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# THE LARGEST EIGENVALUE AND SOME HAMILTONIAN PROPERTIES OF GRAPHS*

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**Abstract.** In this note, sufficient conditions, based on the largest eigenvalue, are presented for some Hamiltonian properties of graphs.

**Key words.** The largest eigenvalue, Hamiltonian property.

**AMS subject classifications.** 05C50, 05C45.

**1. Introduction.** We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G$, we use $n$ to denote its order $|V(G)|$. The complement a graph is denoted by $G^c$. A subset $V_1$ of the vertex set $V(G)$ is independent if no two vertices in $V_1$ are adjacent in $G$. The eigenvalues of a graph $G$, denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, are defined as the eigenvalues of its adjacency matrix $A(G)$. For a square matrix $M$, we use $det(M)$ to denote its determinant. A cycle $C$ in a graph $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is called a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path.

In 2010, Fiedler and Nikiforov [3] obtained the following spectral conditions for the Hamiltonicity and traceability of graphs.

Theorem 1.1. *Let $G$ be a graph of order $n$.*

(1) *If $\lambda_1(G) \geq n-2$, then $G$ contains a Hamiltonian path unless $G = K_{n-1} + v$; if strict inequality holds, then $G$ contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*
(2) *If $\lambda_1(G^c) \leq \sqrt{n-1}$, then $G$ contains a Hamiltonian path unless unless $G = K_{n-1} + v$.*
(3) *If $\lambda_1(G^c) \leq \sqrt{n-2}$, then $G$ contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*

Motivated by the results of Fiedler and Nikiforov, a lot of authors obtained additional spectral conditions for the Hamiltonian properties of graphs. Some of them can be found in [11], [6], [8], [7], [10], and [1]. In this note, we present new spectral conditions based on the largest eigenvalue for the Hamiltonicity and traceability of graphs. The main results are as follows.

Theorem 1.2. *Let $G$ be a graph of order $n \geq 3$ with connectivity $\kappa$ ($\kappa \geq 2$). If $\lambda_1 \leq \delta\sqrt{\frac{\kappa+1}{n-\kappa-1}}$, then $G$ is Hamiltonian or $G$ is $K_{\kappa,\,\kappa+1}$.*

Theorem 1.3. *Let $G$ be a graph of order $n \geq 12$ with connectivity $\kappa$ ($\kappa \geq 1$). If $\lambda_1 \leq \delta\sqrt{\frac{\kappa+2}{n-\kappa-2}}$, then $G$ is traceable or $G$ is $K_{\kappa,\,\kappa+2}$.*

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**2. Lemmas.** We need the following results as our lemmas when we prove Theorems 1.2 and 1.3. Lemma 2.1 below is from [9].

LEMMA 2.1. *Let $G$ be a balanced bipartite graph of order $2n$ with bipartition $(A,\, B)$. If $d(x)+d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then $G$ is Hamiltonian.*

Lemma 2.2 below is from [5].

LEMMA 2.2. *Let $G$ be a 2-connected bipartite graph with bipartition $(A,\, B)$, where $|A| \geq |B|$. If each vertex in $A$ has degree at least $k$ and each vertex in $B$ has degree at least $l$, then $G$ contains a cycle of length at least $2\min(|B|, k+l-1, 2k-2)$.*

**3. Proof of Theorem 1.2.** Let $G$ be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that $G$ is not Hamiltonian. Then $n \geq 2\kappa + 1$ (otherwise $\delta \geq \kappa \geq \frac{n}{2}$ and $G$ is Hamiltonian). Since $\kappa \geq 2$, $G$ has a cycle. Choose a longest cycle $C$ in $G$ and give an orientation on $C$. Since $G$ is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find $s$ ($s \geq \kappa$) pairwise disjoint (except for $u_0$) paths $P_1, P_2, \ldots, P_s$ between $u_0$ and $V(C)$. Let $v_i$ be the end vertex of $P_i$ on $C$, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of $C$. We use $v_i^+$ to denote the successor of $v_i$ along the orientation of $C$, where $1 \leq i \leq s$. Since $C$ is a longest cycle in $G$, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_s^+\}$ is independent (otherwise $G$ would have cycles which are longer than $C$). Set $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+\}$. Then $S$ is independent. Let $u_i = v_i^+$ for each $i$ with $1 \leq i \leq \kappa$. Set $T := V - S = \{\, w_1, w_2, \ldots, w_{n-\kappa-1}\,\}$. We label the vertices of $u_0, u_1, \ldots, u_\kappa, w_1, w_2, \ldots, w_{n-\kappa-1}$ by $1, 2, \ldots, \kappa+1, \kappa+2, \ldots, n$, respectively. Let $d_1(w_i) = |N(w_i) \cap S|$ and $d_2(w_i) = |N(w_i) \cap T|$ for each $i$ with $1 \leq i \leq n-\kappa-1$. Obviously, $\sum_{i=0}^{\kappa+1} d(u_i) = \sum_{i=1}^{n-\kappa-1} d_1(w_i)$.

Define a two by two matrix $B = (\, B_{i\,j}\,)_{2\times 2}$, where

$$B_{11} = 0,\;\; B_{12} = \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1},\;\; B_{21} = \frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1},\;\; B_{22} = \frac{\sum_{i=1}^{n-\kappa-1} d_2(w_i)}{n-\kappa-1}.$$

Then $B$ is a quotient matrix of the adjacency matrix of $G$ with partition $S$ and $T$. Let $\mu_1 \geq \mu_2$ be the eigenvalues of $B$. Then, by Corollary 2.3 on Page 596 in [4], we have that $\lambda_1 \geq \mu_1$ and $\mu_2 \geq \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $G$.

In the proofs below, we use some ideas in the proof of Theorem 3.3 in [4]. We, from Perron-Frobenius theorem, have that $|\lambda_n| \leq \lambda_1$. Thus,

$$\begin{aligned}\lambda_1^2 &\geq -\lambda_1\lambda_n \geq -\mu_1\mu_2 = -det(B) = B_{12}\, B_{21}\\ &= \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1}\;\frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1} = \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1}\;\frac{\sum_{i=0}^{k} d(u_i)}{\kappa+1}\;\frac{\kappa+1}{n-\kappa-1}\\ &\geq \frac{\delta^2(\kappa+1)}{n-\kappa-1} \geq \lambda_1^2.\end{aligned}$$

Therefore, $\lambda_1 = -\lambda_n$, $\lambda_1 = \mu_1$, $\lambda_n = \mu_2$, and $d(u_i) = \delta$ for each $i$ with $0 \leq i \leq \kappa$. Since $0 = \lambda_1 + \lambda_n = \mu_1 + \mu_2 = B_{22}$, $d_2(w_i) = |N(w_i) \cap T| = 0$ for each $i$ with $1 \leq i \leq n-\kappa-1$. Thus, $G$ is a bipartite graph with partition sets $S$ and $T$.

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Notice that

$$\begin{aligned}\delta &= \frac{\sum_{i=0}^{\kappa} d(u_i)}{\kappa+1} = \frac{\sum_{i=1}^{n-\kappa-1} d_1(w_i)}{n-\kappa-1}\,\frac{n-\kappa-1}{\kappa+1} \\ &= \frac{\sum_{i=1}^{n-\kappa-1} d(w_i)}{n-\kappa-1}\,\frac{n-\kappa-1}{\kappa+1} \geq \delta\,\frac{n-\kappa-1}{\kappa+1}.\end{aligned}$$

Therefore, $n \leq 2\kappa+2$. Since $n \geq 2\kappa+1$. We have $n = 2\kappa+1$ or $n = 2\kappa+2$.

When $n = 2\kappa+1$, then $n-\kappa-1 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for $i$ with $0 \leq i \leq \kappa$, $u_i w_j \in E$ for each $i$ with $0 \leq i \leq \kappa$ and for each $j$ with $1 \leq j \leq n-\kappa-1$. Hence, $G$ is $K_{\kappa,\,\kappa+1}$.

When $n = 2\kappa+2$, then $n-\kappa-1 = \kappa+1$ and $G$ is a balanced bipartite graph. From Lemma 2.1, we have $G$ is Hamiltonian, a contradiction.

This completes the proof of Theorem 1.2. □

**4. Proof of Theorem 1.3.** Let $G$ be a graph satisfying the conditions in Theorem 1.3. Suppose, to the contrary, that $G$ is not traceable. Then $n \geq 2\kappa+2$ (otherwise $\delta \geq \kappa \geq \frac{n-1}{2}$ and $G$ is traceable). Choose a longest path $P$ in $G$ and give an orientation on $P$. Let $x$ and $y$ be the two end vertices of $P$. Since $G$ is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find $s$ ($s \geq \kappa$) pairwise disjoint (except for $u_0$) paths $P_1, P_2, \ldots, P_s$ between $u_0$ and $V(P)$. Let $v_i$ be the end vertex of $P_i$ on $P$, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of $P$. Since $P$ is a longest path in $G$, $x \neq v_i$ and $y \neq v_i$, for each $i$ with $1 \leq i \leq s$, otherwise $G$ would have paths which are longer than $P$. We use $v_i^+$ to denote the successor of $v_i$ along the orientation of $P$, where $1 \leq i \leq s$. Since $P$ is a longest path in $G$, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s-1$. Moreover, $\{u_0, v_1^+, v_2^+, \ldots, v_s^+, x\}$ is independent (otherwise $G$ would have paths which are longer than $P$). Set $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+, x\}$. Then $S$ is independent. Let $u_i = v_i^+$ for each $i$ with $1 \leq i \leq \kappa$ and $u_{\kappa+1} = x$. Set $T := V - S = \{\, w_1, w_2, \ldots, w_{n-\kappa-2} \,\}$. We label the vertices of $u_0, u_1, \ldots, u_\kappa, u_{\kappa+1}, w_1, w_2, \ldots, w_{n-\kappa-2}$ by $1, 2, \ldots, \kappa+1, \kappa+2, \ldots, n$, respectively. Let $d_1(w_i) = |N(w_i) \cap S|$ and $d_2(w_i) = |N(w_i) \cap T|$ for each $i$ with $1 \leq i \leq n-\kappa-2$. Obviously, $\sum_{i=0}^{\kappa+1} d(u_i) = \sum_{i=1}^{n-\kappa-2} d_1(w_i)$.

Define a two by two matrix $B = (\, B_{i\,j} \,)_{2\times 2}$, where

$$B_{11} = 0,\;\; B_{12} = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2},\;\; B_{21} = \frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2},\;\; B_{22} = \frac{\sum_{i=1}^{n-\kappa-2} d_2(w_i)}{n-\kappa-2}.$$

Then $B$ is a quotient matrix of the adjacency matrix of $G$ with partition $S$ and $T$. Let $\mu_1 \geq \mu_2$ be the eigenvalues of $B$. Then, by Corollary 2.3 on Page 596 in [4], we have that $\lambda_1 \geq \mu_1$ and $\mu_2 \geq \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $G$.

We, from Perron-Frobenius theorem, have that $|\lambda_n| \leq \lambda_1$. Thus,

$$\begin{aligned}\lambda_1^2 &\geq -\lambda_1\lambda_n \geq -\mu_1\mu_2 = -det(B) = B_{12}\,B_{21} \\ &= \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2}\,\frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2} = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2}\,\frac{\sum_{i=0}^{k+1} d(u_i)}{\kappa+2}\,\frac{\kappa+2}{n-\kappa-2} \\ &\geq \frac{\delta^2(\kappa+2)}{n-\kappa-2} \geq \lambda_1^2.\end{aligned}$$

Therefore, $\lambda_1 = -\lambda_n$, $\lambda_1 = \mu_1$, $\lambda_n = \mu_2$, and $d(u_i) = \delta$ for each $i$ with $0 \leq i \leq \kappa+1$. Since $0 = \lambda_1 + \lambda_n = \mu_1 + \mu_2 = B_{22}$, $d_2(w_i) = |N(w_i) \cap T| = 0$ for each $i$ with $1 \leq i \leq n-\kappa-2$. Thus, $G$ is a bipartite graph with partition sets $S$ and $T$.

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Notice that

$$\delta = \frac{\sum_{i=0}^{\kappa+1} d(u_i)}{\kappa+2} = \frac{\sum_{i=1}^{n-\kappa-2} d_1(w_i)}{n-\kappa-2}\ \frac{n-\kappa-2}{\kappa+2}$$
$$= \frac{\sum_{i=1}^{n-\kappa-2} d(w_i)}{n-\kappa-2}\ \frac{n-\kappa-2}{\kappa+2} \geq \delta\ \frac{n-\kappa-2}{\kappa+2}.$$

Therefore, $n \leq 2\kappa + 4$. Since $n \geq 2\kappa + 2$. We have $n = 2\kappa + 2$, $n = 2\kappa + 3$, or $n = 2\kappa + 4$.

When $n = 2\kappa + 2$, then $n - \kappa - 2 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for $i$ with $0 \leq i \leq \kappa + 1$, $u_i w_j \in E$ for each $i$ with $0 \leq i \leq \kappa + 1$ and for each $j$ with $1 \leq j \leq n - \kappa - 2$. Hence, $G$ is $K_{\kappa,\, \kappa+2}$.

When $n = 2\kappa + 3$, then $n - \kappa - 2 = \kappa + 1$. Notice that $\kappa \geq 5$ since $n = 2\kappa + 3 \geq 12$. Notice further that each vertex in $S$ or $T$ has degree at least $\delta \geq \kappa$. From Lemma 2.2, we have $G$ has a cycle of length $2\kappa + 2$. Since $n = 2\kappa + 3$ and $\kappa \geq 5$, $G$ has a path containing all the vertices of $G$. Namely, $G$ is traceable, a contradiction.

When $n = 2\kappa + 4$, then $n - \kappa - 2 = \kappa + 2$. Notice that $\kappa \geq 4$ since $n = 2\kappa + 4 \geq 12$. Notice further that each vertex in $S$ or $T$ has degree at least $\delta \geq \kappa$. From Lemma 2.2, we have $G$ has a cycle of length $2\kappa + 4$, which implies that $G$ is traceable, a contradiction.

This completes the proof of Theorem 1.3. □

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