3.3, although these terms are widely used in the delay literature [2, 3, 16, 18]. The reason for this is, that in the classical definition in [3], a retarded DDE becomes advanced if it is solved backward in time, an advanced equation becomes retarded and a neutral equation stays neutral. For the classification introduced in Definition 3.3 this is however not true. To see this, we introduce the new variable $\xi(t-\tau) = x(-t)$ such that (1.1a) transforms to

$$E\dot{\xi}(t-\tau) = -D\xi(t) - A\xi(t-\tau) - f(-t).$$

This leads to the following definition.

Definition 3.14. Consider the DDAE (1.1a). Define

$$\mathcal{E} := \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{2n,2n}, \quad \mathcal{A} := \begin{bmatrix} -D & 0 \\ 0 & I_n \end{bmatrix} \in \mathbb{F}^{2n,2n}, \quad \mathcal{B} := \begin{bmatrix} -A & 0 \\ -I_n & 0 \end{bmatrix} \in \mathbb{F}^{2n,2n}.$$

Then we call the DDAE

$$\mathcal{E}\dot{\zeta}(t) = \mathcal{A}\zeta(t) + \mathcal{B}\zeta(t-\tau) + \mathcal{F}(t) \tag{3.33}$$

with $\mathcal{F} : \mathbb{I} \to \mathbb{F}^{2n}$ the *backward system* for the DDAE (1.1a).

Note that the backward system satisfies Assumption 2.2 if and only if $\det(D) \neq 0$. In this case, we can transform the backward system (3.33) to quasi-Weierstraß form via the matrices

$$S = \begin{bmatrix} -D^{-1} & 0 \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad T = I_{2n}.$$

In particular, we have

$$(S\mathcal{E}T)(S\mathcal{B}T) = \begin{bmatrix} 0 & -D^{-1}E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^{-1}A & 0 \\ -I_n & 0 \end{bmatrix} = \begin{bmatrix} -D^{-1}E & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, Theorem 3.9 implies that $E = 0$ is a necessary condition for the backward system (3.33) to be of smoothing type or discontinuity invariant type, which implies that the DDAE (1.1a) cannot be of de-smoothing type.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

Example 3.15. Consider the DDAE given by $\mathbb{F}=\mathbb{R}$, $n=2$, $f\equiv 0$, $\tau=1$, and

$$E=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

Since (E,A) is already in Weierstraß form and $ED\neq 0$, Theorem 3.9 implies that the DDAE is of de-smoothing type. Since $E\neq 0$ also the backward system is of de-smoothing type.

Let us mention that if $\det(B)=0$, then the method of steps (2.6) cannot be used to determine the solution of the backward system. Instead, one may use the shift-index concept defined in [16,17] to make the pencil $(\mathcal{E},\mathcal{A})$ regular.

4. Impact of splicing conditions. In the previous section, we have established algebraic criteria to check whether a discontinuity in the derivative of $\dot{x}$ at $t=0$ is smoothed out, is propagated to $t=\tau$ or is amplified in the sense that x becomes discontinuous at $t=\tau$. While the definition of discontinuity invariant type is valid for all integer multiples of the delay time, the definitions of smoothing type and de-smoothing type are based on single time points, and hence, the question whether the (de-)smoothing continues is imminent. For DDAEs of smoothing type, this can be answered positively (see Remark 3.11). For DDAEs of de-smoothing type the question can be rephrased as follows: If we restrict the set of admissible history functions such that the *splicing condition* (cf. [2])

$$\phi^{(k)}(0^-)=x^{(k)}(0)\quad\text{for } k=0,\ldots,\kappa \tag{4.34}$$

is satisfied for some $\kappa\in\mathbb{N}$, is there an integer $j\in\mathbb{N}$ such that the inital condition

$$x_j(0)=x_{j-1}(\tau^-)$$

is not consistent for the DAE (2.6)? Similarly, we can ask if for DDAEs of discontinuity invariant type the smoothness at integer multiples of the delay time stays invariant? Before we answer these question, we note that in order to check if the splicing condition (4.34) is satisfied, it seems that one has to solve the DDAE (1.1a) first. That is however not necessary, since the splicing condition (4.34) can be checked by solely investigating the history function ϕ with Proposition 3.1.

Lemma 4.1. *Suppose that the DDAE (1.1a) is of discontinuity invariant type and the admissible history function $\phi\in\mathcal{C}^\infty([-\tau,0],\mathbb{F}^n)$ satisfies the splicing condition (4.34). Then*

$$x_i^{(k)}(0)=x_{i-1}^{(k)}(\tau^-)\quad\textit{for all } i\in\mathbb{N},\ k=0,\ldots,\kappa.$$

Proof. Since (1.1a) is of discontinuity invariant type, we have $ND_\mathrm{a}=0$ in (2.19) according to Theorem 3.9. It suffices to show that

$$x_2^{(j)}(0)=x_1^{(j)}(\tau^-)\quad\text{for all } j=0,\ldots,\kappa.$$

Since ϕ is admissible and the DDAE is of discontinuity invariant type, equation (2.19a) implies that

$$\dot{v}_2(0)-\dot{v}_1(\tau^-)=J\left(v_2(0)-v_1(\tau)\right)+D_\mathrm{d}\left(x_1(0)-\phi(0)\right)=0.$$

Iteratively, we obtain

$$v_2^{(k+1)}(0)-v_1^{(k+1)}(\tau^-)=J\left(v_2^{(k)}(0)-v_1^{(k)}(\tau^-)\right)+D_\mathrm{d}\left(x_1^{(k)}(0)-\phi^{(k)}(0)\right)=0$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

for $k = 2, \ldots, \kappa$. For the fast system (2.19b) we infer directly

$$w_2^{(k)}(0) - w_1^{(k)}(\tau^-) = D_\mathrm{a}\left(\phi^{(k)}(0^-) - x_1^{(k)}(0)\right) = 0$$

for $k = 0, 1, \ldots, \kappa$, which completes the proof. ∎

Note that Lemma 4.1 guarantees that the solution of the DDAE is at least as smooth as the initial transition from the history function to the solution. Conversely, assume that the Jordan canonical form of $D_{\mathrm{a},2}$ exists and let $\tilde{w} \in \mathbb{F}^{n_\mathrm{a}} \setminus \{0\}$ be an eigenvector of $D_{\mathrm{a},2}$ for the eigenvalue $\lambda \neq 0$. Then Proposition 3.8 implies (with $m = \kappa + 1$) the existence of an history function ϕ such that the solution of the IVP (1.1) satisfies

$$\begin{aligned} w_2^{(\kappa+1)}(0) - w_1^{(\kappa+1)}(\tau^-) &= D_{\mathrm{a},2}\left(\eta^{(\kappa+1)}(0^-) - w_1^{(\kappa+1)}(0)\right) \\ &= \lambda\tilde{w} \neq 0. \end{aligned}$$

Thus, in general, we cannot expect the solution of a DDAE of discontinuity invariant type to get any smoother, which again justifies the terminology. For DDAEs of de-smoothing type, Example 1.2 might suggest that the solution becomes less and less smooth until it becomes discontinuous. This is however not necessarily the case as the following example demonstrates.

Example 4.2. Suppose that the DDAE (1.1a) satisfies Assumptions 2.1 and 2.2 and additionally satisfies $ND_{\mathrm{a},2} = 0$, $ND_\mathrm{a} \neq 0$, and $N^2 D_\mathrm{a} = 0$, i.e., the DDAE is of de-smoothing type according to Theorem 3.9. Suppose that the history function ϕ satisfies (3.21). Then

$$\begin{aligned} w_2(0) - w_1(0) &= \sum_{k=0}^{\mathrm{ind}(E,A)-1} N^k D_\mathrm{a}\left(\phi^{(k)}(0^-) - x_1^{(k)}(0)\right) \\ &= \sum_{k=0}^{1} N^k D_\mathrm{a}\left(\phi^{(k)}(0^-) - x_1^{(k)}(0)\right) = 0. \end{aligned}$$

However, we have $\dot{v}_2(0) - \dot{v}_1(\tau^-) = 0$ by the definition of the slow system (2.19a), and by induction, we infer

$$w_{i+1}(0) - w_i(\tau^-) = ND_{\mathrm{a},1}\left(\dot{v}_{i-1}(\tau^-) - \dot{v}_i(0)\right) = 0.$$

Thus, the initial condition $x_i(0) = x_{i-1}(\tau^-)$ is consistent for (2.6), and hence, the solution exists for all $t_\mathrm{f} > 0$.

For a general analysis let us assume that the DDAE (1.1a) satisfies Assumptions 2.1 and 2.2, is of de-smoothing type, and the history function ϕ satisfies the splicing condition (4.34) for some $\kappa \in \mathbb{N}$. From (2.19a) we infer inductively

$$v_2^{(k)}(0) = Jv_2^{(k-1)}(0) + D_\mathrm{d}x_1^{(k-1)}(0) + g_2^{(k)}(0) = v_1^{(k)}(\tau^-)$$

for $k = 1, \ldots, \kappa + 1$. For the fast subsystem, the splicing condition (4.34) implies

$$w_2(0) - w_1(\tau^-) = \sum_{k=\kappa+1}^{\mathrm{ind}(E,A)-1} N^k D_\mathrm{a}\left(\phi^{(k)}(0^-) - x_1^{(k)}(0)\right),$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

and hence, a sufficient condition for the initial condition $w_2(0) = w_1(\tau^-)$ to be consistent is to assume $N^k D_\mathrm{a} = 0$ for $k \geq \kappa + 1$. Note that this is immediately satisfied for $\operatorname{ind}(E, A) \leq \kappa + 1$. To analyze the next interval we compute

$$\begin{aligned} w_3(0) - w_2(\tau^-) &= \sum_{k=1}^{\kappa} N^k D_\mathrm{a} T \begin{bmatrix} v_1^{(k)}(\tau^-) - v_2^{(k)}(0) \\ w_1^{(k)}(\tau^-) - w_2^{(k)}(0) \end{bmatrix} \\ &= \sum_{k=1}^{\kappa} N^k D_{\mathrm{a},2} \left(w_1^{(k)}(\tau^-) - w_2^{(k)}(0) \right). \end{aligned}$$

Thus, the assumption $ND_{\mathrm{a},2} = 0$ implies $w_3(0) - w_2(\tau^-) = 0$. Unfortunately, we have

$$v_3^{(2)}(0) - v_2^{(2)}(\tau^-) = D_{\mathrm{d},2} \left(\dot{w}_2(0) - \dot{w}_1(\tau^-) \right),$$

and thus cannot show that the initial condition $w_4(0) = w_3(\tau)$ is consistent without posing further assumptions on the matrices E, A, and D. Since this becomes quite technical, we summarize our findings only for the case $\operatorname{ind}(E, A) \leq 3$.

Theorem 4.3. *Suppose the IVP (1.1) satisfies Assumptions 2.1 and 2.2 and* $\operatorname{ind}(E, A) \leq 3$. *Moreover, assume $ND_{\mathrm{a},2} = 0$ and $N^2 D_{\mathrm{a},1} D_{\mathrm{d},2} = 0$. Then for every admissible history function ϕ that satisfies (3.22) for $\kappa = 2$, the IVP (1.1) has a unique solution.*

Proof. The assumptions on ϕ imply that the splicing condition (4.34) is satisfied for $\kappa = 2$ (see Proposition 3.1). Since $\operatorname{ind}(E, A) \leq 3$, we have $N^3 = 0$. Together with $ND_{\mathrm{a},2} = 0$ the previous discussion guarantees that a solution exists on the interval $[-\tau, 3\tau]$. Using $ND_{\mathrm{a},2} = 0$, we observe (inductively)

$$\begin{aligned} w_{i+1}(0) - w_i(\tau^-) &= \sum_{k=0}^{2} N^k D_{\mathrm{a},1} \left(v_{i-1}^{(k)}(\tau^-) - v_i^{(k)}(0) \right) \\ &= N^2 D_{\mathrm{a},1} D_{\mathrm{d},2} \left(\dot{w}_{i-2}(\tau^-) - \dot{w}_{i-1}(0) \right) = 0, \end{aligned}$$

and thus, the initial condition $x_{i+1}(0) = x_i(\tau^-)$ is consistent for all $i \in \mathbb{N}$. The result follows from Corollary 2.6. ∎

Note that the assumptions in Theorem 4.3 can also be formulated in terms of the underlying DDE (2.16) and the matrices defined in (2.11). More precisely, (3.32) and

$$D_0(I_n - A^{\mathrm{con}}) = T \begin{bmatrix} D_{\mathrm{d},1} & D_{\mathrm{d},2} \\ 0 & 0 \end{bmatrix} T^{-1} T \left(\begin{bmatrix} I_{n_\mathrm{d}} & 0 \\ 0 & I_{n_\mathrm{a}} \end{bmatrix} - \begin{bmatrix} I_{n_\mathrm{d}} & 0 \\ 0 & 0 \end{bmatrix} \right) T^{-1} = T \begin{bmatrix} 0 & D_{\mathrm{d},2} \\ 0 & 0 \end{bmatrix} T^{-1}$$

imply that $ND_{\mathrm{a},2} = 0$ and $N^2 D_{\mathrm{a},1} D_{\mathrm{d},2} = 0$ if, and only if, $D_2(I_n - A^{\mathrm{con}}) = 0$ and $D_3 A^{\mathrm{con}} D_0 (I_n - A^{\mathrm{con}}) = 0$, respectively.

Remark 4.4. The proof of Theorem 4.3 shows that the result can be further improved by requiring different splicing conditions for the history function ψ for the slow state v and for the history function η of the fast state w.

5. Comparison to the existing classification. In [16], the authors replace the delayed argument in the DDAE (1.1a) with a function parameter $\lambda : \mathbb{I} \to \mathbb{F}^n$ and obtain the initial value problem

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + D\lambda(t) + f(t), \\ x(t) &= \phi(0), \end{aligned} \tag{5.35}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

on the time interval $\mathbb{I}$. They call the function parameter λ *consistent* if there exists a consistent initial condition $\phi(0)$ for the IVP (5.35). Based on the function parameter λ the following classification for DDAEs [16] is introduced.

Definition 5.1. The DDAE (1.1a) is called *retarded*, *neutral*, or *advanced*, if the minimum smoothness requirement for a consistent function parameter λ is that $\lambda \in \mathcal{C}(\mathbb{I}, \mathbb{F}^n)$, $\lambda \in \mathcal{C}^1(\mathbb{I}, \mathbb{F}^n)$, or $\lambda \in \mathcal{C}^k(\mathbb{I}, \mathbb{F}^n)$ for some $k \geq 2$.

To compare the classification based on propagation of primary discontinuities (cf. Definition 3.3) with the classification of [16], we need to understand Definition 5.1 in terms of the quasi-Weierstraß form.

Proposition 5.2. *Suppose that the DDAE* (1.1a) *satisfies Assumptions* 2.1 *and* 2.2*. Then the DDAE* (1.1a) *is*

- *retarded if and only if* $D_{\mathrm{a}} = 0$,
- *neutral if and only if* $D_{\mathrm{a}} \neq 0$ *and* $ND_{\mathrm{a}} = 0$, *and*
- *advanced otherwise,*

where D_{a} *and* N *are the matrices from the quasi-Weierstraß form (Theorem* 2.4*) and* (2.19).

Proof. The smoothness requirements for λ can be directly seen from the underlying DDE (2.16). Note that we have

$$D_0 = T \begin{bmatrix} I_{n_{\mathrm{d}}} & 0 \\ 0 & 0 \end{bmatrix} SD = T \begin{bmatrix} D_{\mathrm{d}} \\ 0 \end{bmatrix} \quad \text{and}$$
$$D_k = -T \begin{bmatrix} 0 & 0 \\ 0 & N^{k-1} \end{bmatrix} SD = -T \begin{bmatrix} 0 \\ N^{k-1} D_{\mathrm{a}} \end{bmatrix}$$

for $k = 1, \ldots, \operatorname{ind}(E, A)$. Hence, (1.1a) is retarded if and only if $N^{k-1} D_{\mathrm{a}} = 0$ for all $k = 1, \ldots, \operatorname{ind}(E, A)$, which is equivalent to $D_{\mathrm{a}} = 0$. The DDAE is neutral, if $N^{k-1} D_{\mathrm{a}} = 0$ for all $k = 2, \ldots, \operatorname{ind}(E, A)$, which is equivalent to $ND_{\mathrm{a}} = 0$ and otherwise advanced. ∎

With the characterization, we see immediately that the classification by [16] provides in the following sense an upper bound for the new definition.

Corollary 5.3. *Suppose that the DDAE* (1.1a) *satisfies Assumptions* 2.2 *and* 2.2.

- *If* (1.1a) *is not advanced, then the DDAE* (1.1a) *is not of de-smoothing type.*
- *If the DDAE* (1.1a) *is advanced, then it is of de-smoothing type.*

Since the classification introduced in this paper is based on the worst-case scenario, the numerical method described in [16], which is formulated for DDAEs that are not advanced, is safe to use.

Remark 5.4. The numerical method introduced in [16] is tailored to DDAEs that are not advanced and cannot be used for advanced DDAEs. However, if it is known that the history function satisfies the splicing condition (4.34) for some $\kappa > 0$, then also advanced DDAEs may be solved (cf. Theorem 4.3). Thus, there is a need for numerical integration schemes that can handle such situations. This is subject to further research.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 582-601, November 2018.

6. Summary. In this paper, we have studied the propagation of primary discontinuities in initial value problems for delay differential-algebraic equations. Based on the different possible propagation types we have introduced a new classification for DDAEs and developed a complete characterization in terms of the coefficient matrices. Moreover, the analysis shows that hidden delays may be possible in DDAEs and we have introduced a systematic way to reformulate the DDAE in terms of these delays. As a consequence, we showed that the stability analysis for such DDAEs can be performed by computing the spectral abscissa. Besides, we have studied the impact of splicing conditions on the classification and derived sufficient conditions for DDAEs of index less or equal three to have a unique solution.

Acknowledgments. The author would like to thank the anonymous referee and Prof. Stephan Trenn for valuable feedback that helped to improve the presentation and the formulation of the main results in terms of the matrices D_k (see Corollary 3.12 and the discussion after Theorem 4.3).

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