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EXPLICIT BLOCK-STRUCTURES FOR BLOCK-SYMMETRIC FIEDLER-LIKE PENCILS*

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Abstract. In the last decade, there has been a continued effort to produce families of strong linearizations of a matrix polynomial $P(\lambda)$, regular and singular, with good properties, such as, being companion forms, allowing the recovery of eigenvectors of a regular $P(\lambda)$ in an easy way, allowing the computation of the minimal indices of a singular $P(\lambda)$ in an easy way, etc. As a consequence of this research, families such as the family of Fiedler pencils, the family of generalized Fiedler pencils (GFP), the family of Fiedler pencils with repetition, and the family of generalized Fiedler pencils with repetition (GFPR) were constructed. In particular, one of the goals was to find in these families structured linearizations of structured matrix polynomials. For example, if a matrix polynomial $P(\lambda)$ is symmetric (Hermitian), it is convenient to use linearizations of $P(\lambda)$ that are also symmetric (Hermitian). Both the family of GFP and the family of GFPR contain block-symmetric linearizations of $P(\lambda)$, which are symmetric (Hermitian) when $P(\lambda)$ is. Now the objective is to determine which of those structured linearizations have the best numerical properties. The main obstacle for this study is the fact that these pencils are defined implicitly as products of so-called elementary matrices. Recent papers in the literature had as a goal to provide an explicit block-structure for the pencils belonging to the family of Fiedler pencils and any of its further generalizations to solve this problem. In particular, it was shown that all GFP and GFPR, after permuting some block-rows and block-columns, belong to the family of extended block Kronecker pencils, which are defined explicitly in terms of their block-structure. Unfortunately, those permutations that transform a GFP or a GFPR into an extended block Kronecker pencil do not preserve the block-symmetric structure. Thus, in this paper, the family of block-minimal bases pencils, which is closely related to the family of extended block Kronecker pencils, and whose pencils are also defined in terms of their block-structure, is considered as a source of canonical forms for block-symmetric pencils. More precisely, four families of block-symmetric pencils which, under some generic nonsingularity conditions are block minimal bases pencils and strong linearizations of a matrix polynomial, are presented. It is shown that the block-symmetric GFP and GFPR, after some row and column permutations, belong to the union of these four families. Furthermore, it is shown that, when $P(\lambda)$ is a complex matrix polynomial, any block-symmetric GFP and GFPR is permutationally congruent to a pencil in some of these four families. Hence, these four families of pencils provide an alternative but explicit approach to the block-symmetric Fiedler-like pencils existing in the literature.

Key words. Fiedler pencil, Block-symmetric generalized Fiedler pencil, Block-symmetric generalized Fiedler pencil with repetition, Matrix polynomial, Strong linearization, Symmetric strong linearization, Block Kronecker pencil, Extended block Kronecker pencil, Block minimal bases pencil.

AMS subject classifications. 65F15, 15A18, 15A22, 15A54.

1. Introduction.

The standard approach to numerically solving a *polynomial eigenvalue problem (PEP)* associated with a matrix polynomial (whose matrix coefficients have entries in a field $\mathbb{F}$) of the

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form

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i, \quad \text{with } A_0, A_1, \ldots, A_k \in \mathbb{F}^{m\times n}, \tag{1.1}$$

starts by embedding the coefficients of $P(\lambda)$ into a matrix pencil (that is, a matrix polynomial of grade equal to 1). This process is known as *linearization*, and it transforms the given PEP into a *generalized eigenvalue problem (GEP).* Then, the obtained GEP can be solved by using the QZ algorithm [25] or the staircase algorithm [28, 29], for example.

The literature on linearizations is huge as can be seen, for example, by counting all the references in [5] concerning this topic. The best well-known examples of linearizations of a matrix polynomial $P(\lambda)$ as in (1.1) are the so-called *Frobenius companion forms* given by

$$\begin{bmatrix} \lambda A_k + A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & \lambda I_n & & \\ & \ddots & \ddots & \\ & & -I_n & \lambda I_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda A_k + A_{k-1} & -I_m & & \\ A_{k-2} & \lambda I_m & \ddots & \\ \vdots & & \ddots & -I_m \\ A_0 & & & \lambda I_m \end{bmatrix}.$$

We note that here and throughout the paper, we sometimes omit the block-entries of a matrix polynomial that are equal to zero as we have done above. The algorithm QZ implemented in Matlab to solve the PEP uses the first Frobenius companion form as a linearization by default.

Frobenius companion forms have many desirable properties from a numerical point of view, as i) they are constructed from the matrix coefficients of $P(\lambda)$ without performing any arithmetic operations; ii) they are strong linearizations of $P(\lambda)$ regardless of whether $P(\lambda)$ is regular or singular [11, 13]; iii) the minimal indices of $P(\lambda)$ are related with the minimal indices of the Frobenius companion forms by uniform shifts [10, 11]; iv) eigenvectors of regular matrix polynomials and minimal bases of singular matrix polynomials are easily recovered from those of the Frobenius companion forms [11]; and v) solving PEP's by applying a backward stable eigensolver to the Frobenius companion forms is backward stable [15, 29]. Nonetheless, solving a PEP by solving the GEP associated with a Frobenius companion form presents some significant drawbacks. For instance, if the matrix polynomial $P(\lambda)$ is symmetric (Hermitian), that is $P(\lambda)^T = P(\lambda)$ ($\mathbb{F} = \mathbb{C}$ and $P(\lambda)^* = P(\lambda)$), neither of the Frobenius companion forms is symmetric (Hermitian). Since the preservation of the structure has been recognized as key for obtaining better (and physically more meaningful) numerical results [23], this drawback has motivated an intense research on structure-preserving linearizations; see, for example [2, 3, 4, 6, 7, 12, 17, 23, 26, 30], to name a few recent references on this topic. There are many papers in the literature addressing the problem of constructing symmetric (Hermitian) strong linearizations of symmetric (Hermitian) matrix polynomials. Most of these papers approach the problem by constructing first block-symmetric strong linearizations, as it is done, for example, in [3, 4, 21, 23].

Among the block-symmetric linearizations in the literature, it has been shown that, within the vector space of block-symmetric pencils $\mathbb{DL}(P)$ [21, 22], the first and last pencils in its standard basis, denoted by $D_1(\lambda, P)$ and $D_k(\lambda, P)$, respectively, have almost optimal behavior in terms of conditioning and backward error when used to compute an eigenvalue δ of $P(\lambda)$, as long as $|\delta| \geq 1$ if $D_1(\lambda, P)$ is used or $|\delta| \leq 1$ if $D_k(\lambda, P)$ is used [20, 27]. A natural question is whether a single block-symmetric linearization can be found with good conditioning and backward error regardless of the modulus of δ or if any block-symmetric linearizations outside $\mathbb{DL}(P)$ present a better numerical behavior than $D_1(\lambda, P)$ and $D_k(\lambda, P)$. One possible

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approach to answering these questions consists in replacing some nonzero blocks of the form $\pm A_i$ in the matrix coefficients of these pencils (which can be seen as block-matrices whose blocks are of the form 0, $\pm I_n$, and $\pm A_i$) by zero or identity blocks. But in order to do that, it is necessary to identify which of those blocks are essential to keep a given linearization a linearization of $P(\lambda)$ as they are replaced by zero or identity blocks. Thus, an explicit block-structure of the well-known block-symmetric pencils in the literature that allows to determine easily if they are a linerization of $P(\lambda)$ or not can be useful, for example, to accomplish this goal. In this paper, we focus on the block-symmetric pencils in the families of Fiedler-like pencils presented in [3, 4], which are known as *block-symmetric generalized Fiedler pencils (block-symmetric GFP)* and *block-symmetric generalized Fiedler pencils with repetition (block-symmetric GFPR)*. The block-symmetric GFP are strong linearizations of any $P(\lambda)$. The block-symmetric GFPR are strong linearizations of $P(\lambda)$ modulo some generic nonsingularity conditions. Moreover, all block-symmetric GFP and GFPR are symmetric (Hermitian) when $P(\lambda)$ is. Furthermore, they share some of the desirable properties of the Frobenius companion forms mentioned above. The main disadvantage of these pencils is that they were defined implicitly in terms of products of elementary matrices, which makes it difficult to study their algebraic and numerical properties. Thus, identifying their block structure might solve some of these difficulties.

The family of block minimal bases pencils was recently constructed with the goal of performing a backward stability analysis of PEP's when solved by linearization [15]. These pencils are defined by their explicit block-structure. Moreover, it has been shown that, modulo some generic nonsingularity conditions, Fiedler pencils, generalized Fiedler pencils, Fiedler pencils with repetition (and, thus, the standard basis of the $\mathbb{DL}(P)$ space), and generalized Fiedler pencils with repetition are *permutationally equivalent* to block minimal bases pencils [5, 15]. However, none of these results takes into account any extra structural properties that these pencils might possess. For example, given a block-symmetric GFPR, the results in [5, 15] do not guarantee that this pencil is *permutationally block-congruent*[1] to a block-symmetric block minimal bases pencil. The focus of this paper is not on constructing new families of block-symmetric pencils but on identifying a family of block-symmetric pencils, that under some generic nonsingularity conditions are block minimal bases pencils, and showing that the block-symmetric GFP and the block-symmetric GFPR are permutationally block-congruent to a pencil in that family. This family of block-symmetric minimal bases pencils can be divided into four subfamilies, two associated with odd degree polynomials and two associated with even degree polynomials. Each of these subfamilies is built by applying certain block-congruences to a very simple block-symmetric block minimal bases pencil, the "skeleton" or "generator" of the family. The "skeleton" of each family contains a "minimal" block-structure (in the sense that its matrix coefficients contain more zero blocks and less nonzero nonidentity blocks than any other pencil in the family) that guarantees it being a strong linearization of a given matrix polynomial $P(\lambda)$. Hence, this approach allows to identify the block-entries of the block-structure of strong linearizations based on block-symmetric Fiedler-like pencils (including the basis of $\mathbb{DL}(P)$) that are essential to embed the spectral information of $P(\lambda)$ in the pencil and the block-entries that are not, while preserving the block-symmetry. We expect these "skeletons" to be candidates to have optimal numerical properties among the block-symmetric linearizations in the family they "generate".

The rest of the paper is structured as follows. In Section 2, we review the basic theory of matrix polynomials, linearizations, minimal bases and dual minimal bases needed throughout the paper. In Section 3, we recall the definitions of the family of block minimal bases pencils and the family of extended block

[1] Given two block-symmetric pencils $L_1(\lambda)$ and $L_2(\lambda)$, we say that they are *permutationally block-congruent* if there exists a block-permutation matrix Q such that $L_1(\lambda) = QL_2(\lambda)Q^{\mathcal{B}}$, where $M^{\mathcal{B}}$ denotes the block-transpose of the matrix M.

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Kronecker pencils. By using extended block Kronecker pencils, we introduce in Section 4 four families of block-symmetric pencils which are block minimal bases pencils under generic nonsingularity conditions, and contain infinitely many block-symmetric strong linearizations of a matrix polynomial. These pencils are explicitly defined in terms of their block-entries. For completeness, we also include eigenvector recovery procedures for the left and right eigenvectors of a regular matrix polynomial $P(\lambda)$ from the left and right eigenvectors of any of its linearizations in the four families of block-symmetric pencils presented in this section. In Section 5, we recall the definitions of block-symmetric GFP and block-symmetric GFPR. Finally, in Section 6 we give a result that states that the block-symmetric GFP associated with an odd degree matrix polynomial and any block-symmetric GFPR is permutationally block-congruent to a pencil belonging to some of the four families introduced in Section 4. Thus, these four families of pencils provide an alternative and simplified approach to block-symmetric Fiedler-like pencils by providing their explicit block-structure. The proof of the result for the block-symmetric GFPR turns out to be quite involved, long and highly technical. One reason for this is that the family of block-symmetric GFPR is infinite and we are stating theorems that hold true for all the pencils in this family. The other reason, as we said before, is that these pencils are defined in an implicit way in terms of products of matrices, which makes the work with them quite cumbersome. The implicit definition of these pencils also leads to the use of a very heavy notation. Since the proof of this result is very similar to the proof of Theorem 8.1 in [5], in order to keep the length of this paper within a reasonable number of pages, we are not including it in this manuscript although, for the interested reader, it can be found in an extended version of this paper available in ArXiv. Now that we have an explicit definition of the block-symmetric GFPR in terms of their block entries, all the notation and the original implicit definition can be abandoned. What remains is a simpler description of block-symmetric Fiedler-like linearizations as block-symmetric block minimal bases pencils. This explicit definition of the block-symmetric GFPR has already proven to be useful. In [8, 9], it has been used to identify sparse pencils that outperform numerically (in terms of conditioning and backward error) the block-symmetric linearizations $D_1(\lambda, P)$ and $D_k(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$.

2. Notation and background. Throughout the paper, we use the following notation. If a and b are two integers, we define

$$a : b := \begin{cases} a, a+1, \ldots, b, & \text{if } a \leq b, \\ \emptyset, & \text{if } a > b. \end{cases}$$

In this work, we consider square matrix polynomials whose matrix coefficients have entries in a field $\mathbb{F}$, that is, matrix polynomials as in (1.1) with $m = n$. The number k in (1.1) is called the *grade* of $P(\lambda)$. The *degree* of $P(\lambda)$ is defined as the largest d such that $A_d \neq 0$. Notice that the degree is a number intrinsic to $P(\lambda)$, while the grade is an option (larger than or equal to the degree).

A square matrix polynomial $P(\lambda)$ is said to be *regular* if the scalar polynomial $\det(P(\lambda))$ is not the zero polynomial; otherwise $P(\lambda)$ is said to be *singular*. Furthermore, if $\det P(\lambda) \in \mathbb{F}$, $P(\lambda)$ is called a *unimodular matrix polynomial*. The *complete eigenstructure* of a regular matrix polynomial consists of its finite and infinite elementary divisors. For a singular matrix polynomial, the complete eigenstructure consists of its finite and infinite elementary divisors together with its right and left minimal indices. For more detailed definitions of the complete eigenstructure of matrix polynomials, we refer the reader to [14, Section 2].

By the *polynomial eigenvalue problem (PEP)* associated with a matrix polynomial $P(\lambda)$, we refer to the problem of computing the complete eigenstructure of $P(\lambda)$. If $P(\lambda)$ is a matrix pencil, the associated PEP is referred to as a *generalized eigenvalue problem (GEP)*. A matrix pencil $\mathcal{L}(\lambda)$ is said to be a *linearization*

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of a matrix polynomial $P(\lambda)$ if there exist unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ and some $s \geq 0$ such that

$$U(\lambda)\mathcal{L}(\lambda)V(\lambda) = \begin{bmatrix} I_s & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

A linearization $\mathcal{L}(\lambda) = \lambda L_1 + L_0$ of a matrix polynomial $P(\lambda)$ of grade k is a *strong linearization* of $P(\lambda)$ if $\lambda L_0 + L_1$ is a linearization of $\operatorname{rev} P = \lambda^k P(1/\lambda)$. If $\mathcal{L}(\lambda)$ is a strong linearization of a regular matrix polynomial $P(\lambda)$, then $\mathcal{L}(\lambda)$ has the same finite and infinite elementary divisors as $P(\lambda)$; when $P(\lambda)$ is singular, a strong linearization must also have the same numbers of right and left minimal indices as $P(\lambda)$. Hence, the PEP associated with the polynomial $P(\lambda)$ can be solved by solving the GEP associated with $\mathcal{L}(\lambda)$ provided that the minimal indices of $P(\lambda)$ and $\mathcal{L}(\lambda)$ are related in a simple way.

Given two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ of the same size, we recall the following equivalence relations. The polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be

(i) *unimodularly equivalent* if there are unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$ such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$; and
(ii) *strictly equivalent* if there are nonsingular constant matrices U and V such that $Q(\lambda) = UP(\lambda)V$.

We recall that unimodular equivalence preserves the finite eigenstructure of matrix polynomials, while strict equivalence preserves the whole eigenstructure [19]. In this work, we also use the following concepts extensively.

(i) Given an $s \times t$ block matrix $M = [M_{ij}]$ with $n \times n$ block-entries M_{ij}, the *block-transpose* matrix of M, denoted by $M^{\mathcal{B}}$, is the $t \times s$ block-matrix whose (i, j) block-entry is M_{ji}.
(ii) Given a $k \times k$ block matrix $M = [M_{ij}]$ with $n \times n$ block-entries M_{ij}, we say that M is *block-symmetric* if $M^{\mathcal{B}} = M$.
(iii) A $kn \times kn$ permutation matrix Π is called a *block-permutation* matrix if $\Pi = P \otimes I_n$, for some $k \times k$ permutation matrix P, where $\otimes$ denotes the Kronecker product of two matrices.
(iv) We say that the matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are *permutationally equivalent* if there are permutation matrices Π_1 and Π_2 such that $Q(\lambda) = \Pi_1 P(\lambda)\Pi_2$.
(v) We say that the $kn \times kn$ matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are *permutationally block-congruent* if there exists a block-permutation matrix Π such that $Q(\lambda) = \Pi P(\lambda)\Pi^{\mathcal{B}}$.

We notice that permutational equivalence and, thus, permutational block-congruency are particular instances of strict equivalence. Hence, the matrix polynomials $P(\lambda)$, $\Pi P(\lambda)\Pi^{\mathcal{B}}$ and $\Pi_1 P(\lambda)\Pi_2$ have the same eigenstructure (finite, infinite and singular). Furthermore, since the block-entries of any block permutation matrix Π are either the zero or the identity matrices, $P(\lambda)$ is block-symmetric if and only if $\Pi P(\lambda)\Pi^{\mathcal{B}}$ is block-symmetric.

Here and thereafter, we denote by $\mathbb{F}[\lambda]^{m\times n}$ the set of $m \times n$ matrix polynomials, by $\mathbb{F}(\lambda)$ the field of rational functions over $\mathbb{F}$ and by $\mathbb{F}(\lambda)^n$ the set of n-tuples with entries in $\mathbb{F}(\lambda)$. By $\overline{\mathbb{F}}$ we denote the algebraic closure of $\mathbb{F}$. Any subspace $\mathcal{W} \subseteq \mathbb{F}(\lambda)^n$ is called a *rational subspace*. We recall that any $\mathcal{W} \subseteq \mathbb{F}(\lambda)^n$ has bases consisting entirely of vectors with polynomial entries.

Key for this work are the so-called *minimal bases* and *dual minimal bases*, introduced by Forney [18]. For their definitions, we rely on the concept of *row-degrees vector* of an $m \times n$ matrix polynomial $P(\lambda)$, which is a row vector of length m whose ith component is the maximum of the degrees of the entries in the

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ith row of $P(\lambda)$. For example, the row-degrees vector of the matrix

$$\begin{bmatrix} 1 & \lambda^2 & 1-\lambda \\ 0 & 1 & \lambda \end{bmatrix} \tag{2.2}$$

is $[2,1]$.

Definition 2.1. Let $\mathcal{W}$ be a rational subspace of $\mathbb{F}(\lambda)^n$. We say that a matrix polynomial $L(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a *minimal basis* of $\mathcal{W}$ if its rows form a basis for $\mathcal{W}$ and the sum of the entries of its row-degrees vector is minimal among all the possible polynomial bases for $\mathcal{W}$. Furthermore, the entries of the row-degrees vector of $L(\lambda)$ are called the *minimal indices* of $\mathcal{W}$.

Remark 2.2. For simplicity, we say that "$L(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a minimal basis" to mean that "$L(\lambda)$ is a minimal basis for the subspace of $\mathbb{F}(\lambda)^n$ spanned by its rows".

The following characterization of minimal bases is very useful in practice.

Theorem 2.3. [15, Theorem 2.2] *Let $L(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and let $[d_1,\ldots,d_m]$ be the row-degrees vector of $L(\lambda)$. Then, $L(\lambda)$ is a minimal basis if and only if $L(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and the $m\times n$ constant matrix whose (i,j)th entry is the coefficient of λ^{d_i} in the (i,j)th entry of $L(\lambda)$ has full row rank.*

Example 2.4. The matrix polynomial in (2.2) is a minimal basis because it clearly has full row rank for every $\lambda_0 \in \overline{\mathbb{F}}$ and the matrix $\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ has full row rank as well.

Definition 2.5. Two matrix polynomials $L(\lambda) \in \mathbb{F}[\lambda]^{m_1\times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{m_2\times n}$ are called *dual minimal bases* if $m_1 + m_2 = n$, $L(\lambda)N(\lambda)^T = 0$, and $L(\lambda)$ and $N(\lambda)$ are both minimal bases.

Remark 2.6. We will say that "$N(\lambda)$ is a minimal basis dual to $L(\lambda)$", or vice versa, when referring to matrix polynomials $L(\lambda)$ and $N(\lambda)$ as those in Definition 2.5.

Continuing with the example in (2.2), it is easy to show that the matrix polynomials

$$\begin{bmatrix} 1 & \lambda^2 & 1-\lambda \\ 0 & 1 & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda^3+\lambda-1 & -\lambda & 1 \end{bmatrix}$$

are dual minimal bases.

In the following proposition, we introduce the most important pair of dual minimal bases used in this work.

Proposition 2.7. [15] *Let*

$$L_s(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{s\times(s+1)}, \tag{2.3}$$

and

$$\Lambda_s(\lambda) := \begin{bmatrix} \lambda^s & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(s+1)}. \tag{2.4}$$

Then, for every positive integer p, the matrix polynomials $L_s(\lambda)\otimes I_p$ and $\Lambda_s(\lambda)\otimes I_p$ are dual minimal bases.

The following proposition concerning dual minimal bases will be useful.

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PROPOSITION 2.8. *Let $L(\lambda)$ be a minimal basis. If B is a nonsingular matrix, then $BL(\lambda)$ is also a minimal basis. Further, if $N(\lambda)$ is any minimal basis dual to $L(\lambda)$, $N(\lambda)$ is also dual to $BL(\lambda)$.*

Proof. The proof follows immediately from the characterization of minimal bases in Theorem 2.3, and the definition of dual minimal bases in Definition 2.5. ☐

3. Block minimal bases pencils and extended block Kronecker pencils. We recall in this section the families of block minimal bases pencils and of extended block Kronecker pencils, and state their main properties used in this work.

3.1. Block minimal bases pencils. The block minimal bases pencils were introduced in [15]. The definition of block minimal bases pencil involves the concept of minimal basis and pair of dual minimal bases introduced in the previous section.

DEFINITION 3.1. A matrix pencil

$$C(\lambda) = \left[\begin{array}{c|c} M(\lambda) & G_2(\lambda)^T \\ \hline G_1(\lambda) & 0 \end{array}\right] \tag{3.5}$$

is called a *block minimal bases pencil* if $G_1(\lambda)$ and $G_2(\lambda)$ are both minimal bases. If, in addition, the row-degrees vector of $G_1(\lambda)$ (resp., $G_2(\lambda)$) have all entries equal to 1 and the entries of the row-degrees vector of a minimal basis dual to $G_1(\lambda)$ (resp., $G_2(\lambda)$) are all equal, then $C(\lambda)$ is called a *strong block minimal bases pencil.*

A fundamental property of any strong block minimal bases pencil of the form (3.5) is that it is a strong linearization of some matrix polynomial expressed in terms of the block-entry $M(\lambda)$ and the dual minimal bases of $G_1(\lambda)$ and $G_2(\lambda)$.

THEOREM 3.2. [15, Theorem 3.3] *Let $C(\lambda)$ be a strong block minimal bases pencil as in (3.5). Let $N_1(\lambda)$ (resp., $N_2(\lambda)$) be a minimal basis dual to $G_1(\lambda)$ (resp., $G_2(\lambda)$) whose row-degrees vector has equal entries. Let*

$$Q(\lambda) := N_2(\lambda)M(\lambda)N_1(\lambda)^T. \tag{3.6}$$

Then, $C(\lambda)$ is a strong linearization of $Q(\lambda)$, considered as a polynomial of grade $1+\deg(N_1(\lambda))+\deg(N_2(\lambda))$.

3.2. Extended block Kronecker pencils. Next we recall a family of pencils that has played an important role in the canonical expression of the GFP and GFPR in terms of their block-structure [5]. The pencils in this family are called *extended block Kronecker pencils.* In their definition, we use the dual minimal bases $L_s(\lambda)$ and $\Lambda_s(\lambda)$ introduced, respectively, in (2.3) and (2.4).

DEFINITION 3.3. [5, Definition 3.5] Let $M(\lambda)$ be an arbitrary $(q+1)m \times (p+1)n$ pencil. Let $A \in \mathbb{F}^{np\times np}$ and $B \in \mathbb{F}^{mq\times mq}$ be arbitrary matrices. Then the matrix pencil

$$C(\lambda) = \begin{array}{c} \left[\begin{array}{c|c} M(\lambda) & (L_q(\lambda)^T \otimes I_m)B \\ \hline A(L_p(\lambda)\otimes I_n) & 0 \end{array}\right] \begin{array}{l} \}\,{\scriptstyle (q+1)m} \\ \}\,{\scriptstyle pn} \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{A(L_p(\lambda)\otimes I_n)}}_{(p+1)n} & \underbrace{\hphantom{(L_q(\lambda)^T \otimes I_m)B}}_{qm} \end{array} \end{array} \tag{3.7}$$

where $L_p(\lambda)$ and $L_q(\lambda)$ are as in (2.3), is called an *extended (p,n,q,m)-block Kronecker pencil* or, simply, an *extended block Kronecker pencil.* When $A = I_{np}$ and $B = I_{mq}$, then $C(\lambda)$ is called a block Kronecker pencil. The block $M(\lambda)$ is called the *body of $C(\lambda)$.*

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Note that, if A and B are nonsingular matrices, then $C(\lambda)$ is a (strong) block minimal bases pencil (see Proposition 2.8). However, if either A or B is singular, it is not guaranteed that $C(\lambda)$ is a block minimal bases pencil.

One advantage of the extended block Kronecker pencils with A and B nonsingular over more general strong block minimal bases pencils is that it is easy to give simple characterizations for all the grade-1 solutions $M(\lambda)$ of the equation

$$(\Lambda_q(\lambda)^T \otimes I_m)M(\lambda)(\Lambda_p(\lambda) \otimes I_n) = P(\lambda) \tag{3.8}$$

for a prescribed matrix polynomial $P(\lambda)$ of grade $k = p + q + 1$.

The following definition will be used in one of such characterizations.

Definition 3.4. [5, Definition 3.7] Let $M(\lambda) = \lambda M_1 + M_0 \in \mathbb{F}[\lambda]^{(q+1)m\times(p+1)n}$ be a matrix pencil and set $k := p + q + 1$. Let us denote by $[M_0]_{ij}$ and $[M_1]_{ij}$ the (i, j)th block-entries of M_0 and M_1, respectively, when M_0 and M_1 are partitioned as $(q + 1) \times (p + 1)$ block-matrices with blocks of size $m \times n$. We call the *antidiagonal sum of* $M(\lambda)$ *related to* $s \in \{0 : k\}$ the matrix

$$\mathrm{AS}(M, s) := \sum_{i+j=k+2-s} [M_1]_{ij} + \sum_{i+j=k+1-s} [M_0]_{ij}.$$

Additionally, given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$, we say that $M(\lambda)$ satisfies the *antidiagonal sum condition (AS condition)* for $P(\lambda)$ if

$$\mathrm{AS}(M, s) = A_s, \quad s = 0 : k. \tag{3.9}$$

The AS condition has been used in the construction of large classes of linearizations of a matrix polynomial $P(\lambda)$ easily constructible from the coefficients of $P(\lambda)$; see [15, Theorem 5.4] or [17, Section 3].

Example 3.5. Let $P(\lambda) = \sum_{i=0}^{5} A_i\lambda^i$. The matrix pencil

$$M(\lambda) = \begin{bmatrix} A_5\lambda & 0 & 0 \\ A_4\lambda & 0 & 0 \\ A_3\lambda & A_2\lambda & A_1\lambda + A_0 \end{bmatrix}.$$

satisfies the AS condition for $P(\lambda)$.

Theorem 3.6. *Let* $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{m\times n}$*, and let* $C(\lambda)$ *be an extended block Kronecker pencil as in* (3.7) *with* $p + q + 1 = k$ *and with body* $M(\lambda)$*. The following conditions are equivalent.*

(a) *The pencil* $M(\lambda)$ *satisfies* (3.8).
(b) *The pencil* $M(\lambda)$ *satisfies the AS condition for* $P(\lambda)$.
(c) *The pencil* $M(\lambda)$ *is of the form*

$$M(\lambda) = M_0(\lambda) + C_1(L_p(\lambda) \otimes I_n) + (L_q(\lambda)^T \otimes I_m)C_2,$$

where $M_0(\lambda)$ *is any solution of* (3.8) *and* $C_1 \in \mathbb{F}^{(q+1)m\times pn}$ *and* $C_2 \in \mathbb{F}^{qm\times(p+1)n}$ *are arbitrary matrices.*

Proof. The proof that (a) and (b) are equivalent can be obtained by some simple algebraic manipulations. The proof that parts (a) and (c) are equivalent can be found in [17] (in a paragraph just before Theorem 1). $\square$

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Now, as an immediate consequence of Theorem 3.6, we obtain the following family of strong linearizations of $P(\lambda)$.

THEOREM 3.7. *Let $P(\lambda)$ be a matrix polynomial, and let p, q be nonnegative integers such that $p+q+1 = \deg(P(\lambda))$. Let $M_0(\lambda)$ be a pencil satisfying the AS condition for $P(\lambda)$. Then, any pencil of the form*

$$\left[\begin{array}{c|c} M_0(\lambda) + C_1(L_p(\lambda)\otimes I_n) + (L_q(\lambda)^T \otimes I_m)C_2 & (L_q(\lambda)^T\otimes I_m)B_2 \\ \hline B_1(L_p(\lambda)\otimes I_n) & 0 \end{array}\right], \tag{3.10}$$

where $C_1 \in \mathbb{F}^{(q+1)m\times pn}$ and $C_2 \in \mathbb{F}^{qm\times(p+1)n}$ are arbitrary matrices, and $B_1 \in \mathbb{F}^{pn\times pn}$ and $B_2 \in \mathbb{F}^{qm\times qm}$ are arbitrary nonsingular matrices, is a strong linearization of $P(\lambda)$.

Proof. The result is an immediate consequence of Theorems 3.2 and 3.6, together with the fact that, when the matrices B_1 and B_2 are nonsingular, the extended block Kronecker pencil (3.10) is a strong block minimal bases pencil (see Proposition 2.8). □

REMARK 3.8. Observe that any pencil of the form (3.10) is an extended block Kronecker pencil whose body satisfies the AS condition for $P(\lambda)$ since, given two matrix pencils $M_1(\lambda)$ and $M_2(\lambda)$, $\mathrm{AS}(M_1+M_2, s) = \mathrm{AS}(M_1, s) + \mathrm{AS}(M_2, s)$. Moreover, the pencil in (3.10) can be expressed as follows:

$$\begin{bmatrix} I & C_1 \\ 0 & B_1 \end{bmatrix} \left[\begin{array}{c|c} M_0(\lambda) & L_q(\lambda)^T\otimes I_m \\ \hline L_p(\lambda)\otimes I_n & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ C_2 & B_2 \end{bmatrix}.$$

Theorem 3.7 will be key to provide a simple canonical block-structure for block-symmetric Fiedler-like pencils under permutational block-congruence operations. The description of these block-structures is the main goal of the following section.

4. The four families of block-symmetric minimal bases pencils. We introduce in this section four types of block-symmetric pencils associated with a matrix polynomial $P(\lambda)$, which are block minimal bases pencils, under some generic nonsingularity conditions, and we give their explicit block structure. We will show later that the block-symmetric Fiedler-like pencils known in the literature belong to one of these families, modulo a permutational block-congruence. For completeness, we also include simple procedures for recovering the (left and right) eigenvectors of a regular $P(\lambda)$ from the (left and right) eigenvectors of a linearization of $P(\lambda)$ in any of these families.

Since block-symmetric Fiedler-like pencils are only defined for square matrix polynomials, here and thereafter, we restrict our study to square matrix polynomials $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. As the size of $P(\lambda)$ is always going to be denoted by n, there is no risk of confusion if we introduce the notation

$$K_s(\lambda) := L_s(\lambda) \otimes I_n,$$

with $L_s(\lambda)$ as in (2.3). We note that

$$K_s(\lambda)^T = K_s(\lambda)^{\mathcal{B}} \quad \text{and} \quad (\Lambda_s(\lambda)\otimes I_n)^T = (\Lambda_s(\lambda)\otimes I_n)^{\mathcal{B}}, \tag{4.11}$$

with $\Lambda_s(\lambda)$ as in (2.4), when $K_s(\lambda)$ is seen as an $s\times(s+1)$ block matrix with blocks of size $n\times n$ and $\Lambda_s(\lambda)\otimes I_n$ is seen as a $1\times(s+1)$ block matrix with blocks of size $n\times n$. Moreover, if B is an $s\times s$ block matrix, then

$$(BK_s(\lambda))^{\mathcal{B}} = K_s(\lambda)^{\mathcal{B}} B^{\mathcal{B}} = K_s(\lambda)^T B^{\mathcal{B}}. \tag{4.12}$$

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Additionally, we introduce the block-symmetric pencil

$$M(\lambda;Q) := \begin{bmatrix} \lambda Q_d + Q_{d-1} & & \\ & \ddots & \\ & & \lambda Q_1 + Q_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{\frac{n(d+1)}{2}\times\frac{n(d+1)}{2}} \tag{4.13}$$

associated with a matrix polynomial $Q(\lambda) = \sum_{i=0}^{d} Q_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ of odd degree d, which will play a fundamental role in what follows. Notice that $M(\lambda;Q)$ satisfies the AS condition for $Q(\lambda)$.

Associated with the matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$, we define the following matrix polynomials

$$P^{k-1}(\lambda) := A_{k-1}\lambda^{k-1} + \cdots + \lambda A_1 + A_0, \tag{4.14}$$

$$P^{k-1}_{k-1}(\lambda) := A_{k-1}\lambda^{k-2} + \cdots + A_2\lambda + A_1, \quad \text{and} \tag{4.15}$$

$$P_{k-1}(\lambda) := A_k\lambda^{k-1} + \cdots + \lambda A_2 + A_1, \tag{4.16}$$

which will be used in the definition of the four families of block-symmetric pencils introduced in this section. Note that $P^{k-1}(\lambda)$ is a truncation of degree $k-1$ of $P(\lambda)$ while $P_{k-1}(\lambda)$ is the so-called $(k-1)$th Horner shift polynomial associated with $P(\lambda)$ (see Definition 4.17).

4.1. The first fundamental block-structure. We introduce here the first of the families of block-symmetric pencils. Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of odd degree k, and let $s := (k-1)/2$. We start by defining the pencil

$$\mathcal{O}_1^P(\lambda) := \left[\begin{array}{c|c} M(\lambda;P) & K_s(\lambda)^T \\ \hline K_s(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{nk\times nk}, \tag{4.17}$$

where $M(\lambda;P)$ is defined in (4.13). By Definition 3.3, the pencil $\mathcal{O}_1^P(\lambda)$ is an (s,n,s,n)- block Kronecker pencil and a strong block minimal bases pencil. Furthermore, taking into account (4.11), it is clearly block-symmetric. Notice additionally that, by Theorem 3.7, $\mathcal{O}_1^P(\lambda)$ is a strong linearization of $P(\lambda)$ because $M(\lambda;P)$ satisfies the AS condition for $P(\lambda)$. Thus, the pencil $\mathcal{O}_1^P(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$.

We can obtain many more block-symmetric strong linearizations of $P(\lambda)$ by considering pencils obtained by applying the block-congruence

$$\begin{bmatrix} I_{(s+1)n} & C \\ 0 & B \end{bmatrix} \left[\begin{array}{c|c} M(\lambda;P) & K_s(\lambda)^T \\ \hline K_s(\lambda) & 0 \end{array}\right] \begin{bmatrix} I_{(s+1)n} & 0 \\ C^{\mathcal{B}} & B^{\mathcal{B}} \end{bmatrix}, \tag{4.18}$$

where $B = [B_{ij}]$ is an $s\times s$ block matrix and $C = [C_{ij}]$ is an $(s+1)\times s$ block-matrix, with $n\times n$ block-entries B_{ij} and C_{ij}, respectively. The pencil (4.18) motivates the first fundamental block-structure family associated with the matrix polynomial $P(\lambda)$.

Definition 4.1. Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of odd degree k, and let $s = (k-1)/2$. The *first fundamental block-structure family*, denoted by $\langle\mathcal{O}_1^P\rangle$, is the set of pencils of the form

$$\left[\begin{array}{c|c} M(\lambda;P) + CK_s(\lambda) + K_s^T(\lambda)C^{\mathcal{B}} & K_s(\lambda)^T B^{\mathcal{B}} \\ \hline BK_s(\lambda) & 0 \end{array}\right], \tag{4.19}$$

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where $M(\lambda; P)$ is defined in (4.13), and $B = [B_{ij}]$ and $C = [C_{ij}]$ are, respectively, some arbitrary $s \times s$ block matrix and $(s+1) \times s$ block matrix, with $n \times n$ block-entries B_{ij} and C_{ij}.

Remark 4.2. The matrix pencil in (4.19), which is also the pencil in (4.18), can be expressed as follows:

$$\begin{bmatrix} I_{(s+1)n} & C \\ 0 & B \end{bmatrix} \left[\begin{array}{c|c} M(\lambda;P) & K_s(\lambda)^T \\ \hline K_s(\lambda) & 0 \end{array}\right] \begin{bmatrix} I_{(s+1)n} & C \\ 0 & B \end{bmatrix}^{\mathcal{B}},$$

where the block transpose is applied on the matrix $\begin{bmatrix} I_{(s+1)n} & C \\ 0 & B \end{bmatrix}$ when considered a $k \times k$ block matrix. That is, every pencil in $\langle \mathcal{O}_1^P \rangle$ is block congruent to $\mathcal{O}_1^P$ and, therefore, block-symmetric.

By (4.11) and Definition 3.3, any pencil in the family $\langle \mathcal{O}_1^P \rangle$ is a block-symmetric (s, n, s, n)-extended block Kronecker pencil. Moreover, if B and $B^{\mathcal{B}}$ are nonsingular, each pencil in this family is a strong block minimal bases pencil, which leads to the following theorem.

Theorem 4.3. *Let $P(\lambda) = \sum_{i=0}^k A_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of odd degree k, let $s = (k-1)/2$, and let $\mathcal{L}(\lambda) \in \langle \mathcal{O}_1^P \rangle$, that is, $\mathcal{L}(\lambda)$ is of the form (4.19). If B and $B^{\mathcal{B}}$ are nonsingular, then the pencil $\mathcal{L}(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$. Moreover, if $P(\lambda)$ and all the block-entries B_{ij} are symmetric (resp., Hermitian), then the pencil $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian).*

Proof. The fact that $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$ when B and $B^{\mathcal{B}}$ are nonsingular is an immediate consequence of Theorem 3.7. The pencil $\mathcal{L}(\lambda)$ is block-symmetric as a consequence of (4.11), together with the fact that $M(\lambda;P)$ is block-symmetric. The fact that $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian) when $P(\lambda)$ and all the block-entries B_{ij} of B are symmetric (resp., Hermitian) follows easily from the facts that $M(\lambda;P)$ is symmetric (resp., Hermitian) when $P(\lambda)$ is symmetric (resp., Hermitian), and that $B^{\mathcal{B}} = B^T$ (resp., $B^{\mathcal{B}} = B^*$) and $C^{\mathcal{B}} = C^T$ (resp., $C^{\mathcal{B}} = C^*$) when all the block-entries B_{ij} and C_{ij} are symmetric (resp., Hermitian). ∎

Example 4.4. As mentioned in the introduction, the best well-known block-symmetric pencils in the literature are those in the vector space $\mathbb{DL}(P)$. The pencils in the standard basis of this space are block-symmetric GFPR of special importance. Let $P(\lambda)$ be a matrix polynomial of odd degree k and let m be an odd positive integer. Then, as we will show in Theorem 6.2, the mth pencil $D_m(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$, which is a GFPR with parameter $h = k - m$, is permutationally block-congruent to a pencil in $\langle \mathcal{O}_1^P \rangle$. This holds, in particular, for $D_1(\lambda, P)$ and $D_k(\lambda, P)$.

4.2. The second fundamental block-structure. We introduce in this section the second fundamental family of block-symmetric pencils. This family is also associated with odd-degree matrix polynomials, but describing its block-structure is more involved.

Let $P(\lambda) = \sum_{i=0}^k A_i \lambda^i$ be an $n \times n$ matrix polynomial of odd degree k, and let $s := (k-1)/2$. First, we define the pencil

$$\mathcal{O}_2^P(\lambda) := \left[\begin{array}{c:cc:c|c} -A_k & \lambda A_k & 0 & 0 & 0 \\ \hdashline \lambda A_k & \multicolumn{2}{c:}{\multirow{2}{*}{$M(\lambda; P_{k-1}^{k-1})$}} & 0 & \multirow{2}{*}{$K_{s-1}(\lambda)^T$} \\ 0 & & & A_0 & \\ \hdashline 0 & 0 & A_0 & -\lambda A_0 & 0 \\ \hline 0 & \multicolumn{2}{c:}{K_{s-1}(\lambda)} & 0 & 0 \end{array}\right], \tag{4.20}$$

where $P_{k-1}^{k-1}(\lambda)$ is defined in (4.15) and $M(\lambda; P_{k-1}^{k-1})$ is defined in (4.13). Notice that the pencil $\mathcal{O}_2^P(\lambda)$ is

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a block-symmetric block minimal bases pencil. However, this pencil is not an extended block-Kronecker pencil.

Next we give an example to clarify the block-structure of the pencil $\mathcal{O}_2^P(\lambda)$.

EXAMPLE 4.5. Let $P(\lambda) = \sum_{i=0}^{7} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$. Then,

$$\mathcal{O}_2^P(\lambda) = \left[\begin{array}{c:ccc:c|cc} -A_7 & \lambda A_7 & 0 & 0 & 0 & 0 & 0 \\ \hdashline \lambda A_7 & \lambda A_6 + A_5 & 0 & 0 & 0 & -I_n & 0 \\ 0 & 0 & \lambda A_4 + A_3 & 0 & 0 & \lambda I_n & -I_n \\ 0 & 0 & 0 & \lambda A_2 + A_1 & A_0 & 0 & \lambda I_n \\ \hdashline 0 & 0 & 0 & A_0 & -\lambda A_0 & 0 & 0 \\ \hline 0 & -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right].$$

Notice that, if we denote by Π_2 the block-permutation matrix that permutes the first block-column of $\mathcal{O}_2^P$ with the block-columns in positions 2–5, we have

$$\mathcal{O}_2^P(\lambda)\Pi_2 = \left[\begin{array}{c|c} M(\lambda) & K_3(\lambda)^T B_2 \\ \hline B_1 K_3(\lambda) & 0 \end{array}\right] :=$$

$$\left[\begin{array}{cccc|ccc} \lambda A_7 & 0 & 0 & 0 & -A_7 & 0 & 0 \\ \lambda A_6 + A_5 & 0 & 0 & 0 & \lambda A_7 & -I_n & 0 \\ 0 & \lambda A_4 + A_3 & 0 & 0 & 0 & \lambda I_n & -I_n \\ 0 & 0 & \lambda A_2 + A_1 & A_0 & 0 & 0 & \lambda I_n \\ \hline 0 & 0 & A_0 & -\lambda A_0 & 0 & 0 & 0 \\ -I_n & \lambda I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 & 0 \end{array}\right],$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & -A_0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} A_7 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

Thus, although $\mathcal{O}_2^P(\lambda)$ is not an extended block Kronecker pencil, it is only a column-permutation away from being so. It is easy to see that the body $M(\lambda)$ of $\mathcal{O}_2^P(\lambda)\Pi_2$ satisfies the AS condition for $P(\lambda)$. Hence, by Theorem 3.7 and Remark 3.8, the pencil $\mathcal{O}_2^P(\lambda)\Pi_2$, and therefore $\mathcal{O}_2^P(\lambda)$, is a strong linearization of $P(\lambda)$ if A_0 and A_k are nonsingular matrices.

The procedure used in the previous example can be generalized to matrix polynomials of any odd-degree k. Denoting by Π_2 the block-permutation matrix that permutes the first block-column of $\mathcal{O}_2^P(\lambda)$, defined in (4.20), with the block-columns in positions 2 through $s+2 = \frac{k+3}{2}$, we obtain

$$\mathcal{O}_2^P(\lambda)\Pi_2 := \left[\begin{array}{cc:c|c:c} \lambda A_k & 0 & 0 & -A_k & 0 \\ \hdashline \multicolumn{2}{c:}{\multirow{2}{*}{$M(\lambda; P_{k-1}^{k-1})$}} & 0 & \lambda A_k & \multirow{2}{*}{$K_{s-1}(\lambda)^T$} \\ & & A_0 & 0 & \\ \hline 0 & A_0 & -\lambda A_0 & 0 & 0 \\ \hdashline \multicolumn{2}{c:}{K_{s-1}(\lambda)} & 0 & 0 & 0 \end{array}\right], \tag{4.21}$$

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which is an (s,n,s,n)-extended block Kronecker pencil. Furthermore, if A_0 and A_k are nonsingular, from Theorem 3.7, Remark 3.8, and the fact that

$$\left[\begin{array}{cc:c} \lambda A_k & 0 & 0 \\ \hdashline \multicolumn{2}{c:}{\multirow{2}{*}{$M(\lambda;P_{k-1}^{k-1})$}} & 0 \\ & & A_0 \end{array}\right],$$

satisfies the AS condition for $P(\lambda)$, it is immediately obtained that the pencil in (4.21) is a strong linearization of $P(\lambda)$. In summary, the pencil $\mathcal{O}_2^P(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$ if A_0 and A_k are nonsingular. Moreover, $\mathcal{O}_2^P(\lambda)$ is symmetric (resp., Hermitian) whenever $P(\lambda)$ is symmetric (resp., Hermitian).

Motivated by the block-structure of the pencil (4.21) and by Theorem 3.7, we now consider a subfamily of extended block Kronecker pencils constructed from $\mathcal{O}_2^P(\lambda)\Pi_2$. Note that, among all the possible operations that would transform $\mathcal{O}_2^P(\lambda)\Pi_2$ into another extended block Kronecker pencil, we are only applying some that will preserve the block-symmetry once the $(s+2)$th block column is permuted back to the original position, that is, the first block-column. More precisely, we begin by considering pencils of the form

$$\begin{bmatrix} I_n & 0 & 0 & B \\ 0 & I_{sn} & 0 & C \\ 0 & 0 & I_n & D \\ 0 & 0 & 0 & E \end{bmatrix} \mathcal{O}_2^P(\lambda)\Pi_2 \begin{bmatrix} I_{sn} & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ C^{\mathcal{B}} & D^{\mathcal{B}} & B^{\mathcal{B}} & E^{\mathcal{B}} \end{bmatrix}, \tag{4.22}$$

for some arbitrary $1\times(s-1)$, $s\times(s-1)$, $1\times(s-1)$ and $(s-1)\times(s-1)$ block matrices $B=[B_{ij}]$, $C=[C_{ij}]$, $D=[D_{ij}]$ and $E=[E_{ij}]$, with $n\times n$ block-entries B_{ij}, C_{ij}, D_{ij} and E_{ij}.

Then, permuting the $(s+2)$th block-column of the above pencil back to the first position, we get

$$\begin{bmatrix} I_n & 0 & 0 & B \\ 0 & I_{sn} & 0 & C \\ 0 & 0 & I_n & D \\ 0 & 0 & 0 & E \end{bmatrix} \mathcal{O}_2^P(\lambda) \begin{bmatrix} 0 & 0 & I_n & 0 \\ I_{sn} & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ C^{\mathcal{B}} & D^{\mathcal{B}} & B^{\mathcal{B}} & E^{\mathcal{B}} \end{bmatrix} \Pi_2^{\mathcal{B}},$$

which equals

$$\begin{bmatrix} I_n & 0 & 0 & B \\ 0 & I_{sn} & 0 & C \\ 0 & 0 & I_n & D \\ 0 & 0 & 0 & E \end{bmatrix} \mathcal{O}_2^P(\lambda) \begin{bmatrix} I_n & 0 & 0 & B \\ 0 & I_{sn} & 0 & C \\ 0 & 0 & I_n & D \\ 0 & 0 & 0 & E \end{bmatrix}^{\mathcal{B}}. \tag{4.23}$$

In this way, we obtain the block-structure (4.23) defining the second fundamental family of block-structures associated with the matrix polynomial $P(\lambda)$.

Definition 4.6. Let $P(\lambda)=\sum_{i=0}^k A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of odd degree k, let $s=(k-1)/2$. The *second fundamental block-structure family*, denoted by $\langle \mathcal{O}_2^P\rangle$, is the set of pencils of the form (we are omitting the dependence on λ in the pencil $K_{s-1}(\lambda)$ for lack of space)

$$\left[\begin{array}{c:c:c|c} -A_k & \begin{bmatrix}\lambda A_k & 0\end{bmatrix} + BK_{s-1} & 0 & 0 \\ \hdashline \begin{bmatrix}\lambda A_k \\ 0\end{bmatrix} + K_{s-1}^T B^{\mathcal{B}} & M(\lambda;P_{k-1}^{k-1}) + CK_{s-1} + K_{s-1}^T C^{\mathcal{B}} & \begin{bmatrix} 0 \\ A_0\end{bmatrix} + K_{s-1}^T D^{\mathcal{B}} & K_{s-1}^T E^{\mathcal{B}} \\ \hdashline 0 & \begin{bmatrix} 0 & A_0\end{bmatrix} + DK_{s-1} & -\lambda A_0 & 0 \\ \hline 0 & EK_{s-1} & 0 & 0 \end{array}\right], \tag{4.24}$$

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where $P_{k-1}^{k-1}(\lambda)$ is defined in (4.15) and $M(\lambda; P_{k-1}^{k-1})$ is defined in (4.13), for some arbitrary $1 \times (s-1)$ block-matrix $B = [B_{ij}]$, $s \times (s-1)$ block-matrix $C = [C_{ij}]$, $1 \times (s-1)$ block-matrix $D = [D_{ij}]$, and $(s-1) \times (s-1)$ block-matrix $E = [E_{ij}]$, with $n \times n$ block-entries B_{ij}, C_{ij}, D_{ij} and E_{ij}, respectively.

We note that every pencil in $\langle \mathcal{O}_2^P \rangle$ is a block minimal bases pencil if E and $E^{\mathcal{B}}$ are nonsingular matrices.

The following theorem gives sufficient conditions for pencils in the family $\langle \mathcal{O}_2^P \rangle$ to be strong linearizations of an odd-degree matrix polynomial $P(\lambda)$.

Theorem 4.7. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$ be a matrix polynomial of odd degree k, let $s = (k-1)/2$, and consider a pencil $\mathcal{L}(\lambda) \in \langle \mathcal{O}_2^P \rangle$, that is, a pencil of the form (4.24). If A_0, A_k, E and $E^{\mathcal{B}}$ are nonsingular, then $\mathcal{L}(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$. Furthermore, if $P(\lambda)$, and all the block-entries of B, C, D and E are symmetric (resp., Hermitian), then $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian).*

Proof. When A_0 and A_k are nonsingular, the extended block Kronecker pencil (4.21) and, thus, the pencil $\mathcal{O}_2^P(\lambda)$ are strong linearizations of $P(\lambda)$. In addition, we see from (4.23) that if E and $E^{\mathcal{B}}$ are nonsingular, then the pencil $\mathcal{L}(\lambda)$ is strictly equivalent to the pencil $\mathcal{O}_2^P(\lambda)$. Therefore, in this case, $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$. The pencil $\mathcal{L}(\lambda)$ is block-symmetric as a consequence of (4.11), together with the fact that $M(\lambda; P_{k-1}^{k-1})$ is block-symmetric. The fact that $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian) when $P(\lambda)$ and all the block-entries of B, C, D and E are symmetric (resp., Hermitian) follows easily from the following facts. First, $M(\lambda; P_{k-1}^{k-1})$ is symmetric (resp., Hermitian) when $P(\lambda)$ is symmetric (resp., Hermitian). Secondly, we have $B^{\mathcal{B}} = B^T$ (resp., $B^{\mathcal{B}} = B^*$), $C^{\mathcal{B}} = C^T$ (resp., $C^{\mathcal{B}} = C^*$), $D^{\mathcal{B}} = D^T$ (resp., $D^{\mathcal{B}} = D^*$) and $E^{\mathcal{B}} = E^T$ (resp., $E^{\mathcal{B}} = E^*$) when all the block-entries of B, C, D and E are symmetric (resp., Hermitian). ∎

Example 4.8. Let $P(\lambda)$ be a matrix polynomial of odd degree k and let m be an even positive integer. Then, as we will show in Theorem 6.2, the mth pencil $D_m(\lambda, P)$ in the standard basis of the vector space $\mathbb{DL}(P)$, which is a GFPR with parameter $h = k - m$, is permutationally block congruent to a pencil in $\langle \mathcal{O}_2^P \rangle$.

4.3. The third fundamental block-structure. The third fundamental family of block-symmetric pencils is defined for matrix polynomials of even degree. So, let $P(\lambda)$ be a matrix polynomial of even degree k, and let $s := (k-2)/2$. First, we define the pencil

$$\mathcal{E}_1^P(\lambda) := \left[\begin{array}{cc|c:c} \multicolumn{2}{c|}{M(\lambda; P_{k-1})} & \begin{matrix} 0 \\ A_0 \end{matrix} & K_s(\lambda)^T \\ \hdashline 0 & A_0 & -\lambda A_0 & 0 \\ \hline \multicolumn{2}{c|}{K_s(\lambda)} & 0 & 0 \end{array}\right], \tag{4.25}$$

where $P_{k-1}(\lambda)$ is defined in (4.16) and $M(\lambda; P_{k-1})$ is defined in (4.13). The pencil $\mathcal{E}_1^P(\lambda)$ is an extended $(s, n, s+1, n)$-block Kronecker pencil[2], with the solid lines indicating one of its natural partitions.

Note that the body of $\mathcal{E}_1^P(\lambda)$, regardless of the chosen partition (see [15, Theorem 3.10]), satisfies the AS condition for $P(\lambda)$. Thus, $\mathcal{E}_1^P(\lambda)$ is a strong linearization of $P(\lambda)$, provided that A_0 is nonsingular. Furthermore, the pencil $\mathcal{E}_1^P(\lambda)$ is block-symmetric, and it is symmetric (resp., Hermitian) when $P(\lambda)$ is symmetric (resp., Hermitian).

[2]It can be seen as an extended $(s+1, n, s, n)$-block Kronecker pencil as well.

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EXAMPLE 4.9. Let $P(\lambda)=\sum_{i=0}^{6} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$. Then,

$$\mathcal{E}_1^P(\lambda)=\left[\begin{array}{ccc|c:cc} \lambda A_6+A_5 & 0 & 0 & 0 & -I_n & 0\\ 0 & \lambda A_4+A_3 & 0 & 0 & \lambda I_n & -I_n\\ 0 & 0 & \lambda A_2+A_1 & A_0 & 0 & \lambda I_n\\ \hdashline 0 & 0 & A_0 & -\lambda A_0 & 0 & 0\\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0\\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right].$$

Motivated by the block-structure of the pencil $\mathcal{E}_1^P(\lambda)$, we introduce the third fundamental family of block-structures by applying the following block-congruence

$$\begin{bmatrix} I_{(s+1)n} & 0 & B\\ 0 & I_n & C\\ 0 & 0 & D\end{bmatrix}\mathcal{E}_1^P(\lambda)\begin{bmatrix} I_{(s+1)n} & 0 & 0\\ 0 & I_n & 0\\ B^{\mathcal{B}} & C^{\mathcal{B}} & D^{\mathcal{B}}\end{bmatrix}, \tag{4.26}$$

where $B=[B_{ij}]$ is a $(s+1)\times s$ block-matrix, $C=[C_{ij}]$ is an $1\times s$ block-matrix and $D=[D_{ij}]$ is an $s\times s$ block-matrix, with $n\times n$ block-entries B_{ij}, C_{ij} and D_{ij}, respectively.

DEFINITION 4.10. Let $P(\lambda)=\sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of even degree k, and let $s=(k-2)/2$. The *third fundamental block-structure family*, denoted by $\langle\mathcal{E}_1^P\rangle$, is the set of pencils of the form

$$\left[\begin{array}{c|c:c} M(\lambda;P_{k-1})+BK_s(\lambda)+K_s(\lambda)^T B^{\mathcal{B}} & \begin{bmatrix}0\\ A_0\end{bmatrix}+K_s(\lambda)^T C^{\mathcal{B}} & K_s(\lambda)^T D^{\mathcal{B}}\\ \hdashline \begin{bmatrix}0 & A_0\end{bmatrix}+CK_s(\lambda) & -\lambda A_0 & 0\\ \hline DK_s(\lambda) & 0 & 0\end{array}\right], \tag{4.27}$$

where $P_{k-1}(\lambda)$ is defined in (4.16) and $M(\lambda;P_{k-1})$ is defined in (4.13), for some arbitrary $(s+1)\times s$ block-matrix $B=[B_{ij}]$, $1\times s$ block-matrix $C=[C_{ij}]$ and $s\times s$ block-matrix $D=[D_{ij}]$, with $n\times n$ block-entries B_{ij}, C_{ij} and D_{ij}, respectively.

Note that the pencils in $\langle\mathcal{E}_1^P\rangle$ are block minimal bases pencils if A_0, D and $D^{\mathcal{B}}$ are nonsingunlar.

The following theorem gives sufficient conditions for the pencils in the family $\langle\mathcal{E}_1^P\rangle$ to be strong linearizations of the even-degree matrix polynomial $P(\lambda)$.

THEOREM 4.11. *Let $P(\lambda)$ be a matrix polynomial of even degree k, let $s=(k-2)/2$, and consider a pencil $\mathcal{L}(\lambda)\in\langle\mathcal{E}_1^P\rangle$ of the form (4.27). If A_0, D and $D^{\mathcal{B}}$ are nonsingular, then $\mathcal{L}(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$. Moreover, if $P(\lambda)$ and all the block-entries B_{ij}, C_{ij} and D_{ij} are symmetric (resp., Hermitian), then $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian).*

Proof. If A_0 is nonsingular, the pencil $\mathcal{E}_1^P(\lambda)$ is a strong linearization of $P(\lambda)$. Additionally, if D and $D^{\mathcal{B}}$ are nonsingular, the pencil $\mathcal{L}(\lambda)$ is strictly equivalent to $\mathcal{E}_1^P(\lambda)$ (see (4.26)). Thus, if A_0, D and $D^{\mathcal{B}}$ are nonsingular, $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$. The fact that $\mathcal{L}(\lambda)$ is block-symmetric follows readily from (4.11) and the fact that $M(\lambda;P_{k-1})$ is block-symmetric. Finally, the fact that $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian) when $P(\lambda)$ and all the block-entries B_{ij}, C_{ij} and D_{ij} are symmetric (resp., Hermitian) follows easily from the following facts. First, $M(\lambda;P_{k-1})$ is symmetric (resp., Hermitian) when $P(\lambda)$ is symmetric (resp., Hermitian). Secondly, we have $B^{\mathcal{B}}=B^T$ (resp., $B^{\mathcal{B}}=B^*$), $C^{\mathcal{B}}=C^T$ (resp., $C^{\mathcal{B}}=C^*$) and $D^{\mathcal{B}}=D^T$ (resp., $D^{\mathcal{B}}=D^*$) when all the block-entries B_{ij}, C_{ij} and D_{ij} are symmetric (resp., Hermitian). ∎

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EXAMPLE 4.12. Let $P(\lambda)$ be a matrix polynomial of even degree k and let m be an odd positive integer. Then, as we will show in Theorem 6.2, the mth pencil $D_m(\lambda, P)$ in the standard basis of the vector space $\mathbb{DL}(P)$, which is a block-symmetric GFPR with parameter $h = k - m$, is permutationally block congruent to a pencil in $\langle \mathcal{E}_1^P \rangle$. This holds true, in particular, for $D_1(\lambda, P)$.

4.4. The fourth fundamental block-structure. The fourth fundamental block-structure family is also associated with even-degree matrix polynomials. So, let $P(\lambda)$ be a matrix polynomial of even degree k, and let $s := (k-2)/2$. First, we define the pencil

$$\mathcal{E}_2^P(\lambda) := \left[\begin{array}{c:c|c} -A_k & \lambda A_k \quad 0 & 0 \\ \hdashline \begin{matrix} \lambda A_k \\ 0 \end{matrix} & M(\lambda; P^{k-1}) & K_s(\lambda)^T \\ \hline 0 & K_s(\lambda) & 0 \end{array}\right], \tag{4.28}$$

where $P^{k-1}(\lambda)$ is defined in (4.14) and $M(\lambda; P^{k-1})$ is defined in (4.13). By applying a block column-permutation Π_4 to $\mathcal{E}_2^P(\lambda)$, we obtain the pencil

$$\mathcal{E}_2^P(\lambda)\Pi_4 := \left[\begin{array}{c|c:c} \lambda A_k \quad 0 & -A_k & 0 \\ \hdashline M(\lambda; P^{k-1}) & \begin{matrix} \lambda A_k \\ 0 \end{matrix} & K_s(\lambda)^T \\ \hline K_s(\lambda) & 0 & 0 \end{array}\right],$$

which is an extended $(s, n, s+1, n)$-block Kronecker pencil. Notice that the body of the pencil $\mathcal{E}_2^P(\lambda)\Pi_4$ satisfies the AS condition for $P(\lambda)$. Hence, $\mathcal{E}_2^P(\lambda)\Pi_4$ and, thus, $\mathcal{E}_2^P(\lambda)$, are strong linearizations of $P(\lambda)$ if A_k is nonsingular. Moreover, the pencil $\mathcal{E}_2^P(\lambda)$ is block-symmetric, and it is symmetric (resp., Hermitian) provided that $P(\lambda)$ is symmetric (resp., Hermitian).

EXAMPLE 4.13. Let $P(\lambda) = \sum_{i=0}^{6} A_i \lambda^i \in \mathbb{F}[\lambda]^{n\times n}$. Then,

$$\mathcal{E}_2^P(\lambda) = \left[\begin{array}{cccc|cc} -A_6 & \lambda A_6 & 0 & 0 & 0 & 0 \\ \lambda A_6 & \lambda A_5 + A_4 & 0 & 0 & -I_n & 0 \\ 0 & 0 & \lambda A_3 + A_2 & 0 & \lambda I_n & -I_n \\ 0 & 0 & 0 & \lambda A_1 + A_0 & 0 & \lambda I_n \\ \hline 0 & -I_n & \lambda I_n & 0 & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 \end{array}\right].$$

By permuting the first block-column with the block-columns in positions 2-4, we get

$$\mathcal{E}_2^P(\lambda)\Pi_4 = \left[\begin{array}{ccc|ccc} \lambda A_6 & 0 & 0 & -A_6 & 0 & 0 \\ \lambda A_5 + A_4 & 0 & 0 & \lambda A_6 & -I_n & 0 \\ 0 & \lambda A_3 + A_2 & 0 & 0 & \lambda I_n & -I_n \\ 0 & 0 & \lambda A_1 + A_0 & 0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right],$$

which is clearly an extended block Kronecker pencil.

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Inspired by the block-structure of $\mathcal{E}_2^P(\lambda)\Pi_4$, we consider extended block Kronecker pencils of the form

$$\begin{bmatrix} I_n & 0 & C \\ 0 & I_{(s+1)n} & B \\ 0 & 0 & D \end{bmatrix} \left[\begin{array}{cc|c:c} \lambda A_k & 0 & -A_k & 0 \\ \hdashline \multicolumn{2}{c|}{M(\lambda;P^{k-1})} & \begin{matrix}\lambda A_k \\ 0\end{matrix} & K_s(\lambda)^T \\ \hline \multicolumn{2}{c|}{K_s(\lambda)} & 0 & 0 \end{array}\right] \begin{bmatrix} I_{(s+1)n} & 0 & 0 \\ 0 & I_n & 0 \\ B^{\mathcal{B}} & C^{\mathcal{B}} & D^{\mathcal{B}} \end{bmatrix} \tag{4.29}$$

for arbitrary matrices $C \in \mathbb{F}^{n\times sn}$, $B \in \mathbb{F}^{(s+1)n\times sn}$ and $D \in \mathbb{F}^{sn\times sn}$, or, equivalently,

$$\left[\begin{array}{c|c:c} \begin{bmatrix}\lambda A_k & 0\end{bmatrix} + CK_s(\lambda) & -A_k & 0 \\ \hdashline M(\lambda;P^{k-1}) + BK_s(\lambda) + K_s(\lambda)^T B^{\mathcal{B}} & \begin{bmatrix}\lambda A_k \\ 0\end{bmatrix} + K_s(\lambda)^T C^{\mathcal{B}} & K_s(\lambda)^T D^{\mathcal{B}} \\ \hline DK_s(\lambda) & 0 & 0 \end{array}\right].$$

Reversing the block-permutation we did originally, we obtain the block-structure defining the fourth family of block-structures.

Definition 4.14. Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial of even degree k, and let $s = (k-2)/2$. The *fourth fundamental block-structure family*, denoted by $\langle \mathcal{E}_2^P \rangle$, is the set of pencils of the form

$$\left[\begin{array}{c:c|c} -A_k & \begin{bmatrix}\lambda A_k & 0\end{bmatrix} + CK_s(\lambda) & 0 \\ \hdashline \begin{bmatrix}\lambda A_k \\ 0\end{bmatrix} + K_s(\lambda)^T C^{\mathcal{B}} & M(\lambda;P^{k-1}) + BK_s(\lambda) + K_s(\lambda)^T B^{\mathcal{B}} & K_s(\lambda)^T D^{\mathcal{B}} \\ \hline 0 & DK_s(\lambda) & 0 \end{array}\right], \tag{4.30}$$

where $P^{k-1}(\lambda)$ is defined in (4.14) and $M(\lambda;P^{k-1})$ is defined in (4.13), for an arbitrary $(s+1)\times s$ block-matrix $B = [B_{ij}]$, $1\times s$ block-matrix $C = [C_{ij}]$, and $s\times s$ block-matrix $D = [D_{ij}]$, with $n\times n$ block entries B_{ij}, C_{ij} and D_{ij}, respectively.

Note that all pencils in $\langle \mathcal{E}_2^P \rangle$ are block minimal bases pencils if A_k, D, and $D^{\mathcal{B}}$ are nonsingular.

The following theorem gives necessary and sufficient conditions for pencils in the family $\langle \mathcal{E}_2^P \rangle$ to be strong linearizations of the even-degree matrix polynomial $P(\lambda)$.

Theorem 4.15. *Let $P(\lambda)$ be a matrix polynomial of even degree k, let $s = (k-2)/2$, consider a pencil $\mathcal{L}(\lambda) \in \langle \mathcal{E}_2^P \rangle$ of the form (4.30). If A_k, D and $D^{\mathcal{B}}$ are nonsingular, then $\mathcal{L}(\lambda)$ is a block-symmetric strong linearization of $P(\lambda)$. Moreover, if $P(\lambda)$ and all the block-entries B_{ij}, C_{ij} and D_{ij} are symmetric (resp., Hermitian), then $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian).*

Proof. If A_k is nonsingular, the pencil $\mathcal{E}_2^P(\lambda)$ is a strong linearization of $P(\lambda)$. In addition, notice from (4.29) that if D and $D^{\mathcal{B}}$ are nonsingular, the pencil $\mathcal{L}(\lambda)$ is strictly equivalent to $\mathcal{E}_2^P(\lambda)$. Thus, if A_k, D and $D^{\mathcal{B}}$ are nonsingular, then $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$. The fact that $\mathcal{L}(\lambda)$ is block-symmetric follows easily from (4.12) and the fact that $M(\lambda;P^{k-1})$ is block-symmetric. Finally, notice the following two facts. First, if $P(\lambda)$ is symmetric (resp., Hermitian) so is $M(\lambda;P^{k-1})$. Secondly, when all the block-entries of B, C and D are symmetric (resp., Hermitian), we have $B^{\mathcal{B}} = B^T$ (resp., $B^{\mathcal{B}} = B^*$), $C^{\mathcal{B}} = C^T$ (resp., $C^{\mathcal{B}} = C^*$) and $D^{\mathcal{B}} = D^T$ (resp., $D^{\mathcal{B}} = D^*$). Hence, if $P(\lambda)$ and all the block-entries B_{ij}, C_{ij} and D_{ij} are symmetric (resp., Hermitian), then $\mathcal{L}(\lambda)$ is symmetric (resp., Hermitian). ∎

Example 4.16. Let $P(\lambda)$ be a matrix polynomial of even degree k and let m be an even positive integer. Then, as we will show in Theorem 6.2, the mth pencil $D_m(\lambda, P)$ in the standard basis of the vector space

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$\mathbb{DL}(P)$, which is a block-symmetric GFPR with parameter $h = k - m$, is permutationally block congruent to a pencil in $\langle \mathcal{E}_2^P \rangle$. This holds true, in particular, for $D_k(\lambda, P)$.

4.5. Eigenvector recovery formulas. Given a regular matrix polynomial $P(\lambda)$ as in (1.1), we say that $x \neq 0$ (resp., $y^T \neq 0$) is a right (resp., left) eigenvector of $P(\lambda)$ associated with an eigenvalue $\delta \in \mathbb{F}$ if $P(\delta)x = 0$ (resp., $y^T P(\delta) = 0$). We say that $P(\lambda)$ has an infinite eigenvalue if 0 is an eigenvalue of rev $P(\lambda) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} A_{k-i}\lambda^i$.

When solving the polynomial eigenvalue problem associated with $P(\lambda)$ using a strong linearization $\mathcal{L}(\lambda)$, even though $P(\lambda)$ and $\mathcal{L}(\lambda)$ share the finite and infinite eigenvalues, it is clear that they do not have the same eigenspaces since $P(\lambda)$ and $\mathcal{L}(\lambda)$ are matrices of different size. Thus, when choosing a linearization for $P(\lambda)$, it is important to choose one that allows an easy recovery of the left and right eigenvectors of $P(\lambda)$ from those of $\mathcal{L}(\lambda)$.

We note that abstract eigenvector recovery procedures for eigenvectors of regular matrix polynomials from those of block minimal bases pencils can be found in [16, Section 6.3]. Since the results in that paper are not explicit for block minimal bases pencils that are not block Kronecker, here we provide explicit recovery formulas for the eigenvectors of $P(\lambda)$ from those of any of its linearizations in the four block-structures introduced in this section. In order to do so, we start by providing formulas for the eigenvectors of the linearizations of $P(\lambda)$ in any of the four fundamental block-structures. We first construct the eigenvectors of the four "generators", that is, $\mathcal{O}_1^P(\lambda)$, $\mathcal{O}_2^P(\lambda)$, $\mathcal{E}_1^P(\lambda)$ and $\mathcal{E}_2^P(\lambda)$ and then, we construct the eigenvectors of any linearization of $P(\lambda)$ in a given fundamental block-structure from those of the generator.

In order to provide a compact notation for the block-entries of the eigenvectors of the linearizations, we recall the family of *Horner shifts* associated with a matrix polynomial $P(\lambda)$ as in (1.1). This family is defined recursively as follows.

Definition 4.17. Let $P(\lambda)$ be a matrix polynomial as in (1.1). The family of Horner shifts associated with $P(\lambda)$ is given by

$$\begin{aligned} &P_0(\lambda) = A_k \\ &P_i(\lambda) = \lambda P_{i-1}(\lambda) + A_{k-i} \quad \text{for } i = 1 : k. \end{aligned}$$

Using the Horner shifts, we construct four $kn \times n$ matrix polynomials. They are used to obtain one-sided factorizations of the pencils $\mathcal{O}_1^P(\lambda)$, $\mathcal{O}_2^P(\lambda)$, $\mathcal{E}_1^P(\lambda)$ and $\mathcal{E}_2^P(\lambda)$.

Definition 4.18. Let $P(\lambda)$ be a matrix polynomial as in (1.1) of degree k. If k is odd, we define

$$H_1^P(\lambda) := \begin{bmatrix} \lambda^{(k-1)/2} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ \lambda^{(k-1)/2} P_1(\lambda) \\ \lambda^{(k-3)/2} P_3(\lambda) \\ \vdots \\ \lambda^2 P_{k-4}(\lambda) \\ \lambda P_{k-2}(\lambda) \end{bmatrix} \quad \text{and} \quad H_2^P(\lambda) := \begin{bmatrix} \lambda^{(k+1)/2} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ \lambda^{(k-1)/2} P_2(\lambda) \\ \lambda^{(k-3)/2} P_4(\lambda) \\ \vdots \\ \lambda^3 P_{k-5}(\lambda) \\ \lambda^2 P_{k-3}(\lambda) \end{bmatrix} \tag{4.31}$$

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If k is even, we define

$$R_1^P(\lambda) := \begin{bmatrix} \lambda^{k/2} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ \lambda^{k/2} P_1(\lambda) \\ \lambda^{k/2-1} P_3(\lambda) \\ \vdots \\ \lambda^3 P_{k-5}(\lambda) \\ \lambda^2 P_{k-3}(\lambda) \end{bmatrix} \quad \text{and} \quad R_2^P(\lambda) := \begin{bmatrix} \lambda^{k/2} I_n \\ \vdots \\ \lambda I_n \\ I_n \\ \lambda^{k/2-1} P_2(\lambda) \\ \lambda^{k/2-2} P_4(\lambda) \\ \vdots \\ \lambda^2 P_{k-4}(\lambda) \\ \lambda P_{k-2}(\lambda) \end{bmatrix} \tag{4.32}$$

THEOREM 4.19. *Let $P(\lambda)$ be a matrix polynomial as in (1.1) of degree k, let $H_1^P(\lambda)$, $H_2^P(\lambda)$, $R_1^P(\lambda)$ and $R_2^P(\lambda)$ be the matrix polynomials defined in (4.31) and (4.32), respectively. If k is odd, then*

$$\mathcal{O}_1^P(\lambda) H_1^P(\lambda) = e_{\frac{k+1}{2}} \otimes P(\lambda) \quad \textit{and} \quad \mathcal{O}_2^P(\lambda) H_2^P(\lambda) = e_{\frac{k+1}{2}} \otimes P(\lambda).$$

If k is even, then

$$\mathcal{E}_1^P(\lambda) R_1^P(\lambda) = e_{\frac{k}{2}} \otimes P(\lambda) \quad \textit{and} \quad \mathcal{E}_2^P(\lambda) R_2^P(\lambda) = e_{\frac{k+2}{2}} \otimes P(\lambda),$$

where e_i denotes the ith column of the $k \times k$ identity matrix.

Proof. These results can be checked by performing directly the matrix multiplications $\mathcal{O}_1^P(\lambda)H_1^P(\lambda)$, $\mathcal{O}_2^P(\lambda)H_2^P(\lambda)$, $\mathcal{E}_1^P(\lambda)R_1^P(\lambda)$ and $\mathcal{E}_2^P(\lambda)R_2^P(\lambda)$. ∎

These right-sided factorizations allow us to relate the eigenvectors of the matrix polynomial $P(\lambda)$ to the eigenvectors of $\mathcal{O}_1^P(\lambda)$, $\mathcal{O}_2^P(\lambda)$, $\mathcal{E}_1^P(\lambda)$ and $\mathcal{E}_2^P(\lambda)$, under certain nonsingularity and degree-parity conditions.

THEOREM 4.20. *Let $P(\lambda)$ be a regular matrix polynomial as in (1.1) of degree k. Let δ be a finite eigenvalue of $P(\lambda)$.*

If k is odd, then the following statements hold.

- (a-1) *z is a right eigenvector of $\mathcal{O}_1^P(\lambda)$ associated with δ if and only if $z = H_1^P(\delta)x$ for some right eigenvector x of $P(\lambda)$ associated with δ.*
- (a-2) *If A_0 and A_k are both nonsingular matrices, then z is a right eigenvector of $\mathcal{O}_2^P(\lambda)$ associated with δ if and only if $z = H_2^P(\delta)x$ for some right eigenvector x of $P(\lambda)$.*

If k is even, then the following statements hold.

- (b-1) *If A_0 is nonsingular, then z is a right eigenvector of $\mathcal{E}_1^P(\lambda)$ associated with δ if and only if $z = R_1^P(\delta)x$ for some right eigenvector x of $P(\lambda)$ associated with δ.*
- (b-2) *If A_k is nonsingular, then z is a right eigenvector of $\mathcal{E}_2^P(\lambda)$ associated with δ if and only if $z = R_2^P(\delta)x$ for some right eigenvector x of $P(\lambda)$ associated with δ.*

Proof. We only prove (a-1). The rest of the claims in this theorem can be proven similarly.

Assume that x is an eigenvector of $P(\lambda)$ associated with δ. Then, by Theorem 4.19, we have

$$\mathcal{O}_1^P(\delta) H_1^P(\delta) x = (e_{\frac{k+1}{2}} \otimes P(\delta)) x = e_{\frac{k+1}{2}} \otimes P(\delta) x = 0,$$

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where the last equality follows from the fact that x is a right eigenvector of $P(\lambda)$ associated with δ. Note that $H_1^P(\delta)x \neq 0$ since $x \neq 0$ and the $\left(\frac{k+1}{2}\right)$th block-entry of $H_1^P(\lambda)$ is I_n. Therefore, $H_1^P(\delta)x$ is a right eigenvector of $\mathcal{O}_1^P(\lambda)$ associated with δ.

Now we prove the converse. Assume that $\delta \in \mathbb{F}$ is a finite eigenvalue of $\mathcal{O}_1^P(\lambda)$ with geometric multiplicity m and let z be any right eigenvector of $\mathcal{O}_1^P(\lambda)$ associated with δ. Since $\mathcal{O}_1^P(\lambda)$ is a linearization of $P(\lambda)$, the geometric multiplicity of δ as an eigenvalue of $P(\lambda)$ is also m. Let $\{x_1, \ldots, x_m\}$ be linearly independent eigenvectors of $P(\lambda)$ associated with δ and define $z_i = H_1^P(\delta)x_i$, for $i = 1 : m$. Then, taking into account the definition of $H_1^P(\lambda)$, it is clear that $z_1, \ldots, z_m$ are linearly independent right eigenvectors of $\mathcal{O}_1^P(\lambda)$. Thus, z is a linear combination of $z_1, \ldots, z_m$, which implies that z is of the form $H_1^P(\delta)x$ for some right eigenvector x of $P(\lambda)$. ∎

The next result provides explicit formulas for the recovery of the right eigenvectors of $P(\lambda)$ from the right eigenvectors of any pencil $\mathcal{L}(\lambda)$ in any of the fundamental block-structure families, provided that $\mathcal{L}(\lambda)$ is a strong linearization of $P(\lambda)$. We observe that we do not need to consider recovery formulas for eigenvectors associated with eigenvalues at infinity for pencils in $\langle \mathcal{O}_2^P \rangle$ and $\langle \mathcal{E}_2^P \rangle$, since when $P(\lambda)$ has eigenvalues at infinity, none of them are strong linearizations of $P(\lambda)$.

We will use the following notation: Let z be a column vector of length kn. Then, when z is seen as a column block-vector with k block-entries (each of size n), the vector $z(i)$ denotes the ith block-entry of z, counting from top to bottom.

Theorem 4.21. *Let $P(\lambda)$ be a regular matrix polynomial as in (1.1) of degree k and let δ be an eigenvalue of $P(\lambda)$.*

Let k be odd. The following statements hold.

- *(a) Let $\mathcal{L}(\lambda) \in \langle \mathcal{O}_1^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is finite and z is a right eigenvector of $\mathcal{L}(\lambda)$ associated with δ, then $z(\frac{k+1}{2})$ is a right eigenvector of $P(\lambda)$ associated with δ. In addition, if $\delta \neq 0$, the block-entries $z(1), z(2), \ldots, z(\frac{k-1}{2})$ of z are also right eigenvectors of $P(\lambda)$ associated with δ.*
- *(b) Let $\mathcal{L}(\lambda) \in \langle \mathcal{O}_1^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is infinite and z is a right eigenvector of $\mathcal{L}(\lambda)$ associated with δ, then $z(1)$ is a right eigenvector of $P(\lambda)$ associated with δ.*
- *(c) Let $\mathcal{L}(\lambda) \in \langle \mathcal{O}_2^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is finite and z is a right eigenvector of $\mathcal{L}(\lambda)$, then $z(\frac{k+3}{2})$ is a right eigenvector of $P(\lambda)$ associated with δ. In addition, if $\delta \neq 0$, then the block-entries $z(1), z(2), \ldots, z(\frac{k+1}{2})$ are also right eigenvectors of $P(\lambda)$ associated with δ.*

Let k be even. The following statements hold.

- *(d) Let $\mathcal{L}(\lambda) \in \langle \mathcal{E}_1^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is finite and z is a right eigenvector of $\mathcal{L}(\lambda)$, then $z(\frac{k+2}{2})$ is a right eigenvector of $P(\lambda)$ associated with δ. In addition, if $\delta \neq 0$, the block-entries $z(1), z(2), \ldots, z(\frac{k}{2})$ of z are also right eigenvectors of $P(\lambda)$ associated with δ.*
- *(e) Let $\mathcal{L}(\lambda) \in \langle \mathcal{E}_1^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is infinite and z is a right eigenvector of $\mathcal{L}(\lambda)$ associated with δ, then $z(1)$ is a right eigenvector of $P(\lambda)$ associated with δ.*
- *(f) Let $\mathcal{L}(\lambda) \in \langle \mathcal{E}_2^P \rangle$ be a strong linearization of $P(\lambda)$. If δ is finite and z is a right eigenvector of $\mathcal{L}(\lambda)$, then $z(\frac{k+2}{2})$ is a right eigenvector of $P(\lambda)$ associated with δ. In addition, if $\delta \neq 0$, the block-entries $z(1), z(2), \ldots, z(\frac{k}{2})$ of z are also right eigenvectors of $P(\lambda)$ associated with δ.*

Proof. Let $\mathcal{L}(\lambda)$ be a pencil in any of the four fundamental block-structure families. If we denote by $\mathcal{G}(\lambda)$ the generator of its family, then recall that $\mathcal{L}(\lambda)$ is related with $\mathcal{G}(\lambda)$ via a strict equivalence $\mathcal{L}(\lambda) = U^{\mathcal{B}}\mathcal{G}(\lambda)U$

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(see equations (4.18), (4.23), (4.26), and (4.29)). The pencil $\mathcal{L}(\lambda)$ is regular because it is a strong linearization of a regular matrix polynomial $P(\lambda)$. Hence, the matrix U must be nonsingular (otherwise, $\mathcal{L}(\lambda)$ would be singular). This implies that z is a right eigenvector of $\mathcal{G}(\lambda)$ if and only if $U^{-1}z$ is a right eigenvector of $\mathcal{L}(\lambda)$. So we only need to prove the results for the four generators, since the results for any pencil $\mathcal{L}(\lambda)$ in any of the four families follows from $U^{-1}z$ and the block-structure of U given in (4.18), (4.23), (4.26), and (4.29).

Assume first that δ is a finite eigenvalue. Then, parts (a), (c), (d) and (f) for the generators follow immediately from Definition 4.18 and Theorem 4.20.

Assume now that δ is an infinite eigenvalue. Notice that part (b) for $\mathcal{O}_1^P(\lambda)$ follows from [16, Theorem 6.9], since $\mathcal{O}_1^P(\lambda)$ is a block Kronecker pencil. So we just need to prove part (e). Recall that the right eigenvectors of $P(\lambda)$ associated with the eigenvalue at infinity are just the right null vectors of rev $P(0) = A_k$, the leading coefficient of $P(\lambda)$. Note that A_0 must be nonsingular, because $\mathcal{E}_1^P(\lambda)$ is a strong linearization of a regular $P(\lambda)$. Since the leading coefficient of $\mathcal{E}_1^P(\lambda)$ is

$$\left[\begin{array}{cccc|c|ccc}
A_k & & & & & & & \\
 & A_{k-2} & & & & I_n & & \\
 & & \ddots & & & & \ddots & \\
 & & & A_2 & & & & I_n \\
\hline
 & & & & -A_0 & & & \\
\hline
 & I_n & & & & & & \\
 & & \ddots & & & & & \\
 & & & I_n & & & &
\end{array}\right]$$

and the matrix A_0 is nonsingular, we immediately obtain that z is a right eigenvector of $\mathcal{E}_1^P(\lambda)$ associated with the eigenvalue infinity if and only if $z = [x^T, 0, \ldots, 0]^T$, where x is a right eigenvector of $P(\lambda)$ associated with the eigenvalue infinity. □

The recovery procedures for left eigenvectors are exactly the same as those for right eigenvectors. To see this, recall first that the left eigenvectors of $P(\lambda)$ associated with an eigenvalue δ are the right eigenvectors of $P(\lambda)^T$ associated with δ. Then, from (4.18), (4.23), (4.26), and (4.29), together with the fact that the transpose and the block-transpose operations commute, it follows that, if $\mathcal{L}(\lambda)$ belongs to one of the four fundamental families associated with $P(\lambda)$, then $\mathcal{L}(\lambda)^T$ belongs to the same fundamental family but associated with $P(\lambda)^T$. Hence, we can apply Theorem 4.21 to $\mathcal{L}(\lambda)^T$ for recovering the right eigenvectors of $P(\lambda)^T$, that is, the left eigenvectors of $P(\lambda)$, from the right eigenvectors of $\mathcal{L}(\lambda)^T$, which are the left eigenvectors of $\mathcal{L}(\lambda)$.

5. Block-symmetric generalized Fiedler pencils and block-symmetric generalized Fiedler pencils with repetition. We introduce in this section the block-symmetric generalized Fiedler pencils and the family of block-symmetric generalized Fiedler pencils with repetition. We start with some concepts and basic results needed for those definitions.

5.1. The index tuple notation and matrix assignments. We start by introducing the fundamental definition of an index tuple and some related notions.

Definition 5.1. [4, Definition 3.1] We call an *index tuple* a finite ordered sequence of integer numbers. Each of these integers is called an *index* of the tuple. The number of indices in an index tuple $\mathbf{t}$ is called its

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length and is denoted by $|\mathbf{t}|$. For integers a and b, we call the tuple $(a:b)$ a *string*.

We will use the following notation for some important basic operations with tuples. If $\mathbf{t} = (t_1, \ldots, t_r)$ is an index tuple, we denote $-\mathbf{t} := (-t_1, \ldots, -t_r)$, and, when a is an integer, we denote $a + \mathbf{t} := (a + t_1, a + t_2, \ldots, a + t_r)$. We call the *reversal index tuple* of $\mathbf{t}$ the index tuple $\operatorname{rev}(\mathbf{t}) := (t_r, \ldots, t_2, t_1)$. Additionally, given index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$, we denote by $(\mathbf{t}_1, \ldots, \mathbf{t}_s)$ the index tuple obtained by concatenating the indices in the index tuples $\mathbf{t}_1, \ldots, \mathbf{t}_s$ in the indicated order.

An important property of index tuples used to define the block-symmetric GFPR is the so-called Successor Infix Property, which we introduce in the following definition.

Definition 5.2. [30, Definition 7] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple of either all nonnegative integers or all negative integers. Then, $\mathbf{t}$ is said to satisfy the *Successor Infix Property (SIP)* if for every pair of indices $i_a, i_b \in \mathbf{t}$, with $1 \le a < b \le r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ with $a < c < b$.

Remark 5.3. We note the following basic properties of tuples satisfying the SIP. Any subtuple of consecutive indices of a tuple satisfying the SIP also satisfies the SIP. The reversal of any tuple satisfying the SIP also satisfies the SIP. If the tuple $\mathbf{t}$ has no repeated indices, then $\mathbf{t}$ satisfies the SIP.

The following definitions are motivated by the construction of block-symmetric GFPR in Section 5. For more details, we refer the reader to [2, Section 4] and [7].

Definition 5.4. [2, Definition 4.3] Let h be a nonnegative integer, and let $p = 0$ if h is even, and $p = 1$ is h is odd. Then, we call the index tuple

$$\mathbf{w}_h := (h-1:h, h-3:h-2, \ldots, p+1:p+2, 0:p)$$

the *admissible tuple associated with the integer* $h \ge 0$.

Notice that the tuple $\mathbf{w}_h$ is a permutation of the tuple $(0:h)$.

Definition 5.5. [2, Definition 4.3] Let h be a nonnegative integer, and let $\mathbf{w}_h$ be the admissible tuple associated with h. Then, the symmetric complement of $\mathbf{w}_h$ is the tuple

- $\mathbf{c}_h := (h-1, h-3, \ldots, 2, 0)$ if h is odd;
- $\mathbf{c}_h := (h-1, h-3, \ldots, 1)$ if $h > 0$ is even;
- $\mathbf{c}_h := \emptyset$ if $h = 0$.

Lemma 5.6. [3, Lemma 3.11] *Let h be a nonnegative integer, let $\mathbf{w}_h$ be the admissible tuple associated with h, and let $\mathbf{c}_h$ be the symmetric complement of $\mathbf{w}_h$. Then, the index tuple $(\mathbf{w}_h, \mathbf{c}_h)$ satisfies the SIP.*

The matrix coefficients of the block-symmetric GFPR (and that we review in this section) are products of elementary block-matrices, whose definition we recall next.

Definition 5.7. [4] *Let $k \ge 2$ be an integer and let B be an arbitrary $n \times n$ matrix. We call* elementary matrices *the following $k \times k$ block-matrices partitioned into blocks of size $n \times n$:*

$$M_0(B) := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & B \end{array}\right], \quad M_{-k}(B) := \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right],$$

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(5.33)
$$M_i(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & B & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1:k-1,$$

$$M_{-i}(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & B & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1:k-1,$$

and

$$M_{-0}(B) := M_0(B)^{-1} \quad \textit{and} \quad M_k(B) := M_{-k}(B)^{-1}.$$

assuming that B is nonsingular.

Notice that the notation -0 does not have the usual meaning, that is, in this case $-0 \neq 0$.

Remark 5.8. Notice that, for $i = 1 : k-1$, the elementary matrices $M_i(B)$ and $M_{-i}(B)$ are nonsingular for any B. Furthermore, $(M_i(B))^{-1} = M_{-i}(-B)$. On the other hand, the matrices $M_0(B)$ and $M_{-k}(B)$ are nonsingular if and only if B is nonsingular.

Definition 5.9. [2, Definition 4.6] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with indices contained in $\{-k : k-1\}$ and let $\mathcal{Z} = (Z_1, \ldots, Z_r)$ be a list of r arbitrary $n \times n$ matrices. We define

$$M_{\mathbf{t}}(\mathcal{Z}) := M_{i_1}(Z_1) M_{i_2}(Z_2) \cdots M_{i_r}(Z_r),$$

and say that $\mathcal{Z}$ is a *matrix assignment for* $\mathbf{t}$. If $\mathbf{t}$ (and therefore $\mathcal{Z}$) is empty, then $M_{\mathbf{t}}(\mathcal{Z}) := I_{kn}$. The matrix assignment $\mathcal{Z}$ for $\mathbf{t}$ is said to be *nonsingular* if the matrices assigned to the positions in $\mathbf{t}$ occupied by the 0 and $-k$ indices are nonsingular. If the matrices in $\mathcal{Z}$ are symmetric (resp., Hermitian), then $\mathcal{Z}$ is said to be a *symmetric (resp., Hermitian) matrix assignment* for $\mathbf{t}$.

Given an ordered list of $n \times n$ arbitrary matrices $\mathcal{Z} = (Z_1, \ldots, Z_r)$, we denote by $\mathrm{rev}(\mathcal{Z})$ the list of matrices $(Z_r, \ldots, Z_1)$.

Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \in \mathbb{F}[\lambda]^{n \times n}$, we will use the following abbreviated notation:

$$M_i^P := M_i(-A_i), \quad i = 0 : k-1,$$

and

$$M_{-i}^P := M_{-i}(A_i), \quad i = 1 : k.$$

When the polynomial $P(\lambda)$ is understood from the context, we simply write M_i and M_{-i}, instead of M_i^P and M_{-i}^P to simplify the notation.

5.2. Block-symmetric GFP. Here, we recall the block-symmetric strong linearizations of a matrix polynomial $P(\lambda)$ in the family of generalized Fiedler pencils (GFP). The following GFP was introduced in [1, Theorem 3.1]:

$$\mathcal{T}_P(\lambda) := \lambda M^P_{-1,-3,\ldots,-k+2,-k} - M^P_{0,2,\ldots,k-1},$$

if k is odd, and

$$\mathcal{T}_P(\lambda) := \lambda M^P_{-1,-3,\ldots,-k+1} - M^P_{0,2,\ldots,k},$$

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if k is even and the leading coefficient A_k is nonsingular. The pencil $\mathcal{T}_P(\lambda)$ is explicitly given by

$$\mathcal{T}_P(\lambda) = \begin{bmatrix} \lambda A_k + A_{k-1} & -I_n & & & & & \\ -I_n & 0 & \lambda I_n & & & & \\ & \lambda I_n & \lambda A_{k-2} + A_{k-3} & -I_n & & & \\ & & -I_n & \ddots & & & \\ & & & & -I_n & & \\ & & & -I_n & 0 & \lambda I_n \\ & & & & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix}, \tag{5.34}$$

when k is odd, and by

$$\mathcal{T}_P(\lambda) = \begin{bmatrix} -A_k^{-1} & \lambda I_n & & & & & \\ \lambda I_n & \lambda A_{k-1} + A_{k-2} & -I_n & & & & \\ & -I_n & 0 & \lambda I_n & & & \\ & & \lambda I_n & \ddots & & & \\ & & & & -I_n & & \\ & & & -I_n & 0 & \lambda I_n \\ & & & & \lambda I_n & \lambda A_1 + A_0 \end{bmatrix},$$

when k is even and A_k is nonsingular. We note that this pencil is not a companion form since one of its matrix coefficients contains a block equal to A_k^{-1}. Notice that $\mathcal{T}_P(\lambda)$ is block-symmetric, regardless of the parity of k. Some small variations of these pencils can be found in [24].

5.3. Block-symmetric GFPR. Next, we recall a subfamily of GFPR comprised only of block-symmetric pencils.

DEFINITION 5.10. [2] Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ of degree k, and let h be an integer such that $0 \leq h < k$. Let $\mathbf{w}_h$ and $k + \mathbf{v}_h$ be the admissible tuples associated with h and $k-h-1$, respectively, and let $\mathbf{c}_h$ and $\mathbf{c}_{k-h-1}$ be the symmetric complements of $\mathbf{w}_h$ and $k + \mathbf{v}_h$, respectively. Let $\mathbf{t}_w$ and $k + \mathbf{t}_v$ be index tuples with indices from $\{0 : h-1\}$ and $\{0 : k-h-2\}$, respectively, such that $(\mathbf{t}_w, \mathbf{w}_h, \mathbf{c}_h, \mathrm{rev}(\mathbf{t}_w))$ and $(\mathbf{t}_v, \mathbf{v}_h, -k + \mathbf{c}_{k-h-1}, \mathrm{rev}(\mathbf{t}_v))$ satisfy the SIP. Let $\mathcal{Z}_w$ and $\mathcal{Z}_v$ be matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. Then, the pencil

$$M_{\mathbf{t}_w,\mathbf{t}_v}(\mathcal{Z}_w, \mathcal{Z}_v)(\lambda M^P_{\mathbf{v}_h} - M^P_{\mathbf{w}_h}) M^P_{-k+\mathbf{c}_{k-h-1},\mathbf{c}_h} M_{\mathrm{rev}(\mathbf{t}_w),\mathrm{rev}(\mathbf{t}_v)}(\mathrm{rev}(\mathcal{Z}_w), \mathrm{rev}(\mathcal{Z}_v)) \tag{5.35}$$

is a *block-symmetric generalized Fiedler pencil with repetition (block-symmetric GFPR)* and we denote it by $L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)$. If the matrix assignments $\mathcal{Z}_w$ and $\mathcal{Z}_v$ are chosen so that $M_{\mathbf{t}_w,\mathbf{t}_v}(\mathcal{Z}_w, \mathcal{Z}_v) = M^P_{\mathbf{t}_w,\mathbf{t}_v}$, then the block-symmetric GFPR $L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)$ is a *block-symmetric Fiedler pencil with repetition (block-symmetric FPR)*, which we denote by $L_P(h, \mathbf{t}_w, \mathbf{t}_v)$.

Theorem 5.11 establishes when a block-symmetric GFPR associated with a matrix polynomial $P(\lambda)$ is a strong linearization of $P(\lambda)$.

THEOREM 5.11. [2, Theorem 4.9] *Let $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ be a matrix polynomial, and let $\mathcal{L}(\lambda)$ be the block-symmetric GFPR defined in (5.35). If the following three conditions hold*

(i) *$\mathcal{Z}_w$ and $\mathcal{Z}_v$ are nonsingular matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively,*
(ii) *A_0 is nonsingular if h is odd, and*

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(iii) A_k *is nonsingular if* $k-h$ *is even,*

then, the pencil $\mathcal{L}(\lambda)$ *is a strong linearization of* $P(\lambda)$.

Theorem 5.12 gives sufficient conditions for a block-symmetric GFPR associated with a symmetric (resp., Hermitian) matrix polynomial $P(\lambda)$ to be symmetric (resp., Hermitian).

Theorem 5.12. [2] *Let* $P(\lambda)$ *be a symmetric (resp., Hermitian) matrix polynomial, and let* $\mathcal{L}(\lambda)$ *be the block-symmetric GFPR defined in* (5.35). *If the matrix assignments* $\mathcal{Z}_w$ *and* $\mathcal{Z}_v$ *are symmetric (resp., Hermitian), then* $\mathcal{L}(\lambda)$ *is symmetric (resp., Hermitian).*

6. Block-symmetric GFP and GFPR as block-symmetric block minimal bases pencils. In this section, we start by showing that the block-symmetric pencil $\mathcal{T}_P(\lambda)$ associated with an odd-degree matrix polynomial $P(\lambda)$ is permutationally block-congruent to the pencil $\mathcal{O}_1^P(\lambda)$. The case when $P(\lambda)$ has even degree is not considered, since $\mathcal{T}_P(\lambda)$ is not a companion form, that is, if the matrix coefficients of $\mathcal{T}_P(\lambda)$ are seen as block matrices, one of them contains a block that is not of the form 0, $\pm I_n$ or $\pm A_i$. In fact, since one of the matrix coefficients of $\mathcal{T}_P(\lambda)$ contains a block-entry that is the inverse of A_k, the interest of this pencil in applications is very limited.

Theorem 6.1. *Let* $P(\lambda)=\sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *be an odd-degree matrix polynomial. Let* $\mathcal{T}_P(\lambda)$ *be the block-symmetric GFP associated with* $P(\lambda)$ *defined in* (5.34). *Let* $\mathbf{c}$ *be the permutation of* $\{1:k\}$ *given by* $(1,3,5,\ldots,k,2,4,\ldots,k-1)$. *Then,*

$$(\Pi_{\mathbf{c}}^n)^{\mathcal{B}}\mathcal{T}_P(\lambda)\Pi_{\mathbf{c}}^n=\mathcal{O}_1^P(\lambda).$$

In other words, modulo block-permutations, the block-symmetric GFP $\mathcal{T}_P(\lambda)$ *belongs to the family* $\langle\mathcal{O}_1^P\rangle$.

Proof. Using the explicit expression for $\mathcal{T}_P(\lambda)$ presented in Section 5, the result is easily verified by performing the matrix product $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}}\mathcal{T}_P(\lambda)\Pi_{\mathbf{c}}^n$. ∎

Now we give the main result for block-symmetric GFPR associated with a matrix polynomial $P(\lambda)$, that is, we state that, up to permutations of block-rows and block-columns, every block-symmetric GFPR is in one of the four block-symmetric families introduced in Section 4. This result is stated in Theorem 6.2. We do not include its proof because it is very long and highly technical but very similar to the proof of [5, Theorem 8.1]. However, for the interested reader we have posted a full version of this paper in ArXiv including the proof of Theorem 6.2 in an appendix.

Theorem 6.2. *Let* $P(\lambda)=\sum_{i=0}^{k} A_i\lambda^i \in \mathbb{F}[\lambda]^{n\times n}$ *and let* $s=(k-1)/2$ *if* k *is odd, or* $s=(k-2)/2$ *is* k *is even. Let* $L_P(h,\mathbf{t}_w,\mathbf{t}_v,\mathcal{Z}_w,\mathcal{Z}_v)$ *be the block-symmetric GFPR associated with* $P(\lambda)$ *given in (5.35). Then, there exists a permutation* $\mathbf{c}$ *of* $\{1:k\}$ *such that*

$$(\Pi_{\mathbf{c}}^n)^{\mathcal{B}}L_P(h,\mathbf{t}_w,\mathbf{t}_v,\mathcal{Z}_w,\mathcal{Z}_v)\Pi_{\mathbf{c}}^n \in \begin{cases} \langle\mathcal{O}_1^P\rangle \text{ if } k \text{ is odd and } h \text{ is even,}\\ \langle\mathcal{O}_2^P\rangle \text{ if } k \text{ and } h \text{ are odd,}\\ \langle\mathcal{E}_1^P\rangle \text{ if } k \text{ is even and } h \text{ is odd,}\\ \langle\mathcal{E}_2^P\rangle \text{ if } k \text{ and } h \text{ are even.}\end{cases}$$

Furthermore, if the following conditions hold

(i) $\mathcal{Z}_w$ *and* $\mathcal{Z}_v$ *are nonsingular matrix assignments for* $\mathbf{t}_w$ *and* $\mathbf{t}_v$, *respectively,*
(ii) A_0 *is nonsingular if* h *is odd, and*
(iii) A_k *is nonsingular if* $k-h$ *is even,*

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then $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)\Pi_{\mathbf{c}}^n$ *is a strong linearization of* $P(\lambda)$ *and the following statements hold:*

(a) *If* $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)\Pi_{\mathbf{c}}^n \in \langle \mathcal{O}_1^P \rangle$ *is as in* (4.19), *then* B *and* $B^{\mathcal{B}}$ *are nonsingular.*
(b) *If* $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)\Pi_{\mathbf{c}}^n \in \langle \mathcal{O}_2^P \rangle$ *is as in* (4.24), *then* E *and* $E^{\mathcal{B}}$ *are nonsingular.*
(c) *If* $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)\Pi_{\mathbf{c}}^n \in \langle \mathcal{E}_1^P \rangle$ *is as in* (4.27), *then* D *and* $D^{\mathcal{B}}$ *are nonsingular.*
(d) *If* $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(h, \mathbf{t}_w, \mathbf{t}_v, \mathcal{Z}_w, \mathcal{Z}_v)\Pi_{\mathbf{c}}^n \in \langle \mathcal{E}_2^P \rangle$ *is as in* (4.30), *then* D *and* $D^{\mathcal{B}}$ *are nonsingular.*

The following example illustrates the result for a particular block-symmetric GFPR associated with an odd degree matrix polynomial.

EXAMPLE 6.3. Let $P(\lambda) = \sum_{i=0}^{7} A_i \lambda^i$ be an $n \times n$ matrix polynomial of degree 7. Consider the block-symmetric GFPR

$$L_P(\lambda) := L_P(k-1, \emptyset, \emptyset) = \begin{bmatrix} \lambda A_7 + A_6 & A_5 & -I_n & & & & \\ A_5 & -\lambda A_5 + A_4 & \lambda I_n & A_3 & -I_n & & \\ -I_n & \lambda I_n & 0 & 0 & 0 & & \\ & A_3 & 0 & -\lambda A_3 + A_2 & \lambda I_n & A_1 & -I_n \\ & -I_n & 0 & \lambda I_n & 0 & 0 & 0 \\ & & & A_1 & 0 & -\lambda A_1 + A_0 & \lambda I_n \\ & & & -I_n & 0 & \lambda I_n & 0 \end{bmatrix}.$$

Let $\mathbf{c} = (1, 2, 4, 6, 3, 5, 7)$. Then,

$$(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(\lambda)\Pi_{\mathbf{c}}^n = \left[\begin{array}{cccc|ccc} \lambda A_7 + A_6 & A_5 & 0 & 0 & -I_n & 0 & 0 \\ A_5 & -\lambda A_5 + A_4 & A_3 & 0 & \lambda I_n & -I_n & 0 \\ 0 & A_3 & -\lambda A_3 + A_2 & A_1 & 0 & \lambda I_n & -I_n \\ 0 & 0 & A_1 & -\lambda A_1 + A_0 & 0 & 0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_n & \lambda I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & \lambda I_n & 0 & 0 & 0 \end{array}\right].$$

We note that $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(\lambda)\Pi_{\mathbf{c}}^n \in \langle \mathcal{O}_1^P \rangle$ since

$$(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(\lambda)\Pi_{\mathbf{c}}^n = \begin{bmatrix} I_{4n} & C \\ 0 & I_{3n} \end{bmatrix} \mathcal{O}_1^P \begin{bmatrix} I_{4n} & C \\ 0 & I_{3n} \end{bmatrix}^{\mathcal{B}},$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 \\ -A_5 & 0 & 0 \\ 0 & -A_3 & 0 \\ 0 & 0 & -A_1 \end{bmatrix}.$$

Moreover, $(\Pi_{\mathbf{c}}^n)^{\mathcal{B}} L_P(\lambda)\Pi_{\mathbf{c}}^n$ is a strong linearization of every matrix polynomial $P(\lambda)$.

REMARK 6.4. We note that, when k is odd, by Example 4.4 and Theorem 6.1, the three pencils $D_1(\lambda, P)$, $D_k(\lambda, P)$ and $\mathcal{T}_P(\lambda)$ are permutationally congruent to some pencil in $\langle \mathcal{O}_1^P \rangle$. In fact, $\mathcal{T}_P(\lambda)$ is essentially $\mathcal{O}_1^P(\lambda)$, after permuting some block-rows and some block-columns. Thus, $\mathcal{T}_P(\lambda)$ could be seen, in layman's terms, as the "skeleton" of $D_1(\lambda, P)$ and $D_k(\lambda, P)$, that is, the least information that can be retained from

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these pencils without stopping from being a linearization of $P(\lambda)$. Hence, $\mathcal{T}_P(\lambda)$ is an ideal candidate to outperform numerically the combined use of $D_1(\lambda, P)$ and $D_k(\lambda, P)$ in the solution of the block-symmetric polynomial eigenvalue problem. This problem is studied in [8].

The following example gives the pencils in $\langle \mathcal{O}_1^P \rangle$ permutationally block congruent to $D_1(\lambda, P)$, $D_k(\lambda, P)$ and $\mathcal{T}_P(\lambda)$, when $k = 3$.

Example 6.5. Let $k = 3$. Then, $D_1(\lambda, P)$ is permutationally block congruent to the pencil

$$\left[\begin{array}{cc|c} \lambda A_3 + A_2 & A_1 & A_0 \\ A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ \hline A_0 & -\lambda A_0 & 0 \end{array}\right] \in \langle \mathcal{O}_1^P \rangle.$$

The pencil $D_k(\lambda, P)$ is permutationally block congruent to the pencil

$$\left[\begin{array}{cc|c} \lambda A_3 - A_2 & \lambda A_2 & -A_3 \\ \lambda A_2 & \lambda A_1 - A_0 & \lambda A_3 \\ \hline -A_3 & \lambda A_3 & 0 \end{array}\right] \in \langle \mathcal{O}_1^P \rangle.$$

The pencil $\mathcal{T}_P(\lambda)$ is permutationally block congruent to the pencil

$$\mathcal{O}_1^P(\lambda) = \left[\begin{array}{cc|c} \lambda A_3 + A_2 & 0 & -I_n \\ 0 & \lambda A_1 + A_0 & \lambda I_n \\ \hline -I_n & \lambda I_n & 0 \end{array}\right] = \mathcal{O}_1^P(\lambda).$$

7. Conclusions and future work. In this paper, we have introduced four families of block-symmetric pencils that, under some generic nonsingular conditions, are block-symmetric block minimal bases pencils and strong linearizations of a matrix polynomial $P(\lambda)$. Furthermore, we have shown that every block-symmetric GFP and block-symmetric GFPR is permutationally block-congruent to a pencil in the union of these four families, which provides an alternative approach to the implicit definition of these pencils as products of elementary matrices by providing their block structure. The importance of this result resides in the expectation that the explicit block structure of the block-symmetric GFP and GFPR will provide a venue to explore their numerical properties such as conditioning of eigenvalues and backward error of approximate eigenpairs. In particular, our objective is to find linearizations of $P(\lambda)$ in these famllies with optimal condition number and backward error that can replace the combined used of $D_1(\lambda, P)$ and $D_k(\lambda, P)$ when $P(\lambda)$ is symmetric or Hermitian, as suggested in the current literature.

REFERENCES

[1] E.N. Antoniou and S. Vologiannidis. A new family of companion forms of polynomial matrices. *Electron. J. Linear Algebra*, 11:78–87, 2004.
[2] M.I. Bueno, F.M. Dopico, and S. Furtado. Linearizations of Hermitian matrix polynomials preserving the sign characteristic. *SIAM J. Matrix Anal. Appl.*, 38:73–101, 2017.
[3] M.I. Bueno, K. Curlett, and S. Furtado. Structured strong linearizations from Fiedler pencils with repetition I. *Linear Algebra Appl.*, 460:51–80, 2014.
[4] M.I. Bueno, F.M. Dopico, S. Furtado, and M. Rychnovsky. Large vector spaces of block-symmetric strong linearizations of matrix polynomials. *Linear Algebra Appl.*, 477:165–210, 2015.
[5] M. Bueno, F.M. Dopico, J. Pérez, R. Saavedra, and B. Zykoski. A unified approach to Fiedler-like pencils via strong block minimal bases pencils. *Linear Algebra Appl.*, 547:45–104, 2018.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 472-499, September 2018.

[6] M.I. Bueno and S. Furtado. Palindromic linearizations of a matrix polynomial of odd degree obtained from Fiedler pencils with repetition. *Electron. J. Linear Algebra*, 23:562–577, 2012.

[7] M.I. Bueno and S. Furtado. Structured linearizations from Fiedler pencils with repetition II. *Linear Algebra Appl.*, 463:282–321, 2014.

[8] M.I. Bueno, F.M. Dopico, S. Furtado, and L. Medina. Block-symmetric linearizations of odd degree matrix polynomials with optimal condition number and backward error. Preprint, 2018.

[9] M.I. Bueno, F.M. Dopico, S. Furtado, and L. Medina. Block-symmetric linearizations of even degree matrix polynomials with optimal condition number and backward error. In preparation, 2018.

[10] F. De Terán, F.M. Dopico, and D.S. Mackey. Linearizations of singular matrix polynomials and the recovery of minimal indices. Electron. J. Linear Algebra, 18:371–402, 2009.

[11] F. De Terán, F.M. Dopico and D.S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. *SIAM J. Matrix Anal. Appl.*, 31:2181–2204, 2010.

[12] F. De Terán, F.M. Dopico and D.S. Mackey. Palindromic companion forms for matrix polynomials of odd degree. *J. Comput. Appl. Math.*, 236:1464–1480, 2011.

[13] F. De Terán, F.M. Dopico and D.S. Mackey. Fiedler companion linearizations for rectangular matrix polynomials. *Linear Algebra Appl.*, 437:957–991, 2012.

[14] F. De Terán, F.M. Dopico, and D.S. Mackey. Spectral equivalence of matrix polynomials and the Index Sum Theorem. *Linear Algebra Appl.*, 459:264–333, 2014.

[15] F.M. Dopico, P.W. Lawrence, J. Pérez, and P. Van Dooren. Block Kronecker linearizations of matrix polynomials and their backward errors. *Numerische Mathematik*, 140(2):373–426, 2018.

[16] F.M. Dopico, J. Pérez, and P. Van Dooren. Block minimal bases ℓ-ifications of matrix polynomials. Preprint ArXiv:1803.06306, 2018.

[17] H. Faßbender and P. Saltenberger. Block-Kronecker ansatz spaces for matrix polynomials. *Linear Algebra Appl.*, 542:118–148, 2018.

[18] G.D. Forney, Jr. Minimal bases of rational vector spaces, with applications to multivariable linear systems. *SIAM J. Control*, 13:493–520, 1975.

[19] F.R. Gantmacher. *The Theory of Matrices.* AMS Chelsea Publishing, Providence, RI, 1998.

[20] N.J. Higham, D.S. Mackey, and F. Tisseur, The conditioning of linearizations of matrix polynomials., *SIAM J. Matrix Anal. Appl.*, 28:1005–1028, 2006.

[21] N. Higham, D.S. Mackey, N. Mackey, and F. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159, 2006.

[22] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[23] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.

[24] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432:971–1004, 2010.

[25] C.B. Moler and G.W. Stewart. An algorithm for generalized matrix eigenvalue problems. *SIAM J. Numer. Anal.*, 10(2):241–256, 1971.

[26] L. Robol, R. Vandebril, and P. Van Dooren. A framework for structured linearizations of matrix polynomials in various bases. *SIAM J. Matrix Anal. Appl.*, 38(1):188–216, 2017.

[27] F. Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra Appl.*, 309:339–361, 2000.

[28] P. Van Dooren. The computation of Kronecker's canonical form of a singular pencil. *Linear Algebra Appl.*, 27:103–140, 1979.

[29] P. Van Dooren and P. Dewilde. The eigenstructure of an arbitrary polynomial matrix: computational aspects. *Linear Algebra Appl.*, 50:545–579, 1983.

[30] S. Vologiannidis and E.N. Antoniou. A permuted factors approach for the linearization of polynomial matrices. *Math. Control Signals Syst.*, 22:317–342, 2011.