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PERTURBATION RESULTS AND THE FORWARD ORDER LAW FOR THE MOORE-PENROSE INVERSE OF A PRODUCT*

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Abstract. New expressions are given for the Moore-Penrose inverse of a product AB of two complex matrices. Furthermore, an expression for $(AB)^\dagger - B^\dagger A^\dagger$ for the case where A or B is of full rank is provided. Necessary and sufficient conditions for the forward order law for the Moore-Penrose inverse of a product to hold are established. The perturbation results presented in this paper are applied to characterize some mixed-typed reverse order laws for the Moore-Penrose inverse, as well as the reverse order law.

Key words. Moore-Penrose pseudo-inverse, Generalized inverses of a matrix product, Forward order law, Reverse order law.

AMS subject classifications. 15A09, 15A23, 15A24.

1. Introduction. In numerous applications, such as in the celebrated Karmarkar algorithm [12], one has to find the Moore-Penrose inverse of a matrix product AB, denoted by $(AB)^\dagger$. Traditionally the problem of updating $(AB)^\dagger$ has been attacked by considering AB as a string of rank-one perturbations of A. This is rather cumbersome and poses difficulty in trying to express the final answer in terms of the original matrices A and B. Our goal in this work is to present two formulas for $(AB)^\dagger$ and to show that it allow us to explore the forward order law and mixed-type reverse order laws.

Throughout this paper, $\mathbb{C}^{m\times n}$ is the vector space of $m \times n$ complex matrices and we shall respectively denote column space (range), row space, and null space of a matrix A by $\mathcal{C}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$. The Moore-Penrose inverse of A is the unique matrix satisfying the four Penrose equations

$$(1)\ \ AXA = A, \quad (2)\ \ XAX = X, \quad (3)\ \ (AX)^* = AX, \quad (4)\ \ (XA)^* = XA,$$

and will be denoted by $A^\dagger$. It always exists for complex matrices. For convenience we shorten Moore-Penrose to "M-P". Any solution to the equation (1) is called either a $\{1\}$-inverse or g-inverse of A. The symbol $A\{1\}$ will stand for the set of all g-inverses of A. Any solution to the ith$, \dots, j$th equations of the four Penrose equations is called an $\{i, \dots, j\}$-inverse, denoted by $A^{(i,\dots,j)}$. The group inverse of a square matrix A, is the unique matrix, whenever it exists, satisfying the equations

$$AXA = A, \quad XAX = X, \quad AX = XA,$$

and will be denoted by $A^\sharp$. If $A = A^*$, then the group inverse exists and $A^\sharp = A^\dagger$. We shall assume familiarity with the basic results on the generalized inverses as given in [3].

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The problem of determining an expression for the Moore-Penrose inverse of a matrix product AB has been first attacked by Cline [4], who established a formula which allows one to reduce the problem to a type of matrix product where one of the factors is an orthogonal projector.

LEMMA 1.1 ([4]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then $(AB)^\dagger = K^\dagger R^\dagger$, where $R = ABB^\dagger$ and $K = A^\dagger AB$.*

The reverse order law (ROL) is concerned with the problem of when $B^\dagger A^\dagger$ is the M-P inverse of AB. A solution to this problem has been first given by Greville [9], who showed the following necessary and sufficient conditions for the ROL:

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{C}(A^*AB) \subseteq \mathcal{C}(B) \quad \text{and} \quad \mathcal{R}(ABB^*) \subseteq \mathcal{R}(A). \tag{1.1}$$

These conditions were later expressed by Arghiriade [1] in the single condition that A^*ABB^* is EP, i.e.,

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{C}(A^*ABB^*) = \mathcal{C}(BB^*A^*A). \tag{1.2}$$

Since then, the problem of the reverse order law for the M-P inverse has been widely studied and several other equivalent conditions have been established for the product of two and more matrices or for operators [7, 8, 11, 17, 19].

The ROL for g-inverses has been investigated in [2, 16, 21], for $\{1,2\}$-inverses in [5, 15, 20] and for $\{1,3,4\}$-inverses in [6, 14].

A formulation of the forward order law (FOL) for g-inverses of the product of two matrices deals with the problem of when $A\{1\}B\{1\} \subseteq (AB)\{1\}$. This law has been studied in [10] and for multiple matrix products in [22]. In the present work, we shall consider the problem of when $(AB)^\dagger = A^\dagger B^\dagger$, and we solve it by giving, among others conditions, a set of necessary and sufficient conditions for this forward order law to hold in terms of the matrices A and B.

The paper is organized as follows. In Section 2, we give two alternative formulas to Cline's formula for the M-P of the product of two matrices shown in Lemma 1.1. These formulas lead to another expression for $(AB)^\dagger$ involving the M-P inverse of the product of two orthogonal projectors. In addition, an expression for $(AB)^\dagger - B^\dagger A^\dagger$ for the case where A or B is of full rank is provided.

These perturbation formulas for $(AB)^\dagger$ are utilized to derive necessary and sufficient conditions for $(AB)^\dagger$ to be equal any of the convenient choices, say Y, such as $Y = A^\dagger B^\dagger, B^\dagger R^\dagger, K^\dagger A^\dagger$ or $B^\dagger A^\dagger$. In Section 3, several characterizations are established for the forward order law for the M-P inverse, $(AB)^\dagger = A^\dagger B^\dagger$, to hold. In Section 4, it is shown that the perturbation results presented in Section 2 can be utilized to examine some mixed-typed reverse order laws for the M-P inverse, as well as the reverse order law.

We shall need the following results.

LEMMA 1.2. *Let $X, Y \in \mathbb{C}^{m\times n}$ and let F and G be two idempotent matrices of orders m and n, respectively. Then, the following hold:*

(i) $(I - F)X = Y \Leftrightarrow FY = 0$ *and* $\mathcal{C}(X - Y) \subseteq C(F)$.
(ii) $X(I - G) = Y \Leftrightarrow YG = 0$ *and* $\mathcal{R}(X - Y) \subseteq R(G)$.

Proof. Part (i). Pre-multiplying $(I - F)X = Y$ by F, yields $FY = 0$. Pre-multiplying $(I - F)X = Y$ by $I - F$, leads to $(I - F)(X - Y) = 0$, which is equivalent to $\mathcal{C}(X - Y) \subseteq \mathcal{N}(I - F) = \mathcal{C}(F)$. Likewise, the converse holds.

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Part (ii) follows from part (i) applied to $(I-G^*)X^* = Y^*$. ◻

LEMMA 1.3. ([13, Proposition 4]). *Let $A \in \mathbb{C}^{m\times n}$ and let $F \in \mathbb{C}^{n\times n}$ be idempotent. Then*

$$\mathcal{N}(AF) = (\mathcal{N}(A) \cap \mathcal{C}(F)) \oplus \mathcal{N}(F).$$

2. Perturbation formulas for the Moore-Penrose inverse of a product. In this section, two formulas are derived for $(AB)^\dagger$, which show that $(AB)^\dagger = B^\dagger R^\dagger + \varrho$ and $(AB)^\dagger = K^\dagger A^\dagger + \theta$, where the expressions of ϱ and θ involve the matrices $B^\dagger, R^\dagger$ and $A^\dagger, K^\dagger$, respectively. Furthermore, a formula is established for $(AB)^\dagger$ that involves the Moore-Penrose of the product of two orthogonal projectors, $(A^\dagger ABB^\dagger)^\dagger$, in which there is certain symmetry like in the Cline's formula shown in Lemma 1.1.

We are now ready for our main theorem.

THEOREM 2.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$.*

(a) If $R = ABB^\dagger$, then

$$(AB)^\dagger = (I-\varepsilon\varepsilon^\dagger)B^\dagger R^\dagger = B^\dagger(I-\varepsilon^\dagger B^\dagger)R^\dagger = B^\dagger(I-U^\dagger\varepsilon^* B^\dagger)R^\dagger, \tag{2.1}$$

$$AB(AB)^\dagger = RR^\dagger \quad \text{and} \quad (AB)^\dagger AB = B^\dagger B - \varepsilon\varepsilon^\dagger, \tag{2.2}$$

where $\varepsilon = B^\dagger(I - R^\dagger R)$ and $U = R^\dagger R + \varepsilon^*\varepsilon$.

(b) If $K = A^\dagger AB$, then

$$(AB)^\dagger = K^\dagger A^\dagger(I-\delta^\dagger\delta) = K^\dagger(I-A^\dagger\delta^\dagger)A^\dagger = K^\dagger(I-A^\dagger\delta^* V^\dagger)A^\dagger, \tag{2.3}$$

$$AB(AB)^\dagger = AA^\dagger - \delta^\dagger\delta \quad \text{and} \quad (AB)^\dagger AB = K^\dagger K, \tag{2.4}$$

where $\delta = (I-KK^\dagger)A^\dagger$ and $V = KK^\dagger + \delta\delta^*$.

Proof. Part (a). Let $X = (I-\varepsilon\varepsilon^\dagger)B^\dagger R^\dagger$. We shall prove that X satisfies the four Penrose equations of $(AB)^\dagger$. We begin by observing that $R^\dagger = B(AB)^\dagger$, which can be easily checked. Also note that $AB\varepsilon = 0$, and hence, that (3): $ABX = AB(I-\varepsilon\varepsilon^\dagger)B^\dagger R^\dagger = RR^\dagger = AB(AB)^\dagger$, which is Hermitian. It is now clear that (1): $ABXAB = AB$ and (2): $XABX = (I-\varepsilon\varepsilon^\dagger)B^\dagger R^\dagger RR^\dagger = X$. Lastly,

$$XAB = (I-\varepsilon\varepsilon^\dagger)B^\dagger R^\dagger AB = (I-\varepsilon\varepsilon^\dagger)(B^\dagger R^\dagger R)B = (I-\varepsilon\varepsilon^\dagger)(B^\dagger - \varepsilon)B = (I-\varepsilon\varepsilon^\dagger)B^\dagger B.$$

Next we observe that $B^\dagger B\varepsilon = \varepsilon$, and thus, $B^\dagger B\varepsilon\varepsilon^\dagger = \varepsilon\varepsilon^\dagger$. This gives (4): $XAB = B^\dagger B - \varepsilon\varepsilon^\dagger$, which is Hermitian, completing the proof of the first identity in (2.1) and the proof of (2.2).

To show the equivalence of the second and third expressions in (2.1), we note that $(I-R^\dagger R)\varepsilon^\dagger = \varepsilon^\dagger$, and hence, $\varepsilon\varepsilon^\dagger B^\dagger = B^\dagger(I-R^\dagger R)\varepsilon^\dagger B^\dagger = B^\dagger\varepsilon^\dagger B^\dagger$. Next we observe that $U = R^\dagger R + (\varepsilon^*\varepsilon)$ is Hermitian, and since $R^\dagger R\varepsilon^* = 0$ and $\varepsilon R^\dagger R = 0$, we have

$$U^\dagger = U^\# = R^\dagger R + (\varepsilon^*\varepsilon)^\dagger. \tag{2.5}$$

As such $U\varepsilon^* B^\dagger R^\dagger =^\dagger (R^\dagger R + (\varepsilon^*\varepsilon)^\dagger)\varepsilon^* B^\dagger R^\dagger = \varepsilon^\dagger B^\dagger R^\dagger$, which establishes the equivalence of the last two equalities in (2.1).

Part (b) follows by left-right symmetry. ◻

Of particular use is the full-rank case since then we recover an expression for $(AB)^\dagger - B^\dagger A^\dagger$.

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Corollary 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following hold:*

(a) If $BB^\dagger = I$, then

$$(AB)^\dagger = (I - \varepsilon\varepsilon^\dagger)B^\dagger A^\dagger = B^\dagger(I - U^{-1}\varepsilon^* B^\dagger)A^\dagger,$$
$$AB(AB)^\dagger = AA^\dagger \quad \text{and} \quad (AB)^\dagger AB = B^\dagger B - \varepsilon\varepsilon^\dagger,$$

where $\varepsilon = B^\dagger(I - A^\dagger A)$ and $U = A^\dagger A + \varepsilon^*\varepsilon$.

(b) If $A^\dagger A = I$, then

$$(AB)^\dagger = B^\dagger A^\dagger(I - \delta\delta^\dagger) = B^\dagger(I - A^\dagger\delta^* V^{-1})A^\dagger,$$
$$AB(AB)^\dagger = AA^\dagger - \delta^\dagger\delta \quad \text{and} \quad (AB)^\dagger AB = B^\dagger B,$$

where $\delta = (I - BB^\dagger)A^\dagger$ and $V = BB^\dagger + \delta\delta^*$.

Proof. Part (a). If $BB^\dagger = I$ then U will be invertible. Indeed, if $Q = B^\dagger$ then $[A^\dagger A + (I - A^\dagger A)Q^*Q(I - A^\dagger A)]\mathbf{x} = \mathbf{0} \Rightarrow A^\dagger A\mathbf{x} = \mathbf{0}$ and $(I - A^\dagger A)Q^*Q(I - A^\dagger A)\mathbf{x} = \mathbf{0}$. The latter says that $Q(I - A^\dagger A)\mathbf{x} = \mathbf{0}$, and since $Q^\dagger Q = I$, we arrive at $(I - A^\dagger A)\mathbf{x} = \mathbf{0}$, forcing $\mathbf{x} = \mathbf{0}$. In particular, the expression (2.5) takes the form

$$U^{-1} = A^\dagger A + [(I - A^\dagger A)Q^*Q(I - A^\dagger A)]^\dagger.$$

Thus, (a) holds by referring to part (a) of Theorem 2.1.

Part (b) follows by left-right symmetry. ∎

Combining parts (a) and (b) of Theorem leads to the following formula for $(AB)^\dagger$.

Corollary 2.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then*

$$(AB)^\dagger = (I - \varepsilon\varepsilon^\dagger)B^\dagger J^\dagger A^\dagger(I - \sigma^\dagger\sigma),$$

where $J = A^\dagger ABB^\dagger$, $\varepsilon = B^\dagger(I - J^\dagger J)$, and $\sigma = (I - JJ^\dagger)A^\dagger$.

Proof. By the first identity in (2.1) of Theorem 2.1, $(AB)^\dagger = (I - \varepsilon\varepsilon^\dagger)B^\dagger R^\dagger$, where $R = ABB^\dagger$ and $\varepsilon = B^\dagger(I - R^\dagger R)$. Now, we write $R = AC$, where $C = BB^\dagger$. Then by first identity in (2.3), we obtain

$$R^\dagger = J^\dagger A^\dagger(I - \sigma^\dagger\sigma), \tag{2.6}$$

where $J = A^\dagger AC$ and $\sigma = (I - JJ^\dagger)A^\dagger$. Moreover, $R^\dagger R = J^\dagger J$, and therefore $\varepsilon = B^\dagger(I - R^\dagger R) = B^\dagger(I - J^\dagger J)$, which completes the proof. ∎

The perturbation formulas can be used to explore the necessary and sufficient conditions needed for $(AB)^\dagger$ to be equal to Y, where Y denotes any of the convenient choices of expressions for the M-P of the product AB. This will be our task in the next section, but here we derive a general characterization result.

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$, $R = ABB^\dagger$, and $K = A^\dagger AB$. The following statements are equivalent:*

(a) $Y = (AB)^\dagger$.
(b) $\mathcal{C}(BY) \subseteq \mathcal{C}(BB^*A^*)$ and $\mathcal{C}(B^\dagger R^\dagger - Y) \subseteq \mathcal{C}\left(B^\dagger(I - R^\dagger R)\right)$.
(c) $\mathcal{R}(YA) \subseteq \mathcal{R}(B^*A^*A)$ and $\mathcal{R}(K^\dagger A^\dagger - Y) \subseteq \mathcal{R}\left((I - KK^\dagger)A^\dagger\right)$.

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Proof. (a) $\Leftrightarrow$ (b). Let $Y = (AB)^\dagger$. Then by Theorem 2.1, first identity in (2.1), $Y = (I - \varepsilon\varepsilon^\dagger)B^\dagger R^\dagger$, which according to part (i) of Lemma 1.2 is equivalent to

$$\text{(i)}\ \ \varepsilon^* Y = 0 \quad \text{and} \quad \text{(ii)}\ \ \mathcal{C}(B^\dagger R^\dagger - Y) \subseteq \mathcal{C}(\varepsilon), \tag{2.7}$$

where $\varepsilon = B^\dagger(I - R^\dagger R)$. Now (2.7)-(i) reduces to $(I - R^\dagger R)B^{*\dagger}Y = 0$ or $\mathcal{C}(B^{*\dagger}Y) \subseteq \mathcal{C}(R^*) = \mathcal{C}(BB^\dagger A^*)$. This in turn is equivalent to $\mathcal{C}(B^*B^{*\dagger}Y) \subseteq \mathcal{C}(B^*A^*)$, i.e.,

$$\text{(i)}\ \ \Leftrightarrow \mathcal{C}(BY) \subseteq \mathcal{C}(BB^*A^*) \quad \text{or} \quad \mathcal{R}(Y^*B^\dagger B) \subseteq \mathcal{R}(AB), \tag{2.8}$$

and thus, completing the proof of (a) $\Leftrightarrow$ (b).

(a) $\Leftrightarrow$ (c). Likewise by the first identity in (2.3), $Y = K^\dagger A^\dagger(I - \delta^\dagger\delta)$, which according to part (ii) of Lemma 1.2 is equivalent to

$$\text{(i)}\ Y\delta^* = 0 \quad \text{and} \quad \text{(ii)}\ \ \mathcal{R}(K^\dagger A^\dagger - Y) \subseteq \mathcal{R}(\delta), \tag{2.9}$$

where $\delta = (I - KK^\dagger)A^\dagger$. Now (2.9)-(i), reduces to $YA^{*\dagger}(I - KK^\dagger) = 0$ or $\mathcal{R}(YA^{*\dagger}) \subseteq \mathcal{R}(K^*) = \mathcal{R}(B^*A^\dagger A)$. This in turn is equivalent to $\mathcal{R}(YA^{*\dagger}A^*) \subseteq \mathcal{R}(B^*A^*)$, i.e.,

$$\text{(i)}\ \ \Leftrightarrow \mathcal{R}(YA) \subseteq \mathcal{R}(B^*A^*A) \quad \text{or} \quad \mathcal{C}(AA^\dagger Y^*) \subseteq \mathcal{C}(AB). \tag{2.10}$$

This concludes the proof that (a) $\Leftrightarrow$ (c). ∎

3. Forward order laws. In this section, we investigate the forward order law (FOL) $(AB)^\dagger = A^\dagger B^\dagger$. It is clear that both matrices must necessarily be square of the same size. For two invertible matrices the answer is precisely when $AB = BA$. In general the conditions are considerably more difficult since they must involve a generalization of commutativity. Our next aim is to apply the perturbation result to obtain a characterization in terms of A and B for the FOL of Moore-Penrose inverse to hold.

First we give two auxiliary results.

LEMMA 3.1. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

(i) $(AB)^\dagger = A^\dagger B^\dagger$.
(ii) $(BA)^* = A^*A(AB)^\dagger BB^*$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, *and* $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$.

Proof. (i) $\Rightarrow$ (ii). Suppose $(AB)^\dagger = A^\dagger B^\dagger$. Then

$$A^*B^* = A^*AA^\dagger(B^\dagger BB^*) = A^*A(AB)^\dagger BB^*.$$

Also $\mathcal{C}\left((AB)^*\right) = \mathcal{C}\left((AB)^\dagger\right) \subseteq \mathcal{C}(A^\dagger) \subseteq \mathcal{C}(A^*)$, and thus, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$. Likewise $\mathcal{R}\left((AB)^*\right) = \mathcal{R}\left((AB)^\dagger\right) \subseteq \mathcal{R}(B^\dagger) = \mathcal{R}(B^*)$, and thus, $(AB) \subseteq \mathcal{C}(B)$.

(ii) $\Rightarrow$ (i). The equality $A^*B^* = A^*A(AB)^\dagger BB^*$ is equivalent to

$$A^\dagger B^\dagger = A^\dagger A(AB)^\dagger BB^\dagger. \tag{3.1}$$

Next, from $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ we see that $BB^\dagger AB = AB$ and so $(AB)^*BB^\dagger = (AB)^*$ or $(AB)^\dagger BB^\dagger = (AB)^\dagger$. Likewise $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ ensures that $(AB)^\dagger = A^\dagger A(AB)^\dagger$. Substituting these in (3.1) reduces to the FOL. ∎

LEMMA 3.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Then, the following hold:*

$$\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB) \Leftrightarrow \mathcal{R}(AB) \subseteq \mathcal{R}(A) \quad \textit{and} \quad \mathcal{R}(BA) = \mathcal{R}(ABA^*A). \tag{3.2}$$

$$\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB) \Leftrightarrow \mathcal{C}(AB) \subseteq \mathcal{C}(B) \quad \textit{and} \quad \mathcal{C}(BA) = \mathcal{C}(BB^*AB). \tag{3.3}$$

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Proof. The first necessary condition in (3.2) is clear. Post-multiplying the matrices involved in $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$ by A^*A, we conclude the second part of the necessity. Conversely, post-multiplying the matrices involved in $\mathcal{R}(ABA^*A) = \mathcal{R}(BA)$ by $A^\dagger A^{\dagger *}$ we obtain $\mathcal{R}(ABA^*AA^\dagger A^{\dagger *}) = \mathcal{R}(BAA^\dagger A^{\dagger *})$ or, equivalently, $\mathcal{R}(ABA^\dagger A) = \mathcal{R}(BA^{\dagger *})$. Now, the assumption $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ yields $\mathcal{R}(AB) = \mathcal{R}(BA^{\dagger *})$.

The equivalence (3.3) is settled in a similar way. ∎

THEOREM 3.3. *Let $A, B \in \mathbb{C}^{n\times n}$ and $R = ABB^\dagger$. Then the following conditions are equivalent:*

(a) $(AB)^\dagger = A^\dagger B^\dagger$.
(b) $\mathcal{C}\left(B^\dagger(R^\dagger - BA^\dagger B^\dagger)\right) \subseteq \mathcal{C}[B^\dagger(I - R^\dagger R)]$ *and* $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$.
(c) $R^{(1,3)} = BA^\dagger B^\dagger$ *and* $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$.
(d) $(AB)^* = (AB)^*ABA^\dagger B^\dagger$ *and* $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$.
(e) $(AB)^*BB^* = (AB)^*ABA^\dagger B^*$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, *and* $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$.
(f) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$, *and* $\mathcal{R}([BA,\ BB^*AB]) \subseteq \mathcal{R}([A^*A,\ (AB)^*AB])$.
(g) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, *and* $\mathcal{R}([BA,\ BB^*AB]) \subseteq \mathcal{R}([A^*A(AB)^*,\ (AB)^*AB(AB)^*])$.

Proof. (a) ⇒ (b). Set $Y = A^\dagger B^\dagger$ in Theorem 2.4 (a) ⇔ (b). This gives $(AB)^\dagger = Y$ if and only if

$$\mathcal{C}(BA^\dagger B^\dagger) \subseteq \mathcal{C}(BB^*A^*) \quad \text{and} \quad \mathcal{C}(B^\dagger R^\dagger - A^\dagger B^\dagger) \subseteq \mathcal{C}\left(B^\dagger(I - R^\dagger R)\right). \tag{3.4}$$

From the latter it follows that $(I - B^\dagger B)(B^\dagger R^\dagger - A^\dagger B^\dagger) = 0$, which reduces to $(I - B^\dagger B)A^\dagger B^\dagger = 0$. Then we can write $B^\dagger R^\dagger - A^\dagger B^\dagger = B^\dagger(R^\dagger - BA^\dagger B^\dagger)$ and first condition in (b) follows. Since $(I - B^\dagger B)A^\dagger B^\dagger = 0$, then first condition in (3.4) reduces to $\mathcal{C}(A^\dagger B^\dagger) \subseteq \mathcal{C}(B^*A^*)$ or $\mathcal{R}(BA^{\dagger *}) \subseteq \mathcal{R}(AB)$.

(b) ⇒ (c). Let $X = BA^\dagger B^\dagger$. We will check that X is a $\{1,3\}$-inverse of R. From the first requirement in (b) it follows that $R(R^\dagger - BA^\dagger B^\dagger) = 0$ and, hence, $RR^\dagger = ABA^\dagger B^\dagger = RX$. Therefore RX is Hermitian. Now $RXR = RR^\dagger R = R$.

(c) ⇒ (d). By Theorem 2.1, first identity in (2.2), we have $(AB)(AB)^\dagger = RR^\dagger$. On account of the expression $R^\dagger = R^{(1,4)}RR^{(1,3)}$, we have $(AB)(AB)^\dagger = RR^\dagger = RR^{(1,3)}$. Substituting $R^{(1,3)} = BA^\dagger B^\dagger$ into latter equality leads to the condition $(AB)(AB)^\dagger = ABA^\dagger B^\dagger$. Pre-multiplying by $(AB)^*$ we get $(AB)^* = (AB)^*ABA^\dagger B^\dagger$, which is the desired result.

(d) ⇒ (e). First equality in (e) follows post-multiplying the equality $(AB)^* = (AB)^*ABA^\dagger B^\dagger$ by BB^*.

(e) ⇒ (a). We will show that the conditions (ii) in Lemma 3.1 are satisfied. From $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$ it follows that $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ and

$$\begin{aligned} A^\dagger B^* &= (AB)^\dagger(AB)A^\dagger B^* = (AB)^\dagger(AB)^{\dagger *}(AB)^*ABA^\dagger B^* \\ &= (AB)^\dagger(AB)^{\dagger *}(AB)^*BB^* = (AB)^\dagger BB^*, \end{aligned} \tag{3.5}$$

where the third equality follows from the assumption $(AB)^*BB^* = (AB)^*ABA^\dagger B^*$. Then $A^*B^* = A^*A(AB)^\dagger BB^*$.

(e) ⇒ (f). It remains to prove that $\mathcal{R}([BA,\ BB^*AB]) \subseteq \mathcal{R}([A^*A,\ (AB)^*AB])$. Since $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, the equality $(AB)^*BB^* = (AB)^*ABA^\dagger B^*$ can be written as

$$\begin{bmatrix} -(AB)^*AB(A^*A)^\dagger & A^\dagger A \end{bmatrix} \begin{bmatrix} (BA)^* \\ (AB)^*BB^* \end{bmatrix} = 0. \tag{3.6}$$

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Let $T \in \mathbb{C}^{n\times 2n}$ denote the matrix $T = \begin{bmatrix} -(AB)^*AB(A^*A)^\dagger & A^\dagger A \end{bmatrix}$. It can be easily verified that $T^- = \begin{bmatrix} 0 \\ A^\dagger A \end{bmatrix}$ is an inner inverse of T and

$$I - T^-T = \begin{bmatrix} I & 0 \\ (AB)^*AB(A^*A)^\dagger & I - A^\dagger A \end{bmatrix}.$$

On account of $\mathcal{N}(T) = \mathcal{C}(I - T^-T)$ and $A^\dagger A(AB)^* = (AB)^*$, if the condition (3.6) is fulfilled then $\mathcal{C}\left(\begin{bmatrix} (BA)^* \\ (AB)^*BB^* \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} I \\ (AB)^*AB(A^*A)^\dagger \end{bmatrix}\right)$. Applying now $A^\dagger A$ on the left leads to $\mathcal{C}\left(\begin{bmatrix} (BA)^* \\ (AB)^*BB^* \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} A^*A \\ (AB)^*AB \end{bmatrix}(A^*A)^\dagger\right)$, which shows the inclusion $\mathcal{R}\left([BA,\ BB^*AB]\right) \subseteq \mathcal{R}\left([A^*A,\ (AB)^*AB]\right)$.

(f) $\Rightarrow$ (g). From the third condition in (f) it follows that for any $\mathbf{x} \in \mathbb{C}^n$ there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\begin{bmatrix} (BA)^* \\ (AB)^*BB^* \end{bmatrix}\mathbf{x} = \begin{bmatrix} A^*A \\ (AB)^*AB \end{bmatrix}\mathbf{u}$. Hence, $(BA)^*\mathbf{x} = A^*A\mathbf{u}$ or $A^\dagger B^*\mathbf{x} = A^\dagger A\mathbf{u}$. Now, the requirement $\mathcal{R}(BA^{\dagger *}) = \mathcal{R}(AB)$ implies that $ABA^\dagger A = AB$ and there exists $\mathbf{z} \in \mathbb{C}^n$ such that $A^\dagger B^*\mathbf{x} = (AB)^*\mathbf{z}$. Consequently,

$$\begin{bmatrix} (BA)^* \\ (AB)^*BB^* \end{bmatrix}\mathbf{x} = \begin{bmatrix} A^*A \\ (AB)^*AB \end{bmatrix}(AB)^*\mathbf{z}, \tag{3.7}$$

which shows that $\mathcal{R}\left([BA,\ BB^*AB]\right) \subseteq \mathcal{R}\left([A^*A(AB)^*,\ (AB)^*AB(AB)^*]\right)$.

(g) $\Rightarrow$ (e). From the third condition in (g) it follows that for any $\mathbf{x} \in \mathbb{C}^n$ there exists $\mathbf{z} \in \mathbb{C}^n$ such that (3.7) holds. Furthermore, under the assumption $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, we obtain

$$\left((AB)^*BB^* - (AB)^*ABA^\dagger B^*\right)\mathbf{x} = \begin{bmatrix} (AB)^*AB(A^*A)^\dagger & A^\dagger A \end{bmatrix}\begin{bmatrix} A^*A \\ (AB)^*AB \end{bmatrix}(AB)^*\mathbf{z} = \mathbf{0},$$

showing that $(AB)^*BB^* = (AB)^*ABA^\dagger B^*$. In view of (3.2), it remains to show that $\mathcal{R}(BA) = \mathcal{R}(ABA^*A)$ is fulfilled. But (3.7) clearly implies that $\mathcal{R}(BA) \subseteq \mathcal{R}(ABA^*A)$, which, on account of rank equality, concludes the proof. ∎

The following equivalences follow by left-right symmetry.

THEOREM 3.4. *Let $A, B \in \mathbb{C}^{n\times n}$ and $K = A^\dagger AB$. Then the following statements are equivalent:*

(a) $(AB)^\dagger = A^\dagger B^\dagger$.
(b) $\mathcal{R}[(K^\dagger - A^\dagger B^\dagger A)A^\dagger] \subseteq \mathcal{R}[(I - KK^\dagger)A^\dagger]$ *and* $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB)$.
(c) $K^{(1,4)} = A^\dagger B^\dagger A$ *and* $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB)$.
(d) $AB = AB(AB)^*B^{\dagger *}A^{\dagger *}$ *and* $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB)$.
(e) $ABA^*A = AB(AB)^*B^{\dagger *}A$, $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB)$, *and* $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.
(f) $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, *and* $\mathcal{C}\left(\begin{bmatrix} BA \\ ABA^*A \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} BB^* \\ AB(AB)^* \end{bmatrix}\right)$.
(g) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, *and* $\mathcal{C}\left(\begin{bmatrix} BA \\ ABA^*A \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} BB^*AB \\ AB(AB)^*AB \end{bmatrix}\right)$.

We now derive some necessary and sufficient conditions for the forward order law on commuting matrices.

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Corollary 3.5. *Let $A, B \in \mathbb{C}^{n\times n}$ such that $AB = BA$. Then the following statements are equivalent:*

(a) $(AB)^\dagger = A^\dagger B^\dagger$.
(b) $\mathcal{R}(AB) \subseteq \mathcal{R}(ABA^*A)$ *and* $\mathcal{R}((AB)^*B) \subseteq (\mathcal{N}(B) \cap \mathcal{N}(A^*)) \oplus \mathcal{C}(A)$.
(c) $\mathcal{R}([BA,\ BB^*AB]) \subseteq \mathcal{R}([A^*A(AB)^*,\ (AB)^*AB(AB)^*])$.
(d) $\mathcal{C}(AB) \subseteq \mathcal{C}(BB^*AB)$ *and* $\mathcal{C}(A(AB)^*) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B)) \oplus \mathcal{C}(B^*)$.
(e) $\mathcal{C}\left(\begin{bmatrix} BA \\ ABA^*A \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} BB^*AB \\ AB(AB)^*AB \end{bmatrix}\right)$.

Proof. (a) $\Leftrightarrow$ (b). This equivalence follows from the part (a) $\Leftrightarrow$ (e) of Theorem 3.3 combined with (3.2). Indeed, if $AB = BA$, then the equality in Theorem 3.3 (e) reduces to $(AB)^*B(I - AA^\dagger)B^* = 0$ or, equivalently, $\mathcal{R}((AB)^*B) \subseteq \mathcal{N}((I - AA^\dagger)B)$, which according to Lemma 1.3 shows that $\mathcal{R}((AB)^*B) \subseteq (\mathcal{N}(B) \cap \mathcal{N}(A^*)) \oplus \mathcal{C}(A)$.

(a) $\Leftrightarrow$ (c). This equivalence is immediate from the part (a) $\Leftrightarrow$ (e) of Theorem 3.3.

(a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). These equivalences follow by left-right symmetry from (a) $\Leftrightarrow$ (e) $\Leftrightarrow$ (g) of Theorem 3.4. $\square$

Corollary 3.6. *Let $A, B \in \mathbb{C}^{n\times n}$ such that $A^*B = BA^*$. Then the following statements are equivalent:*

(a) $(AB)^\dagger = A^\dagger B^\dagger$.
(b) $(AB)^*(BA - AB)(AB)^* = 0$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, *and* $\mathcal{R}(BA) = \mathcal{R}(A^*BA)$.
(c) $AB((BA)^* - (AB)^*)AB = 0$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, *and* $\mathcal{C}(BA) = \mathcal{C}(BAB^*)$.

Proof. (a) $\Leftrightarrow$ (b). This equivalence follows from the part (a) $\Leftrightarrow$ (e) of Theorem 3.3. Since $A^*B = BA^*$ and $A^* = A^*AA^\dagger$, the equality $(AB)^*BB^* = (AB)^*ABA^\dagger B^*$ in Theorem 3.3 (e) reduces to

$$(B^*BA^*A - (AB)^*AB)A^\dagger B^* = 0,$$

which on account of the identity $\mathcal{C}(A^\dagger B^*) = \mathcal{C}((AB)^*)$ is equivalent to

$$(AB)^*(BA - AB)(AB)^* = 0.$$

Moreover, we also have $B^*A = AB^*$, which leads to $\mathcal{R}(ABA^*A) = \mathcal{R}(A^*BA)$. Hence, by (3.2) the requirement $\mathcal{C}(A^\dagger B^*) = \mathcal{C}((AB)^*)$ can be replaced by $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(BA) = \mathcal{R}(A^*BA)$.

(a) $\Leftrightarrow$ (c). This equivalence follows by left-right symmetry from the part (a) $\Leftrightarrow$ (e) of Theorem 3.4. $\square$

We note that if $AB = BA$ and $A^*B = BA^*$, then all conditions in part (b) and (c) of Corollaries 3.5 and 3.6 are fulfilled and, thus, $(AB)^\dagger = A^\dagger B^\dagger = B^\dagger A^\dagger$ and both the forward order law and reverse order law hold. In particular if $A^* = A$ and $AB = BA$ this follows.

When A or B is invertible we may give several characterizations for the FOL to hold.

Theorem 3.7. *Let $A, B \in \mathbb{C}^{n\times n}$. If A is invertible, then the following statements are equivalent:*

(a) $(AB)^\dagger = A^{-1}B^\dagger$.
(b) $AB^* = B^\dagger(AB)B^*$ *and* $\mathcal{C}(AB) = \mathcal{C}(B)$.
(c) $(BA)(AB)^* = AB(AB)^*$, $\mathcal{C}(AB) = \mathcal{C}(B)$, *and* $\mathcal{C}(AB^*) = \mathcal{C}(B^*)$.
(d) $A^{-1}B = BA^{-1}B^\dagger B$, $\mathcal{C}(AB) = \mathcal{C}(B)$, *and* $\mathcal{C}(AB^*) = \mathcal{C}(B^*)$.
(e) $BB^\dagger A^{-1}B = BA^{-1}B^\dagger B$, $\mathcal{C}(AB) = \mathcal{C}(B)$, *and* $\mathcal{C}(AB^*) = \mathcal{C}(B^*)$.

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Proof. (a) $\Leftrightarrow$(b). If A is invertible then $\mathcal{C}(B^{\dagger *}A) = \mathcal{C}(B)$ and the equality $AB = AB(AB)^*B^{\dagger *}A^{*-1}$ is equivalent to $BA^* = B(AB)^*B^{\dagger *}$, or, $AB^* = B^{\dagger}(AB)B^*$. Consequently, this equivalence follows by Theorem 3.4, equivalence between (a) and (d).

(b) $\Leftrightarrow$ (c). The requirement that the first equality in (b) holds is equivalent to

$$B^{\dagger}BAB^* = B^{\dagger}(AB)B^* \quad \text{and} \quad (I - B^{\dagger}B)AB^* = 0. \tag{3.8}$$

If we assume that $\mathcal{C}(AB) = \mathcal{C}(B)$, then the first equality in (3.8) is equivalent to $BAB^* = ABB^*$. Multiplying this with A^* on the right, yields $(BA)(AB)^* = (AB)(AB)^*$ as desired. On account of part (a) of Lemma 1.2, the second condition in (3.8) is equivalent to $\mathcal{C}(AB^*) \subseteq \mathcal{C}(B^*)$. This concludes the proof that (b) $\Leftrightarrow$ (c).

(c) $\Leftrightarrow$ (d). The first equality in (c) reduces to $BAB^* = ABB^*$, which is equivalent to $(A^{-1}B - BA^{-1})AB^* = 0$. The latter is equivalent to $A^{-1}B - BA^{-1}BB^{\dagger} = 0$ under the assumption that $\mathcal{C}(AB^*) = \mathcal{C}(B^*)$.

(d) $\Leftrightarrow$ (e). The first equality in (d) is equivalent to $BB^{\dagger}A^{-1}B = BA^{-1}B^{\dagger}B$ under the assumption that $\mathcal{C}(AB) = \mathcal{C}(B)$ or, equivalently, $\mathcal{C}(B) = \mathcal{C}(A^{-1}B)$. This completes the proof. ∎

By symmetry, we have the analog of Theorem 3.7 in the case that B is invertible.

It can be pointed out that the part (a) $\Leftrightarrow$ (e) in Theorems 3.3 or 3.4 leads to quite interesting characterization when either A or B belongs to the class of orthogonal projectors. If both A and B are orthogonal projectors, by the part (a) $\Leftrightarrow$ (d) in Theorems 3.3, it follows that $(AB)^{\dagger} = AB$ if and only if $\mathcal{C}(AB) = \mathcal{C}(BA)$, because the first equality in (d) takes the form $BA = (BA)^2B$ and it is redundant under the requirement that $\mathcal{C}(AB) = \mathcal{C}(BA)$. According with Arghiriade's result shown in (1.2), $(AB)^{\dagger} = AB$ if and only if the reverse order law $(AB)^{\dagger} = BA$ holds.

4. Reverse order laws. From our perturbation results we readily obtain a variety of necessary and sufficient conditions for several reverse order laws to hold.

We start by examining the two cases where $(AB)^{\dagger} = B^{\dagger}R^{\dagger}$ and $(AB)^{\dagger} = K^{\dagger}A^{\dagger}$.

COROLLARY 4.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $R = ABB^{\dagger}$. Then*

$$(AB)^{\dagger} = B^{\dagger}R^{\dagger} \Leftrightarrow \mathcal{R}(AB^{*\dagger}) \subseteq \mathcal{R}(AB) \Leftrightarrow \mathcal{R}(AB) \subseteq \mathcal{R}(ABB^*B).$$

Proof. From Theorem 2.4 (a) $\Leftrightarrow$ (b), we see that $(AB)^{\dagger} = B^{\dagger}R^{\dagger}$ if and only if $\mathcal{C}(BB^{\dagger}R^{\dagger}) \subseteq \mathcal{C}(BB^*A^*)$, which can be reduced to $\mathcal{C}(BB^{\dagger}A^*) \subseteq \mathcal{C}(BB^*A^*)$, or equivalently to $\mathcal{R}(AB) \subseteq \mathcal{R}(ABB^*B)$. On the other hand, if we replace the range condition by the row space condition in (2.8) get that $(AB)^{\dagger} = B^{\dagger}R^{\dagger}$ if and only if $\mathcal{R}((B^{\dagger}R^{\dagger})^*B^{\dagger}B) \subseteq \mathcal{R}(AB)$, which reduces to $\mathcal{R}(RB^{*\dagger}) \subseteq \mathcal{R}(AB)$, i.e., $\mathcal{R}(AB^{*\dagger}) \subseteq \mathcal{R}(AB)$. ∎

COROLLARY 4.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $K = A^{\dagger}AB$. Then*

$$(AB)^{\dagger} = K^{\dagger}A^{\dagger} \Leftrightarrow \mathcal{C}(A^{*\dagger}B) \subseteq \mathcal{C}(AB) \Leftrightarrow \mathcal{C}(AB) \subseteq \mathcal{C}(AA^*AB)).$$

Proof. From Theorem 2.4 (a) $\Leftrightarrow$ (c), we see that $(AB)^{\dagger} = K^{\dagger}A^{\dagger}$ if and only if $\mathcal{C}(A^{*\dagger}K^{*\dagger}) \subseteq \mathcal{C}(AB)$, which reduces to $\mathcal{C}(A^{*\dagger}B) \subseteq \mathcal{C}(AB)$. On the other hand, if we replace the range condition by the row space condition in (2.10) we get that $(AB)^{\dagger} = K^{\dagger}A^{\dagger}$ if and only if $\mathcal{R}(K^{\dagger}A^{\dagger}A) \subseteq \mathcal{R}(B^*A^*A)$, which reduces to $\mathcal{R}(B^*A^{\dagger}A) \subseteq \mathcal{R}(B^*A^*A)$, i.e., $\mathcal{C}(AB) \subseteq \mathcal{C}(AA^*AB)$. ∎

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In [18], Tian studied when the expression for $(AB)^\dagger$ is

$$(AB)^\dagger = B^\dagger A^\dagger - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger. \tag{4.1}$$

Necessary and sufficient conditions for this equality to hold were established in Theorems 1 and 8 in [18]. It has been showed the equivalence between (4.1) and the mixed type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$. A characterization to this law is proved here. Our proof is based on the perturbation formula given in Corollary 2.3, while the proof given in [18] involves block matrix decompositions and rank formulas.

Corollary 4.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and $J = A^\dagger ABB^\dagger$. Then*

$$(AB)^\dagger = B^\dagger J^\dagger A \Leftrightarrow \mathcal{C}(AB) \subseteq \mathcal{C}(AA^*AB) \quad \textit{and} \quad \mathcal{R}(AB) \subseteq \mathcal{R}(ABB^*B).$$

Proof. By Corollary 2.3, $(AB)^\dagger = B^\dagger J^\dagger A$ if and only if

$$\varepsilon\varepsilon^\dagger B^\dagger J^\dagger A^\dagger(I - \sigma^\dagger\sigma) + B^\dagger J^\dagger A^\dagger \sigma^\dagger\sigma = 0,$$

which in turn is equivalent to

$$\text{(i) } B^\dagger J^\dagger A^\dagger \sigma^\dagger \sigma = 0, \quad \text{(ii) } \varepsilon\varepsilon^\dagger B^\dagger J^\dagger A^\dagger(I - \sigma^\dagger\sigma) = 0.$$

Since $BB^\dagger J^\dagger = J^\dagger$, it follows that (i) is equivalent to $B^*A^\dagger\sigma^* = 0$ or $B^*A^\dagger A^{\dagger *}(I-JJ^\dagger) = 0$. By Lemma 1.2, the last condition holds if and only if $\mathcal{R}(B^*A^\dagger A^{\dagger *}) \subseteq \mathcal{R}(J^*) = \mathcal{R}(BB^\dagger A^\dagger A) = \mathcal{R}(B^*A^\dagger A)$. Then (i) holds if and only if $\mathcal{R}(B^*A^\dagger) \subseteq \mathcal{R}(B^*A^*)$, which is equivalent to $\mathcal{R}(B^*A^*) \subseteq \mathcal{R}(B^*A^*AA^*)$ or $\mathcal{C}(AB) \subseteq \mathcal{C}(AA^*AB)$.

Now, in view of (2.6) we have that (ii) is equivalent to $\varepsilon^* B^\dagger R^\dagger = 0$ or $(I - J^\dagger J)B^{\dagger *}B^\dagger A^* = 0$. By Lemma 1.2, this holds if and only if $\mathcal{C}(B^{\dagger *}B^\dagger A^*) \subseteq \mathcal{C}(J) = \mathcal{C}(BB^\dagger A^\dagger A) = \mathcal{C}(BB^\dagger A^*)$, which is equivalent to $\mathcal{C}(B^*A^*) \subseteq \mathcal{C}(B^*BB^*A^*)$ or $\mathcal{R}(AB) \subseteq \mathcal{R}(ABB^*B)$. □

The next well known result will be needed in the proof of Theorem 4.5.

Lemma 4.4. *Let F and G be two orthogonal projectors of a same order. Then*

$$(GF)^\dagger = FG \Leftrightarrow GF = FG \Leftrightarrow GFG = FG \Leftrightarrow (GF)^2 = GF.$$

Now we derive the perturbation conditions under which the reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ holds.

Theorem 4.5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $R = ABB^\dagger$, and $K = A^\dagger AB$. The following statements are equivalent:*

(a) $(AB)^\dagger = B^\dagger A^\dagger$.
(b) $\mathcal{C}(BB^\dagger A^*) = \mathcal{C}(BB^*A^*)$ *and* $\mathcal{C}\left(R^\dagger - BB^\dagger A^\dagger\right) \subseteq \mathcal{C}\left((I - R^\dagger R)BB^\dagger\right)$.
(c) $R^\dagger = BB^\dagger A^\dagger$ *and* $\mathcal{C}(BB^\dagger A^*) = \mathcal{C}(BB^*A^*)$.
(d) $\mathcal{C}(A^\dagger AB) = \mathcal{C}(A^*AB)$ *and* $\mathcal{R}(K^\dagger - B^\dagger A^\dagger A) \subseteq \mathcal{R}\left((I - KK^\dagger)A^\dagger A\right)$.
(e) $K^\dagger = B^\dagger A^\dagger A$ *and* $\mathcal{C}(A^\dagger AB) = \mathcal{C}(A^*AB)$.
(f) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$ *and* $\mathcal{C}(A^\dagger ABB^\dagger) \subseteq \mathcal{C}(A^*AB) \cap \mathcal{C}(BB^*A^*)$.
(g) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$ *and* $\mathcal{C}(A^\dagger ABB^\dagger) \subseteq \mathcal{C}(A^*AB) \cap \mathcal{C}(BB^*A^*)$.

Proof. (a) $\Leftrightarrow$ (b). Set $Y = B^\dagger A^\dagger$ in Theorem 2.4 (a)$\Leftrightarrow$(b). This gives

$$\mathcal{C}(BB^\dagger A^*) \subseteq \mathcal{C}(BB^*A^*) \quad \text{or} \quad \mathcal{R}(AB) \subseteq \mathcal{R}(ABB^*),$$

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in addition to $\mathcal{C}\left(B^\dagger(R^\dagger - A^\dagger)\right) \subseteq \mathcal{C}\left(B^\dagger(I - R^\dagger R)\right)$. The latter is equivalent to

$$\mathcal{C}\left(BB^\dagger(R^\dagger - A^\dagger)\right) \subseteq \mathcal{C}\left(BB^\dagger(I - R^\dagger R)\right),$$

in which we use the fact that $BB^\dagger R^\dagger = R^\dagger$, and we then arrive at $\mathcal{C}\left(R^\dagger - BB^\dagger A^\dagger\right) \subseteq \mathcal{C}\left((I - R^\dagger R)BB^\dagger\right)$. On account of rank equality we may say that $\mathcal{C}(BB^\dagger A^*) = \mathcal{C}(BB^* A^*)$.

(b) $\Leftrightarrow$ (c). From $\mathcal{C}\left(R^\dagger - BB^\dagger A^\dagger\right) \subseteq \mathcal{C}\left((I - R^\dagger R)BB^\dagger\right)$ it follows that $R(R^\dagger - BB^\dagger A^\dagger) = 0$. On the other hand, we also obtain $(I - R^\dagger R)(R^\dagger - BB^\dagger A^\dagger) = (I - R^\dagger R)BB^\dagger A^\dagger = 0$ because $\mathcal{C}(R^*) = \mathcal{C}(BB^\dagger A^*)$. Therefore, $R^\dagger = BB^\dagger A^\dagger$. The converse part is clear.

(a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). These equivalences follow by symmetry.

(a) $\Rightarrow$ (f). From Corollary 2.3 it follows that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if

$$B^\dagger A^\dagger = (I - \varepsilon\varepsilon^\dagger)B^\dagger J^\dagger A^\dagger(I - \sigma^\dagger\sigma), \tag{4.2}$$

where $J = A^\dagger ABB^\dagger$, $\varepsilon = B^\dagger(I - J^\dagger J)$, and $\sigma = (I - JJ^\dagger)A^\dagger$. Pre-multiplying (4.2) by $A^\dagger AB$ and post-multiplying it by $ABB^\dagger$ we obtain $J^2 = J$. Now, Lemma 4.4 asserts that $J^\dagger = BB^\dagger A^\dagger A$ and, thus, the first identity in (f) holds.

On the other hand, from the equivalence of (a) and (b) it follows that $\mathcal{C}(BB^\dagger A^*) = \mathcal{C}(BB^* A^*)$ or, equivalently, $\mathcal{C}(BB^\dagger A^\dagger A) = \mathcal{C}(BB^* A^*)$, while the equivalence of (a) and (d) implies that $\mathcal{C}(A^\dagger AB) = \mathcal{C}(A^* AB)$ or, equivalently, $\mathcal{C}(A^\dagger ABB^\dagger) = \mathcal{C}(A^* AB)$. We now recall Lemma 4.4, which tells us that $BB^\dagger A^\dagger A = A^\dagger ABB^\dagger$, to conclude $\mathcal{C}(A^\dagger ABB^\dagger) \subseteq \mathcal{C}(A^* AB) \cap \mathcal{C}(BB^* A^*)$.

(f) $\Rightarrow$ (g). This implication is clear.

(g) $\Rightarrow$ (a). We will prove that the Arghiriade requirement for the FOL shown in (1.2) holds. From (g) it follows that $\mathcal{C}(A^\dagger AB) = \mathcal{C}(BB^\dagger A^*) \subseteq \mathcal{C}(A^* AB) \cap \mathcal{C}(BB^* A^*)$. On account of rank equality we conclude $\mathcal{C}(A^* AB) = \mathcal{C}(BB^* A^*)$ or, equivalently, $\mathcal{C}(A^* ABB^*) = \mathcal{C}(BB^* A^* A)$. ∎

When we have the product AF where F is an orthogonal projector we obtain the following useful result.

Lemma 4.6. *If F is an orthogonal projector, then the following are equivalent:*

(i) $(AF)^\dagger = FA^\dagger$. (ia) $[A(I - F)]^\dagger = (I - F)A^\dagger$.
(ii) $A^\dagger AF = FA^\dagger A$ *and* $\mathcal{C}(A^\dagger AF) \subseteq \mathcal{C}(A^* AF)$.
(iii) $A^\dagger AF = FA^\dagger A$ *and* $AFA^\dagger$ *is Hermitian.*
(iv) $A^* AF = FA^* A$. (iva) $A^* A(I - F) = (I - F)A^* A$.

In which case, $AFA^\dagger = AA^\dagger - \delta^\dagger\delta$ *with* $\delta = (I - F)A^\dagger$.

Proof. By Theorem 4.5, equivalence between (a) and (g), it follows that (i) $\Leftrightarrow$ (ii). The equivalence between (ii) and (iii) is clear.

(ii) $\Leftrightarrow$ (iv). If $\mathcal{C}(A^\dagger AF) \subseteq \mathcal{C}(A^* AF)$ then $\mathcal{C}\left((A^* A)^\dagger F\right) \subseteq \mathcal{C}(A^\dagger AF)$. If $A^\dagger A$ and F commute then $A^\dagger AF$ is idempotent and $(A^* A)^\dagger F = (A^\dagger AF)(A^* A)^\dagger F = FA^\dagger A(A^* A)^\dagger F = F(A^* A)^\dagger F = F(A^* A)^\dagger$. It thus follows that $A^* A$ and F also commute, which prove the necessity. Conversely, pre-multiplying the equality (iv) by $(A^* A)^\dagger$ we obtain $A^\dagger AF = (A^* A)^\dagger FA^* A$. Hence, $\mathcal{C}(A^\dagger AF) \subseteq \mathcal{C}(A^* AF)$ and also we get $A^\dagger AF = FA^\dagger A$ because $(A^* A)^\dagger F = F(A^* A)^\dagger$.

Finally, by Theorem 2.1 (2.4), it follows that $AFA^\dagger = AF(AF)^\dagger = A^\dagger A - \delta^\dagger\delta$, where $\delta = (I - A^\dagger AF(A^\dagger AF)^\dagger)^\dagger A = (I - F)A^\dagger$. ∎

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