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BOUNDS ON THE SUM OF MINIMUM SEMIDEFINITE RANK OF A GRAPH AND ITS COMPLEMENT*

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Abstract. The minimum semi-definite rank (*msr*) of a graph is the minimum rank among all positive semi-definite matrices associated to the graph. The graph complement conjecture gives an upper bound for the sum of the *msr* of a graph and the *msr* of its complement. It is shown that when the *msr* of a graph is equal to its independence number, the graph complement conjecture holds with a better upper bound. Several sufficient conditions are provided for the *msr* of different classes of graphs to equal to its independence number.

Key words. Minimum semidefinite rank, Matrix of a graph, Independence number, Graph complement conjecture.

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1. Introduction. We will use V to denote the set of vertices of a graph G. The cardinality of V represents the *order* of G, denoted $|G|$. An edge is an unordered pair of vertices and we use E to denote the set of edges in a graph G. A graph is said to be *simple* if it has no loops or multiple edges. Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, we can associate a graph $G(A)$ to the matrix A in such a way that the set of vertices is $V = \{v_1, v_2, \dots, v_n\}$ and the set of edges is $E = \{\{v_i, v_j\} : a_{ij} \neq 0, i \neq j\}$. The diagonal entries of A do not affect the structure of $G(A)$. The graph $G(A)$ is an undirected simple graph. A Hermitian matrix $A \in M_n(\mathbb{C})$ is called *positive semidefinite* (psd) if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ [7].

Suppose G is a simple connected graph with vertex set $\{v_1, v_2, \dots, v_n\}$. We associate a set of vectors $\overrightarrow{V} = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \dots, \overrightarrow{v}_n\}$ in $\mathbb{C}^m$ to the vertices such that, for $i \neq j$, $\langle \overrightarrow{v}_i, \overrightarrow{v}_j \rangle \neq 0$ if and only if v_i and v_j are adjacent vertices in G. The set $\overrightarrow{V}$ is called a vector representation (or orthogonal representation) of G. If $X = [\overrightarrow{v}_1, \overrightarrow{v}_2, \dots, \overrightarrow{v}_n]$ is an $m \times n$ matrix then $A = X^*X$ is a psd matrix associated with G. Since every psd matrix A associated with G can be written as $A = B^*B$ for some matrix B, we can always find a vector representation $\overrightarrow{V}$ that produces the matrix A with $rank(A) = dim(span\overrightarrow{V})$ [7].

If E is a block matrix given by $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is nonsingular and D is square, then the matrix $D - CA^{-1}B$ is called the Schur complement of A in E. If E is Hermitian (i.e, $C = B^*$) and A and $D - B^*A^{-1}B$ are both psd, then E is also psd, and $rank(E) = rank(A) + rank(D - B^*A^{-1}B)$ [7]. If the upper-block B consists of zeros, we call E a *block lower triangular* matrix.

A symmetric matrix A satisfies the Strong Arnold Hypothesis if the only real symmetric matrix X such that $AX = A \circ X = X \circ I = 0$ is the zero matrix, where $\circ$ denotes the entrywise product of matrices, and I is the $n \times n$ identity matrix [5].

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Let $\mathcal{P}(G)$ denote the set of all complex Hermitian psd matrices A whose graph $G(A) = G$. Let $S_+(G)$ denote the set of all real symmetric psd matrices A whose graph $G(A) = G$.

The *complex minimum semidefinite rank* of G is denoted by $mr_+^{\mathbb{C}}(G)$ and is defined as $mr_+^{\mathbb{C}}(G) = min\{rank(A) : A \in \mathcal{P}(G)\}$ and the real minimum semidefinite rank of G is denoted $mr_+^{\mathbb{R}}(G)$ and is defined as $mr_+^{\mathbb{R}}(G) = min\{rank(A) : A \in S_+(G)\}$. Since $S_+(G) \subseteq \mathcal{P}(G)$ we conclude $mr_+^{\mathbb{C}}(G) \leq mr_+^{\mathbb{R}}(G)$. An example where strict inequality holds is given in [1]. When $mr_+^{\mathbb{C}}(G) = mr_+^{\mathbb{R}}(G)$ we write $mr_+(G)$.

The paper is organized as follows: In Section 2, we present graph theory preliminaries and some known results about the *msr* that will be used throughout the paper. In Section 3, we show that when the *msr* of a graph is equal to its independence number, we get an upper bound for the sum of the *msr* of a graph and its complement that is better than the one given in the graph complement conjecture. We present different classes of graphs for which the *msr* is equal to the independence number.

2. Preliminaries. In this section, we present some graph theory definitions and some known results related to the $msr(G)$ which will be used throughout the paper.

We will use $N(v)$ to denote the set of vertices that are adjacent to a vertex v in a simple graph G. The *degree* of a vertex v in G, denoted $deg_G(v)$, is the cardinality of $N(v)$. Let $\delta(G)$ denote the minimum degree of the vertices in G, and $\Delta(G)$ denote the maximum degree of the vertices in G. A graph G is *regular* if $\delta(G) = \Delta(G)$, and it is k-regular if $\delta(G) = \Delta(G) = k$.

A graph is a *tree* if it is a connected graph with n vertices and $n-1$ edges. A *star graph* on n vertices is a tree with one vertex having degree $n-1$, called the center, and the other $n-1$ vertices having degree 1. A *cycle* of length n is the graph C_n on n vertices $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ with n edges $\{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_0\}\}$. A *complete graph* or *clique* is a connected graph in which the vertices are pairwise adjacent, we use K_n to denote a complete graph on n vertices. The size of the largest complete subgraph of G is called the *clique number*, denoted $\omega(G)$ ([12], p. 192).

An *independent set* in a graph G is a set of vertices where no two of the vertices are adjacent. The cardinality of a largest independent set in G is called the *independence number* of G and is denoted by $\alpha(G)$ ([12], p. 113). It was shown in [4] that, for a connected graph G, $msr(G) \geq \alpha(G)$.

We use $G = (R \cup L, E)$ to denote a bipartite graph, where R is the set of vertices on the right side and L is the set of vertices on the left side in a pictorial representation of G. In a bipartite graph G with $|R| = m \geq |L| = n$, the set of vertices in both R and L are pairwise disjoint, so they form two independent sets. When G is connected, the independence number of G is m. Hence, $msr(G) \geq m$. Equality is achieved for a complete bipartite graph $K_{m,n}$, $m \geq n$ [3].

An *induced subgraph* H of G is a subgraph with $V(H) \subseteq V(G)$ and $E(H) = \{\{i, j\} \in E(G) : i, j \in V(H)\}$. Since a principal submatrix of a psd matrix is psd, and the rank of a submatrix can never be greater than that of a matrix [7], minimum semi-definite rank of any induced subgraph of a given graph G gives a lower bound for the minimum semi-definite rank of G. Moreover, for a psd matrix $A = [a_{ij}]$, the diagonal entry a_{ii} is a principal submatrix, and hence, $a_{ii} \geq 0$. If there is a nonzero off-diagonal entry in the *ith* row of A then $a_{ii} > 0$.

The *complement* $\overline{G}$ of a simple graph G on n vertices is the simple graph with vertex set $V(G)$ and edge set $E(\overline{G})$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Since cliques become independent sets (and vise versa) under complementation, we have $\alpha(G) = \omega(\overline{G})$.

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A subset S of vertices of a connected graph G is a *cut-set* if the graph induced by the set of vertices $V(G) - S$ is disconnected. The *vertex connectivity* of a graph G, denoted $k(G)$, is the minimum cardinality of a vertex set S such that $G - S$ is disconnected or has only one vertex. Since deleting the neighbors of any vertex in a graph G disconnects G, $k(G) \leq \delta(G)$ ([12], p. 149). A graph G is *k-connected* if every cut set of G has at least k vertices.

A graph G is a *superposition* of two graphs, G_1 and G_2 if G is obtained by identifying G_1 and G_2 at a set of vertices, keeping all the edges that are present in either G_1 or G_2 [3].

An *isomorphism* from G to H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G is *isomorphic* to H if there is an isomorphism from G to H.

The *Laplacian* matrix of a graph G is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose ith diagonal entry is the degree of vertex v_i in G, and $A(G)$ is the adjacency matrix of G. Since the Laplacian matrix $L(G)$ is psd and has rank $|G| - 1$ [8] when G is a connected graph, it follows that $msr(G) \leq |G| - 1$ for every connected graph G [6]. Moreover, $msr(G) = n - 1$ if and only if G is a tree on n vertices [11]. It is well known that for a connected graph G with $|G| \geq 2$, $msr(G) = 1$ if and only if $G = K_n$. Thus, for any connected graph G with $|G| \geq 2$ where G is neither a tree nor a complete graph, we have $2 \leq msr(G) \leq |G| - 2$. If a graph G is disconnected, then the direct sum of psd matrices for the connected components G_i of G gives a psd matrix for the entire graph. In this case, $msr(G) = \sum_i msr(G_i)$. Hence, we assume in this paper all graphs are simple and connected.

3. The graph complement conjecture. It has been conjectured in [2] that $mr_+^{\mathbb{R}}(G)+m_+^{\mathbb{R}}(\overline{G}) \leq |G|+2$ for all graphs G, which is known as the "graph complement conjecture (GCC+)". Proposition 3.2 below gives a better upper bound for the sum of the msr of a graph and its complement.

LEMMA 3.1. [9] *If a graph G contains a clique of size k, then there exists a matrix $A \in S_+(G)$ that satisfies the Strong Arnold Hypothesis with $rank(A) = |G| - k + 1$.*

PROPOSITION 3.2. *Let G be a connected graph. If $mr_+^{\mathbb{R}}(G) = \alpha(G)$, then $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) \leq |G| + 1$.*

Proof. Since $mr_+^{\mathbb{R}}(G) = \alpha(G)$ and $\alpha(G) = \omega(\overline{G})$, where $\omega(\overline{G})$ is the size of the largest clique in $\overline{G}$, using Lemma 3.1, we get $mr_+^{\mathbb{R}}(\overline{G}) \leq |G| - \omega(\overline{G}) + 1 = |G| - mr_+^{\mathbb{R}}(G) + 1$. Hence, $mr_+^{\mathbb{R}}(G) + m_+^{\mathbb{R}}(\overline{G}) \leq |G| + 1$. ∎

In this section, we investigate classes of graphs whose msr is equal to their independence number and hence satisfy Proposition 3.2. In some cases, we give a better upper bound.

PROPOSITION 3.3. *If G is a connected graph obtained from a complete graph K_m by deleting all the edges of the subgraph K_r with $r \leq m - 1$, then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r$ and $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r + 1$.*

Proof. We show that $mr_+^{\mathbb{R}}(G) = r$. Since edges of K_r are removed from K_m, there is a set of r vertices in G no two of which are adjacent. Since this is a maximum independent set, $\alpha(G) = r$ and so $mr_+^{\mathbb{R}}(G) \geq r$ by [3]. To show that $mr_+^{\mathbb{R}}(G) \leq r$, we will exhibit a vector representation of G in $\mathbb{R}^r$. Label the vertices of G as $\{v_1, v_2, \ldots, v_r, \ldots, v_m\}$. Assume that the set of the vertices $\{v_1, v_2, \ldots, v_r\}$ is the maximal independent set. Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_r}\}$ be the standard orthonormal basis in $\mathbb{R}^r$. We assign to the vertices $v_1, v_2, \ldots, v_r$ the vectors $\overrightarrow{v_1} = \overrightarrow{e_1}$, $\overrightarrow{v_2} = \overrightarrow{e_2}, \ldots, \overrightarrow{v_r} = \overrightarrow{e_r}$. For the remaining $m - r$ vertices, we assign vectors of the form $\overrightarrow{v_i} = \sum_{j=1}^{r} \overrightarrow{e_j}, i = r + 1, r + 2, \ldots, m$. The set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_r}, \ldots, \overrightarrow{v_m}\}$ is a vector representation of the graph G in $\mathbb{R}^r$, so $mr_+^{\mathbb{R}}(G) \leq r$. Hence, $mr_+^{\mathbb{R}}(G) = r$. It is clear that $\overline{G}$ consists of $m - r$ isolated vertices and the complete graph K_r, so $mr_+^{\mathbb{R}}(\overline{G}) = 1$. Therefore, $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r + 1$. ∎

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PROPOSITION 3.4. *If G is a connected graph obtained from a complete graph K_m, $m \geq 5$ by deleting all but s edges of the subgraph K_r, where $1 \leq s \leq r-1$, $3 \leq r \leq m-1$, such that all the s edges share a common vertex in G, then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r-1$.*

Proof. Label the vertices of G as $\{v_1, v_2, \ldots, v_r, \ldots, v_m\}$. Assume that the set of the s edges of K_r are $\{\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_{s+1}\}\} \subset E(G)$. Then, the set of vertices $\{v_2, v_3, \ldots, v_r\}$ is a maximal independent set and so, the independence number of G is $\alpha(G) = r-1$ and $mr_+^{\mathbb{R}}(G) \geq r-1$. To show that $mr_+^{\mathbb{R}}(G) \leq r-1$, we will exhibit a vector representation of G in $\mathbb{R}^{r-1}$. Let $\{\overrightarrow{e}_2, \ldots, \overrightarrow{e}_s, \overrightarrow{e}_{s+1}, \ldots, \overrightarrow{e}_r\}$ be the standard orthonormal basis in $\mathbb{R}^{r-1}$. To the vertices $v_2, \ldots, v_r$ we assign the vectors $\overrightarrow{v}_2 = \overrightarrow{e}_2$, $\overrightarrow{v}_3 = \overrightarrow{e}_3, \ldots, \overrightarrow{v}_{s+1} = \overrightarrow{e}_{s+1}, \ldots, \overrightarrow{v}_r = \overrightarrow{e}_r$, and to the vertex v_1, we assign the vector $\overrightarrow{v}_1 = \sum_{j=2}^{s+1} \overrightarrow{e}_j$. To the remaining $m-r$ vertices, we assign the vectors $\overrightarrow{v}_i = \sum_{j=2}^{r} \overrightarrow{e}_j$ where $i = r+1, r+2, \ldots, m$. The set of the vectors $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m\}$ represents G in $\mathbb{R}^{r-1}$ and so, $mr_+^{\mathbb{R}}(G) \leq r-1$. Therefore, $mr_+^{\mathbb{R}}(G) = r-1$. ◻

COROLLARY 3.5. *If G is a connected graph obtained from a complete graph $K_m, m \geq 5$ by deleting all but one edge of the subgraph $K_r, r \leq m-1$, then $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r+1$.*

Proof. The graph $\overline{G}$ consists of $m-r$ isolated vertices and the complete graph K_r minus one edge. Hence, $mr_+^{\mathbb{R}}(\overline{G}) = 2$ by Proposition 3.3. Since $mr_+^{\mathbb{R}}(G) = r-1$ using Proposition 3.4, we get $mr_+^{\mathbb{R}}(G) + mr_+^{\mathbb{R}}(\overline{G}) = r+1$. ◻

LEMMA 3.6. [3] *If G is a superposition of two graphs G_1 and G_2, then $msr(G) \leq msr(G_1) + msr(G_2)$.*

COROLLARY 3.7. *Let G be a connected graph obtained from a complete graph K_m, $m \geq 7$ by deleting all but s edges of the subgraph K_r, where $2 \leq s < r-1$ and $4 \leq r \leq m-1$. If all the s edges share a common vertex v in G, then $mr_+^{\mathbb{R}}(\overline{G}) \leq r-s$.*

Proof. In this case, the graph $\overline{G}$ consists of $m-r$ isolated vertices and a connected component, call it $\overline{G}'$. Since $mr_+^{\mathbb{R}}$ of an isolated vertex is zero, we may only consider the connected component of $\overline{G}$. Since all the s edges share a common vertex in G, the connected component $\overline{G}'$ can be viewed as the superposition of the complete graph K_{r-1} and the star graph on $r-s$ vertices whose center is the vertex v. The two graphs are identified at the set of vertices $V(\overline{G}') - S$, where S is the set of vertices on which the s edges are incident. Using Lemma 3.6, we get $mr_+^{\mathbb{R}}(\overline{G}) = mr_+^{\mathbb{R}}(\overline{G}') \leq 1 + r - s - 1 = r-s$. ◻

PROPOSITION 3.8. *If G is a connected graph obtained from a complete graph K_m, $m \geq 5$ by deleting all but the s edges of the subgraph K_r, where $2 \leq s \leq \lfloor \frac{r}{2} \rfloor$ and $3 \leq r \leq m-1$ such that no two of the s edges share a vertex, then $mr_+^{\mathbb{R}}(G) = \alpha(G) = r-s$.*

Proof. Label the vertices of G as $\{v_1, v_2, \ldots, v_r, \ldots, v_m\}$. Since no two of the s edges share a vertex, they will be incident on $2s$ different vertices and the graph G will have $m-r$ vertices of degree $m-1$. Let $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \ldots, \overrightarrow{e}_{r-s}\}$ be the standard orthonormal basis in $\mathbb{R}^{r-s}$.

Case I. Suppose r is even and $s = \frac{r}{2}$. Assume that $\{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{r-1}, v_r\}\} \subset E(G)$ is the set of s edges of K_r. By taking one vertex from each of the s existing edges in K_r we get a maximal set of $\frac{r}{2}$ independent vertices, so the independence number of G is $\alpha(G) = \frac{r}{2} = r-s$. Hence, $mr_+^{\mathbb{R}}(G) \geq r-s$. To show that $mr_+^{\mathbb{R}}(G) \leq r-s$, we will exhibit a vector representation of G in $\mathbb{R}^{r-s}$. To the vertices $v_1, v_2, \ldots, v_r$ we assign the vectors $\overrightarrow{v}_{2i} = \overrightarrow{v}_{2i-1} = \overrightarrow{e}_i$, where $1 \leq i \leq s$. To the remaining $m-r$ vertices, we assign the vectors $\overrightarrow{v}_t = \sum_{i=1}^{s} \overrightarrow{e}_i$, where $t = r+1, r+2, \ldots, m$. The set of the vectors $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m\}$ represents G in $\mathbb{R}^{r-s}$ and so, $mr_+^{\mathbb{R}}(G) \leq r-s$. Therefore, $mr_+^{\mathbb{R}}(G) = r-s$.

Case II. Suppose r is odd and $s = \frac{r-1}{2}$. In this case, we assume the vertex v_r is not joined to any of the vertices incident on the s edges. Assume that the s edges of K_r are $\{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{r-2}, v_{r-1}\}\} \subset$

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$E(G)$. By taking one vertex from each of the s existing edges in K_r and the vertex v_r we get a maximal set of $\frac{r-1}{2}+1$ independent vertices, so the independence number of G is $\alpha(G)=\frac{r+1}{2}=r-s$. Hence, $mr_+^{\mathbb{R}}(G)\geq r-s$. To show that $mr_+^{\mathbb{R}}(G)\leq r-s$, we will exhibit a vector representation of G in $\mathbb{R}^{r-s}$. To the vertices $v_1,v_2,\ldots,v_{r-1}$ we assign the vectors $\overrightarrow{v}_{2i}=\overrightarrow{v}_{2i-1}=\overrightarrow{e}_i$, where $1\leq i\leq s$. To the vertex v_r, we assign the vector $\overrightarrow{v}_r=\overrightarrow{e}_{s+1}$. To the remaining $m-r$ vertices, we assign the vectors $\overrightarrow{v}_t=\sum_{i=1}^{s+1}\overrightarrow{e}_i$, where $t=r+1,r+2,\ldots,m$. The set of the vectors $\{\overrightarrow{v}_1,\ldots,\overrightarrow{v}_m\}$ represents G in $\mathbb{R}^{s+1}$ and so, $mr_+^{\mathbb{R}}(G)\leq s+1=r-s$. Therefore, $mr_+^{\mathbb{R}}(G)=r-s$.

Case III. Suppose $2\leq s<\lfloor\frac{r}{2}\rfloor$. In this case, the graph G has $r-2s$ pairwise disjoint vertices other than the vertices that the s edges are incident on. By taking one vertex from each of the s existing edges in K_r and the $r-2s$ pairwise disjoint vertices we get a maximal set I of independent vertices. Since $|I|=s+r-2s=r-s$, the independence number of G is $\alpha(G)=r-s$. Hence, $mr_+^{\mathbb{R}}(G)\geq r-s$. Assume that the s edges of K_r are $\{\{v_1,v_2\},\{v_3,v_4\},\ldots,\{v_{2s-1},v_{2s}\}\}\subset E(G)$. To show that $mr_+^{\mathbb{R}}(G)\leq r-s$, we will exhibit a vector representation of G in $\mathbb{R}^{r-s}$. To the vertices $v_1,v_2,\ldots,v_{2s}$ we assign the vectors $\overrightarrow{v}_{2i}=\overrightarrow{v}_{2i-1}=\overrightarrow{e}_i$, where $1\leq i\leq s$. To the $r-2s$ pairwise disjoint vertices, we assign the vectors $\overrightarrow{v}_j=\overrightarrow{e}_j$, where $2s+1\leq j\leq r$. To the remaining $m-r$ vertices, we assign the vectors $\overrightarrow{v}_t=\sum_{i=1}^{s}\overrightarrow{e}_i+\sum_{j=2s+1}^{r}\overrightarrow{e}_j$ where $t=r+1,r+2,\ldots,m$. The set of the vectors $\{\overrightarrow{v}_1,\ldots,\overrightarrow{v}_m\}$ represents G in $\mathbb{R}^{r-s}$, from this we get $mr_+^{\mathbb{R}}(G)\leq r-s$. Therefore, $mr_+^{\mathbb{R}}(G)=r-s$. ∎

Since the *msr* of a complete bipartite graph is equal to its independence number, complete bipartite graphs satisfy Proposition 3.2. Next, we present upper bounds for the sum of the *msr* of a bipartite graph and its complement.

PROPOSITION 3.9. *Let G be a connected bipartite graph which is not a tree. Then* $mr_+^{\mathbb{R}}(G)+mr_+^{\mathbb{R}}(\overline{G})\leq\frac{3}{2}|G|-1$

Proof. Since the set of vertices in R forms an independent set and $|R|\geq\frac{|G|}{2}$, we get $\alpha(G)=\omega(\overline{G})\geq\frac{|G|}{2}$, where $\omega(\overline{G})$ is the size of the largest clique in $\overline{G}$. Using Lemma 3.1, we get $mr_+^{\mathbb{R}}(\overline{G})\leq|G|-\omega(\overline{G})+1\leq\frac{|G|}{2}+1$. Since $mr_+^{\mathbb{R}}(G)\leq|G|-2$, it follows that $mr_+^{\mathbb{R}}(G)+mr_+^{\mathbb{R}}(\overline{G})\leq|G|-2+\frac{|G|}{2}+1=\frac{3}{2}|G|-1$. ∎

In the next proposition, we give an upper bound for the *msr* of complements of two classes of bipartite graphs.

PROPOSITION 3.10. *Let $G=K_{m,n}\backslash p$ edges, $2\leq p<n\leq m$. Then, $mr_+(\overline{G})\leq p+2$ if one of the following conditions is satisfied:*

(i) All the missing edges in G share a common vertex $v\in V(G)$.
(ii) No two of the missing edges in G share a common vertex.

Proof. To prove (i), assume that all the missing edges share a common vertex $v\in V(G)$. We consider two cases:

Case I. Suppose $v\in R$. In this case, the graph $\overline{G}$ can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the cut vertex v, where $G_1=K_m$ and G_2 is the superposition of K_n and $K_{1,p}$ identified at the p vertices of $K_{1,p}$. By Lemma 3.6, we get $mr_+(\overline{G})=mr_+(G_1)+mr_+(G_2)\leq 1+1+p=2+p$.

Case II. If $v\in L$, then the graph $\overline{G}$ can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the cut vertex v, where $G_1=K_n$ and G_2 is the superposition of K_m and $K_{1,p}$ that are identified at the p vertices of $K_{1,p}$. By Lemma 3.6, we get $mr_+(\overline{G})=mr_+(G_1)+mr_+(G_2)\leq 1+1+p=2+p$.

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To prove (ii), assume that no two of the missing edges share a common vertex. Then, the graph $\overline{G}$ can be viewed as a superposition of two subgraphs, say, G_1 and G_2 identified at the p vertices in L from which the edges are missing, where $G_1 = K_n$ and G_2 is the superposition of $\lfloor \frac{p}{2} \rfloor$ copies of C_4 and another graph G_2' identified at the p vertices in R from which the edges are missing, where $G_2' = K_m$ if p is even and $G_2' = K_m + e$ if p is odd. We explain this as follows: since both sets of vertices L and R form independent sets in G, they form cliques in $\overline{G}$. Since no two of the missing edges share a common vertex, each two of those missing edges will be incident on four different vertices in $\overline{G}$, two of them, say y_1 and y_2 are in R and the other two, say x_1 and x_2 are in L where $y_1 \in N(y_2)$ and $x_1 \in N(x_2)$ in $\overline{G}$. So if p is even, then G_2 can be viewed as a superposition of K_m and $\frac{p}{2}$ copies of C_4 identified at the p vertices in R from which the edges are missing, and if p is odd then G_2 can be viewed as a superposition of $K_m + e$ and $\frac{p-1}{2}$ copies of C_4 identified at the p vertices in R from which the edges are missing. Since $mr_+(G_2) \leq 2\lfloor \frac{p}{2} \rfloor + mr_+(G_2')$, $2\lfloor \frac{p}{2} \rfloor + mr_+(G_2') = 1 + p$ if p is even, and $2\lfloor \frac{p}{2} \rfloor + mr_+(G_2') = \frac{2(p-1)}{2} + 2 = p + 1$ if p is odd, we obtain $mr_+(\overline{G}) \leq mr_+(G_1) + mr_+(G_2) \leq 1 + 1 + p = 2 + p$. □

Corollary 3.11. *Let $G = K_{m,n} \backslash p$ edges, where $m \geq n + 1$ and $2 \leq p < n$ such that all the missing edges share a common vertex $v \in R$, $R \subset V(G)$ with $|R| = m$. Then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$.*

Proof. In this case, the graph G can be viewed as the superposition of the two subgraphs $G - v = K_{m-1,n}$ and $K_{n-p,1}$ identified at the p vertices in L from which the edges are missing. So, by Lemma 3.6 we get $mr_+(G) \leq m - 1 + n - p = |G| - 1 - p$, and by Proposition 3.10 we get $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq |G| - 1 - p + 2 + p = |G| + 1$. □

Corollary 3.12. *Let $G = K_{m,n} \backslash p$ edges, where $m \geq n$ and $2 \leq p < n$ such that all the missing edges share a common vertex $v \in L$, $L \subset V(G)$ with $|L| = n \leq m$. Then $mr_+(G) + mr_+(\overline{G}) \leq m + 2p + 2$.*

Proof. Since G is obtained from $K_{m,n}$ by deleting p edges, we get $mr_+(G) \leq mr_+(K_{m,n}) + p = m + p$ [3]. By Proposition 3.10, $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq m + p + 2 + p = m + 2p + 2$. □

Corollary 3.13. *Let $G = K_{m,n} \backslash p$ edges, where $m \geq n$ and $p \geq 2$ such that no two edges share an end point. Then $mr_+(G) + mr_+(\overline{G}) \leq m + 2p + 2$. In addition, if $p \leq \frac{n}{2}$ then G satisfies GCC+.*

Proof. Since G is modified from $K_{m,n}$ by deleting p edges, we get $mr_+(G) \leq mr_+(K_{m,n}) + p = m + p$ [3]. By Proposition 3.10, $mr_+(\overline{G}) \leq 2 + p$. Hence, $mr_+(G) + mr_+(\overline{G}) \leq m + p + 2 + p = m + 2p + 2$. If $p \leq \frac{n}{2}$ then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$. □

Corollary 3.14. *Let $G = K_{m,n} \backslash p$ edges, where $2 \leq p < n$ such that all the missing edges share a common vertex $v \in L$, $L \subset V(G)$ with $|L| = n = |R| = m$. Then $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$.*

Proof. In this case, the graph G can be viewed as a superposition of the two graphs $K_{m,n-1}$ and $K_{m-p,1}$ identified at the $m - p$ vertices in R. Hence, $mr_+(G) \leq m + m - p = |G| - p$. Using Proposition 3.10, we get $mr_+(G) + mr_+(\overline{G}) \leq |G| + 2$. □

Lemma 3.15. *Let M be an $m \times n$ matrix. If N is the $m \times m$ lower triangular matrix defined by $N[j,j] = 1$, $N[j,1] = -\sum_{s=1}^{n} M[1,s] \cdot M[j,s]$ for $j = 2, \ldots, m$, and $N[j,k] = -\sum_{s=1}^{n} M[k,s] \cdot M[j,s] - \sum_{s=1}^{k-1} N[k,s] \cdot N[j,s]$, where $k = 2, \ldots, m$ and $j = k+1, \ldots, m$, then the rows of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ are mutually orthogonal.*

Proof. We have $\langle \text{row } i, \text{row } j \rangle = \sum_{s=1}^{n} M[i,s] \cdot M[j,s] + \sum_{k=1}^{i} N[i,k] \cdot N[j,k]$. Note that $N[i,k] = 0$ for $k > i$, and $N[j,k] = 0$ for $k > j$. Now $\sum_{s=1}^{n} M[i,s] \cdot M[j,s] = -N[j,i] - \sum_{s=1}^{i-1} N[i,s] \cdot N[j,s] = -\sum_{s=1}^{i} N[i,s] \cdot N[j,s]$ since $N[i,i] = 1$. Hence, $\langle \text{row } i, \text{row } j \rangle = 0$ for all $i \neq j$ where $1 \leq i \leq m$ and $1 \leq j \leq m + n$. □

Theorem 3.16. *Consider the matrix $C = \begin{bmatrix} M & N \end{bmatrix}^T$, where $\begin{bmatrix} M & N \end{bmatrix}$ is the matrix constructed in*

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Lemma 3.15. Let $A = \begin{bmatrix} I_{m+n} & D \\ D^* & I_m \end{bmatrix}$, $m \geq 3$, $n \geq 0$, *where* I_{m+n} *and* I_m *are the* $(m+n)\times(m+n)$ *and the* $m \times m$ *identity matrices, respectively, and the matrix* D *is an* $(m+n)\times m$ *matrix whose column vectors are given by* $\frac{\overrightarrow{c_i}}{\|\overrightarrow{c_i}\|}$, *where* $\overrightarrow{c_i}$ *are the column vectors of the matrix* C *for all* i, $1 \leq i \leq m$. *Then the graph* G *associated to* A *is a connected bipartite graph whose* $mr_+^{\mathbb{C}}(G)$ *is equal to* $\alpha(G) = m+n$.

Proof. Since the rows of the matrix $\begin{bmatrix} M & N \end{bmatrix}$ are mutually orthogonal, the columns of C are mutually orthogonal. Taking Schur Complement with respect to the $(1,1)$ entry of A we get that the matrix A is psd with $rank(A) = m+n$. Since the columns of A are mutually orthogonal, we can associate a graph to the matrix A whose vertices associated to the column vectors of A can be partitioned into two sets of vertices, call them L and R with $|R| = m+n$ and $|L| = m$ such that the vertices in each set are pairwise disjoint. In this case, $\alpha(G) = m+n$ and since $rank(A) = m+n$, we get $mr_+^{\mathbb{C}}(G) = m+n = \alpha(G)$. □

In the next theorem, we show that $mr_+^{\mathbb{C}}(G)$ of a class of bipartite graphs is equal to their independence number by associating a psd matrix to the graph and showing that the rank of the associated matrix is the independence number of the given graph.

THEOREM 3.17. *Let* G *be a connected bipartite graph with* $V(G) = L \cup R$, $|L| = m$ *and* $|R| = m+n, n \geq 0$. *Label the vertices of* L *as* $v_1, v_2, \ldots, v_m$ *and the vertices of* R *as* $u_1, u_2, \ldots, u_{m+n}$. *If* v_1 *is adjacent to* u_i *for all* $1 \leq i \leq n+1, i \neq 2$ *and for* $2 \leq j \leq m$, *the vertex* v_j *is adjacent to* u_k *for all* $1 \leq k \leq n+j$, *the vertex* v_t *is not adjacent to* u_{n+s} *for all* $1 \leq t \leq m$ *and all* $t+1 \leq s \leq m+n$, *then* $mr_+^{\mathbb{C}}(G) = m+n = \alpha(G)$.

Proof. Since $|R| = m+n$, the independence number of G is $\alpha(G) = m+n$. Consider the matrix $C = \begin{bmatrix} M & N \end{bmatrix}^T$, where M is an $m \times n$ matrix that has the entry $M[2,1] = 0$ and all other entries are positive and the matrix N is constructed as in Lemma 3.15. To the graph G, associate the matrix $A = \begin{bmatrix} I_{m+n} & D \\ D^* & I_m \end{bmatrix}$, where I_{m+n} and I_m are the $(m+n)\times(m+n)$ and the $m \times m$ identity matrices respectively, the matrix D is an $(m+n)\times m$ matrix whose column vectors are given by $\frac{\overrightarrow{c_i}}{\|\overrightarrow{c_i}\|}$, where $\overrightarrow{c_i}$ are the column vectors of the matrix C for all i, $1 \leq i \leq m$. Since $\alpha(G) = m+n$ and A is psd with $rank(A) = m+n$, we get $mr_+^{\mathbb{C}}(G) = m+n$. □

PROPOSITION 3.18. *Let* G *be a* k*-regular graph on* $2k$ *vertices with* $k \geq 2$, *such that for each* $v \in V(G)$ *the set* $N(v)$ *forms an independent set. Then* $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$

Proof. We will show that G is isomorphic to the complete bipartite graph $K_{k,k}$. Let $v \in V(G)$. We claim that $V(G) - N(v) = N(w)$ for some $w \in N(v)$. If not, then $N(w) \cap N(v) \neq \phi$ for every $w \in N(v)$. This contradicts the assumption that $N(v)$ is an independent set. Therefore, G is isomorphic to $K_{k,k}$, and so $mr_+(G) + mr_+(\overline{G}) \leq |G| + 1$. □

The k-*connected Harary* graph $H_{k,n}$, $k < n$, is constructed as follows [12]: Place n vertices in circular order. If $k = 2r$, form $H_{k,n}$ by making each vertex adjacent to the nearest r vertices in each direction around the circle. If $k = 2r+1$ and n is even, then form $H_{k,n}$ by making each vertex adjacent to the nearest r vertices in each direction and to the vertex opposite it on the circle. In each case, the graph $H_{k,n}$ is regular. If $k = 2r+1$ and n is odd, then index the vertices by the integers modulo n. Construct $H_{k,n}$ from $H_{2r,n}$ by adding the edges $\{i, i + \frac{(n+1)}{2}\}$ for $0 \leq i \leq \frac{(n-1)}{2}$. In this case, the graph $H_{k,n}$ cannot be regular since both k and n are odd numbers.

LEMMA 3.19. [10] *Let* G *be a* k*-connected Harary graph* $H_{k,n}$, *where* $k = 2r, r \geq 1$. *Then,* $mr_+^{\mathbb{C}}(G) = |G| - \delta(G)$.

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Let G be a connected graph and let $S = \{v_1, \dots, v_m\}$ be an ordered set of vertices of G. Denote by G_k the subgraph induced by $v_1, v_2, \dots, v_k$ for each k, $1 \leq k \leq m$. Let H_k be the connected component of G_k such that $v_k \in V(H_k)$. If for each k, there exists $w_k \in V(G)$ such that $w_k \neq v_l$ for $l \leq k$, $w_k v_k \in E(G)$, and $w_k v_l \notin E(G)$, for all $v_l \in V(H_k)$ with $l \neq k$, then S is called a *vertex set of ordered subgraphs (or OS-vertex set)*. The *OS-number* of a graph G, denoted $OS(G)$, is the maximum cardinality among all $OS-$vertex sets of G [6].

The next theorem shows that the class of k-regular Harary graphs where k is even satisfies the GCC+. Its proof uses the notion of $OS-$number.

THEOREM 3.20. *Let G be a k-regular Harary graph, $H_{k,n}$ which is k-connected. If $k = 2r, r \geq 1$, then* $mr_+^{\mathbb{C}}(G) + mr_+^{\mathbb{C}}(\overline{G}) \leq |G| + 2$.

Proof. By Proposition 9 in [9], we have $OS(G)+mr_+^{\mathbb{C}}(\overline{G}) \leq |G|+2$. Using Proposition 4.11 in [10], we get $mr_+^{\mathbb{C}}(\overline{G}) \leq |G|+2-OS(G) = \delta(G)+2$. But since $\delta(G) = |G|-1-\delta(\overline{G})$, it follows that $mr_+^{\mathbb{C}}(\overline{G}) \leq |G|-\delta(\overline{G})+1$. Since G is k-regular, $\overline{G}$ is $(|G| - k - 1)$-regular. So, $mr_+^{\mathbb{C}}(\overline{G}) \leq k + 2$. Combining this and Lemma 3.19, we achieve the result. ☐

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