Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 561-565, October 2018.

INEQUALITIES BETWEEN $\mid A \mid + \mid B \mid$ AND $\mid A^* \mid + \mid B^* \mid$*

YUN ZHANG†

Abstract. Let A and B be complex square matrices. Some inequalities between $\mid A \mid + \mid B \mid$ and $\mid A^* \mid + \mid B^* \mid$ are established. Applications of these inequalities are also given. For example, in the Frobenius norm,

$$\| A + B \|_F \leq \sqrt[4]{2} \, \|| A | + | B |\|_F .$$

Key words. Unitarily invariant norms, Frobenius norm, Singular values.

AMS subject classifications. 15A60, 47A30, 15A42, 15A18.

1. Introduction. We denote by M_n the vector space of all complex $n \times n$ matrices with the inner product $\langle X, Y \rangle = \operatorname{tr}(Y^*X)$, where $\operatorname{tr} X$ denotes the trace of X and Y^* is the conjugate transpose of Y. Let the eigenvalues of $A \in M_n$ be $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. We denote $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ and $|\lambda(A)| = (|\lambda_1|, \ldots, |\lambda_n|)$. The *singular values* of $A \in M_n$ are the nonnegative square roots of the eigenvalues of A^*A. The absolute value of $A \in M_n$ is $\mid A \mid = (A^*A)^{\frac{1}{2}}$. Thus, the singular values of A are the eigenvalues of $|A|$. We denote the singular values of $A \in M_n$ by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ and denote $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$. The operator norm on M_n induced by the Euclidean norm $\| \cdot \|$ on $\mathbb{C}^n$ is the *spectral norm*:

$$\| A \|_\infty = \max\{\| Ax \| : \| x \| = 1, \, x \in \mathbb{C}_n\}.$$

The Euclidean norm on M_n is the *Frobenius norm*:

$$\| A \|_F := \left(\sum_{i,j} \mid a_{ij} \mid^2 \right)^{\frac{1}{2}} = (\operatorname{tr}(A^*A))^{\frac{1}{2}} = \left(\sum_{i=1}^{n} s_i^2(A) \right)^{\frac{1}{2}}, \quad A = (a_{ij}) \in M_n.$$

A norm on M_n is *unitarily invariant* if $\| UAV \| = \| A \|$ for any $A \in M_n$ and any unitary $U, V \in M_n$. The spectral norm and the Frobenius norm are unitarily invariant.

Our work is motivated by the following inequalities due to Lee in [5]. Let $A, B \in M_n$. Then for every unitarily norm,

$$\| A + B \| \leq \|\mid A \mid + \mid B \mid\|^{\frac{1}{2}} \|\mid A^* \mid + \mid B^* \mid\|^{\frac{1}{2}} \tag{1.1}$$

$$\leq \max\{\|\mid A \mid + \mid B \mid\|, \|\mid A^* \mid + \mid B^* \mid\|\}. \tag{1.2}$$

For the topic of norm inequalities and singular value inequalities, see [3, 6].

In this note, we focus on establishing inequalities between $|A| + |B|$ and $|A^*| + |B^*|$. Applications of these inequalities are also given.

*Received by the editors on September 4, 2018. Accepted for publication on September 20, 2018. Handling Editor: Roger A. Horn.

†School of Mathematical Sciences, Huaibei Normal University, Huaibei, China (zhangyunmaths@163.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 561-565, October 2018.

2. Auxilliary results and proofs. We now list some lemmas that are used in our proofs.

LEMMA 2.1. *If $A, B \in M_n$ and $s_i(A) \le s_i(B)$ for all $i = 1, \ldots, n$, then for every unitarily invariant norm, $\|A\| \le \|B\|$.*

LEMMA 2.2. [7, Theorem 1.27] *Let A, $B \in M_n$. Then AB and BA have the same eigenvalues (multiplicities counted).*

LEMMA 2.3. (Fan, [2]) *Let $A, B \in M_n$, $1 \le i, j \le n$, $i + j - 1 \le n$. Then*

$$s_{i+j-1}(AB) \le s_i(A)s_j(B).$$

In particular, $s_j(AB) \le s_1(A)s_j(B)$, $s_j(AB) \le s_1(B)s_j(A)$.

LEMMA 2.4. [7, p. 101] *Let $A, B \in M_n$ be positive semidefinite. Then*

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| \le \| A + B \|$$

for all unitarily invariant norms.

PROPOSITION 2.5. *Let $A, B \in M_n$. Then for $1 \le j \le n$,*

$$s_j(| A^* | + | B^* |) \le 2s_j(| A | \oplus | B |). \tag{2.3}$$

These inequalities are sharp.

Proof. Let $A = U \mid A \mid$ and $B = V \mid B \mid$ be polar decompositions with U, V unitary. Then we have

$$| A^* |= U \mid A \mid U^*, \quad | B^* |= V \mid B \mid V^*.$$

Denote

$$P_0 = \begin{pmatrix} I & U^*V \\ V^*U & I \end{pmatrix}, \quad Q = \begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}.$$

Then $P_0^* = P_0 = \frac{1}{2}P_0^2$ and P_0 is positive semidefinite with $s_1(P_0) = 2$. Applying Lemma 2.2 and $P_0 = \frac{1}{2}P_0^2$, we have

$$\begin{aligned}
\lambda\left((| A^* | + | B^* |) \oplus 0\right) &= \lambda\left(\begin{pmatrix} U & V \\ 0 & 0 \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\begin{pmatrix} U^* & 0 \\ V^* & 0 \end{pmatrix}\right) \\
&= \lambda\left(\begin{pmatrix} U^* & 0 \\ V^* & 0 \end{pmatrix}\begin{pmatrix} U & V \\ 0 & 0 \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\right) \\
&= \lambda\left(\begin{pmatrix} I & U^*V \\ V^*U & I \end{pmatrix}\begin{pmatrix} | A | & 0 \\ 0 & | B | \end{pmatrix}\right) \\
&= \lambda\left(P_0 Q\right) \\
&= \lambda\left(\frac{1}{2}P_0^2 Q\right) \\
&= \lambda\left(\frac{1}{2}P_0 Q P_0\right).
\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 561-565, October 2018.

Note that both $|A^*| + |B^*|$ and P_0QP_0 are positive semidefinite. Since for positive semidefinite matrices singular values and eigenvalues are the same, applying Lemma 2.3 we obtain for $1 \le j \le n$,

$$\begin{aligned} s_j \left(| A^* | + | B^* |) \oplus 0\right) &= s_j \left(| A^* | + | B^* |\right)) \\ &= s_j \left(\frac{1}{2} P_0 Q P_0\right) \\ &\le s_j \left(QP_0\right) \\ &\le 2s_j \left(Q\right) \\ &= 2s_j \left(| A | \oplus | B |\right). \end{aligned}$$

Next we show that equality is possible in the inequalities (2.3) for some nonzero square matrices A and B. Consider

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

where I is the identity matrix of order n. A calculation indicates that

$$| A | \oplus | B | = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad | A^* | + | B^* | = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $s_j(| A^* | + | B^* |) = 2s_j \left(| A | \oplus | B |\right) = 2$, for $1 \le j \le n$. □

Applying Proposition 2.5 and Lemmas 2.1 and 2.4 we deduce the following corollary.

Corollary 2.6. *Let $A, B \in M_n$. Then we have*

$$\||\, | A^* | + | B^* | \,\||\le 2 \,\||\, | A | + | B | \,\|| \tag{2.4}$$

for every unitarily invariant norm.

Remark 2.7. The inequality (2.4) is sharp for the spectral norm. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.5}$$

Then

$$| A | + | B | = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad | A^* | + | B^* | = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

and so,

$$\||\, | A^* | + | B^* | \,\||_\infty = 2 \,\||\, | A | + | B | \,\||_\infty = 2.$$

Bourin and Uchiyama [1] proved the following triangle inequality: Let A, $B \in M_n$ be normal. Then for all unitarily invariant norms,

$$\| A + B \| \le \||\, | A | + | B | \,\|| . \tag{2.6}$$

In the general case, Lee [5] proved for all $A, B \in M_n$ and all unitarily invariant norms,

$$\| A + B \| \le \sqrt{2} \,\||\, | A | + | B | \,\|| . \tag{2.7}$$

The following result interpolates Lee's inequality (2.7).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 561-565, October 2018.

THEOREM 2.8. *Let $A,B \in M_n$. Then for all unitarily invariant norms,*

$$\| A+B \| \leq \sqrt{2} \, \||\, |A| + |B| \,\||^{\frac{1}{2}} \||\, |A| \oplus |B| \,\||^{\frac{1}{2}} \leq \sqrt{2} \, \||\, |A| + |B| \,\|| \,. \tag{2.8}$$

Proof. Use Lee's inequality (1.1) to compute

$$\begin{aligned} \| A + B \| &\leq \| \, |A| + |B| \, \|^{\frac{1}{2}} \| \, |A^*| + |B^*| \, \|^{\frac{1}{2}} \\ &\leq \sqrt{2} \, \||\, |A| + |B| \,\||^{\frac{1}{2}} \||\, |A| \oplus |B| \,\||^{\frac{1}{2}} \\ &\leq \sqrt{2} \, \||\, |A| + |B| \,\||, \end{aligned}$$

where the second inequality follows from Proposition 2.5 and Lemma 2.1 and the last inequality follows from Lemma 2.4. ∎

In the case of the Frobenius norm, we can improve the inequality (2.4).

PROPOSITION 2.9. *Let $A, B \in M_n$. Then*

$$\||\, |A^*| + |B^*| \,\||_F \leq \sqrt{2} \, \||\, |A| + |B| \,\||_F \tag{2.9}$$

and this inequality is sharp.

Proof. Let $A = U|A|$ and $B = V|B|$ be polar decompositions with U, V unitary. Then

$$|A^*| = U|A|U^*, \quad |B^*| = V|B|V^*.$$

Since $tr(XY) = tr(YX)$, we deduce that

$$tr(|A|) = tr(|A^*|), \quad tr(|B|) = tr(|B^*|)$$

and

$$tr(|A||B|) = tr(|B||A|) = tr(|A|^{\frac{1}{2}}|B||A|^{\frac{1}{2}}) \geq 0.$$

Compute

$$\begin{aligned} \||\, |A^*| + |B^*| \,\||_F^2 &= tr(|A^*|^2 + |B^*|^2) + 2tr(|A^*||B^*|) \\ &\leq tr(|A^*|^2 + |B^*|^2) + 2tr(|A^*|^2)^{\frac{1}{2}} \, tr(|B^*|^2)^{\frac{1}{2}} \\ &\leq tr(|A^*|^2 + |B^*|^2) + tr(|A^*|^2) + tr(|B^*|^2) \\ &= 2tr(|A^*|^2 + |B^*|^2) \\ &= 2tr(|A|^2 + |B|^2) \\ &\leq 2tr[(|A| + |B|)^2] \\ &= 2 \, \||\, |A| + |B| \,\||_F^2 \,. \end{aligned}$$

For the matrices in (2.5), we have $\||\, |A| + |B| \,\||_F = \sqrt{2}$ and $\||\, |A^*| + |B^*| \,\||_F = 2$, which shows that equality is possible in (2.9). ∎

THEOREM 2.10. *Let $A, B \in M_n$. Then*

$$\|| A+B \||_F \leq \| \sqrt[4]{2} \, \||\, |A| + |B| \,\||_F \tag{2.10}$$

Proof. Apply Lee's inequality (1.1) and Proposition 2.9 to obtain

$$\begin{aligned} \| A+B \|_F &\leq \||\, |A| + |B| \,\||_F^{\frac{1}{2}} \||\, |A^*| + |B^*| \,\||_F^{\frac{1}{2}} \\ &\leq \sqrt[4]{2} \, \||\, |A| + |B| \,\||_F \,. \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 561-565, October 2018.

E.Y. Lee [4, 5] conjectured that for the Frobenius norm, the inequality

$$\||\, A+B\,|\|_F \leq \sqrt{\frac{1+\sqrt{2}}{2}}\, \||\, A\,|+|\,B\,|\|_F$$

holds. Note that $\sqrt{\frac{1+\sqrt{2}}{2}} \approx 1.099$ and $\sqrt[4]{2} \approx 1.189$. Although the factor $\sqrt[4]{2}$ in Theorem 2.10 is close to $\sqrt{\frac{1+\sqrt{2}}{2}}$, Lee's conjecture is still open.

Acknowledgment. The author is grateful to the referee and Professor Roger Horn for their valuable comments and suggestions. The work was supported by Anhui Provincial Natural Science Foundation (1708085QA05) and the Key Program in the Youth Elite Support Plan in Universities of Anhui Province (gxyqZD2018047).

REFERENCES

[1] J.C. Bourin and M. Uchiyama. A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$. *Linear Algebra Appl.*, 423:512–518, 2007.
[2] K. Fan. Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proc. Natl. Acad. Sci. USA*, 37:760–766, 1951.
[3] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.
[4] E.Y. Lee. Rotfel'd type inequalities for norms. *Linear Algebra Appl.*, 433:580–584, 2010.
[5] E.Y. Lee. How to compare the absolute values of operator sums and the sums of absolute values? *Oper. Matrices*, 6(3):613–619, 2012.
[6] X. Zhan. *Matrix Inequalities.* Lecture Notes in Mathematics, Vol. 1790, Springer, Berlin, 2002.
[7] X. Zhan. *Matrix Theory.* Graduate Studies in Mathematics, Vol. 147, American Mathematical Society, Providence, RI, 2013.