Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 343-355, July 2018.

INERTIA SETS ALLOWED BY MATRIX PATTERNS*

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Abstract. Motivated by the possible onset of instability in dynamical systems associated with a zero eigenvalue, sets of inertias $\mathbb{S}_n$ and $\mathbb{S}_n^*$ for sign and zero-nonzero patterns, respectively, are introduced. For an $n \times n$ sign pattern $\mathcal{A}$ that allows inertia $(0, n-1, 1)$, a sufficient condition is given for $\mathcal{A}$ and every superpattern of $\mathcal{A}$ to allow $\mathbb{S}_n$, and a family of such irreducible sign patterns for all $n \geq 3$ is specified. All zero-nonzero patterns (up to equivalence) that allow $\mathbb{S}_3^*$ and $\mathbb{S}_4^*$ are determined, and are described by their associated digraphs.

Key words. Sign pattern, Zero-nonzero pattern, Inertia, Digraph.

AMS subject classifications. 15B35, 05C20, 15A18, 15C50.

1. Introduction. The *inertia* of a real matrix A_n of order n is an ordered triple (n_+, n_-, n_0) of nonnegative integers summing to n, where n_+, n_-, n_0 are the number of eigenvalues of A_n with positive, negative, zero real parts, respectively. In a dynamical system, the presence of a zero eigenvalue of the Jacobian matrix at an equilibrium may herald the onset of instability. With the variation of a parameter, the eigenvalues may move from all having negative real parts (linear stability) to having a simple zero eigenvalue, that then moves to have a positive real part (instability), while the remaining eigenvalues continue to have negative real parts. This corresponds to the inertia going from $(0, n, 0)$ to $(0, n-1, 1)$ to $(1, n-1, 0)$. An example from a dynamical system in ecology is given in Section 6.

With this motivation, we define the inertia set $\mathbb{S}_n$ with $n \geq 2$ as

$$\mathbb{S}_n = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0)\}.$$

We are interested in the inertias of irreducible patterns, both sign patterns having entries in $\{+, -, 0\}$, and zero-nonzero patterns with entries in $\{0, *\}$. If $\mathcal{A}_n$ denotes a sign pattern of order n, then $\mathcal{A}_n$ has inertia $i(\mathcal{A}_n) = \{i(A_n) : A_n \in Q(\mathcal{A}_n)\}$, where $Q(\mathcal{A}_n)$ denotes the set of all real matrices having sign pattern $\mathcal{A}_n$, i.e., matrix A_n is a *realization* of $\mathcal{A}_n$. The inertia of a zero-nonzero pattern is defined analogously. We identify some irreducible sign patterns that *allow* $\mathbb{S}_n$, i.e., $i(\mathcal{A}_n) \supseteq \mathbb{S}_n$, or *require* $\mathbb{S}_n$, i.e., $i(\mathcal{A}_n) = \mathbb{S}_n$.

If $\mathcal{A}_n$ is a zero-nonzero pattern, then $(n_+, n_-, n_0) \in i(\mathcal{A}_n)$ if and only if its *reversal* $(n_-, n_+, n_0) \in i(\mathcal{A}_n)$. Thus, we define the inertia set

$$\mathbb{S}_n^* = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0), (n, 0, 0), (n-1, 0, 1), (n-1, 1, 0)\},$$

whose elements are the elements of $\mathbb{S}_n$ and their reversals. Using analogous definitions, we identify some irreducible zero-nonzero patterns that allow $\mathbb{S}_n^*$. If sign pattern $\mathcal{A}_n$ allows $\mathbb{S}_n$, then obviously the associated

*Received by the editors on October 24, 2017. Accepted for publication on May 22, 2018. Handling Editor: Leslie Hogben. Corresponding Author: Adam H. Berliner.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 343-355, July 2018.

zero-nonzero pattern allows $\mathbb{S}_n^*$. However, the corresponding result with "allows" replaced by "requires" may not be true (see, e.g., Example 1.1). For both sign and zero-nonzero patterns, we identify them up to *equivalence*, i.e., up to any combination of transposition and permutation similarity, and signature similarity for sign patterns, since these leave the spectrum unchanged.

It is often convenient to refer to a zero-nonzero pattern $\mathcal{A} = [\alpha_{ij}]$ (or a matrix $A = [a_{ij}]$) of order n by its associated digraph, a digraph on n vertices where there is an arc from vertex i to vertex j if and only if $\alpha_{ij} = *$ (or $a_{ij} \neq 0$). In the case that $\mathcal{A}$ is a sign pattern, the arc from vertex i to vertex j is signed as α_{ij}. Two digraphs are equivalent if and only if their associated patterns are equivalent. In Sections 4 and 5, we use the same naming conventions and notation for digraphs as in [1].

For $n = 2$, there are two nonequivalent sign patterns that allow $\mathbb{S}_2$, namely $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$ and $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$ (see [15]). The first sign pattern requires $\mathbb{S}_2$, whereas the second is spectrally arbitrary (i.e., it can attain every spectrum allowed by a real matrix). Thus, the order 2 zero-nonzero pattern with each entry nonzero allows but does not require $\mathbb{S}_2^*$.

To illustrate the inertia sets for $n = 3$, consider the following sign pattern.

Example 1.1. Sign pattern $\mathcal{B}_3$ and any matrix realization B_3 are given by

$$\mathcal{B}_3 = \begin{bmatrix} - & 0 & + \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} -a & 0 & 1 \\ c & 0 & -b \\ 0 & 1 & 0 \end{bmatrix},$$

with $a, b, c > 0$. Note that without loss of generality, two entries on the 3-cycle in the digraph associated with $\mathcal{B}_3$ can be set equal to 1. The characteristic polynomial of B_3 is

$$p_{B_3}(z) = z^3 + az^2 + bz + ab - c.$$

If $c < ab$, then $i(B) = (0, 3, 0)$, if $c = ab$, then $i(B_3) = (0, 2, 1)$, and if $c > ab$, then $i(B_3) = (1, 2, 0)$. Thus, $\mathcal{B}_3$ allows $\mathbb{S}_3$. There are ten possible inertias for a matrix of order 3, and the remaining seven can be eliminated for $\mathcal{B}_3$ as follows:

- Since $\operatorname{tr}(B_3) < 0$, inertias $(3, 0, 0), (2, 0, 1)$ are not possible.
- Since $b \neq 0$, the only eigenvalue allowed on the imaginary axis by B_3 is a simple 0 eigenvalue, thus eliminating inertias $(0, 0, 3), (1, 0, 2)$, and $(0, 1, 2)$.
- If one eigenvalue of B_3 is 0, then the other two have negative real parts, thus eliminating inertia $(1, 1, 1)$.
- If $c < ab$ in B_3, then the Routh-Hurwitz condition (see, for example, [7, p. 194]) $ab > ab - c$ is satisfied, eliminating inertia $(2, 1, 0)$.

Thus, in fact, sign pattern $\mathcal{B}_3$ requires $\mathbb{S}_3$. The associated zero-nonzero pattern $\mathcal{B}_3^*$ has a realization

$$F_3 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

with $i(F_3) = (1, 1, 1)$. Thus, the zero-nonzero pattern $\mathcal{B}_3^*$ allows but does not require $\mathbb{S}_3^*$. Note that F_3 is not equivalent to a realization of the sign pattern $\mathcal{B}_3$. Up to equivalence, $\mathcal{B}_3$ is the only signing of $\mathcal{B}_3^*$ that

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A publication of the International Linear Algebra Society
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requires or allows $\mathbb{S}_3$, since it is the only such sign pattern that allows inertia $(0,3,0)$, i.e., is potentially stable [14].

A related set of *refined inertias*, in which n_0 is split into zero eigenvalues denoted by n_z and nonzero pure imaginary eigenvalues denoted by $2n_p$, giving the 4-tuple $(n_+, n_-, n_0, 2n_p)$, was introduced for sign patterns in [2]. The set $\mathbb{H}_n$ is defined as

$$\mathbb{H}_n = \{(0,n,0,0),(0,n-2,0,2),(2,n-2,0,0)\},$$

and can herald the onset of instability by a pair of nonzero pure imaginary eigenvalues. For further results on $\mathbb{H}_n$, see [8, 9, 10, 11], and for the related set $\mathbb{H}_n^*$ pertaining to zero-nonzero patterns, see [1]. For some of our results we use techniques that are found in these cited papers.

We begin by considering sign patterns, giving sufficient conditions for a sign pattern and its superpatterns to allow $\mathbb{S}_n$. Recall that $\mathcal{B}_n$ is a *superpattern* of a sign pattern $\mathcal{A}_n$ if $\mathcal{B}_n$ is obtained from $\mathcal{A}_n$ by replacing some (or possibly none) of the zero entries of $\mathcal{A}_n$ with $-$ or $+$. In Section 3, we use this result and a bordering technique to construct a family of irreducible sign patterns that allows $\mathbb{S}_n$. In Sections 4 and 5, we identify all zero-nonzero patterns (up to equivalence) that allow $\mathbb{S}_n^*$ for $n=3$ and $n=4$, respectively. In Section 6, we give an ecological example as an application of our sign pattern results.

2. Preliminary results on sign patterns allowing $\mathbb{S}_n$. In this section, we begin with some observations, give an example of a family of irreducible sign patterns that allows $\mathbb{S}_n$ and requires $\mathbb{S}_n$ for small orders, and conclude with a theorem on superpatterns that is used to obtain results in the following sections.

Observation 2.1. *If sign pattern $\mathcal{A}_n$ allows $\mathbb{S}_n$, then any superpattern $\hat{\mathcal{A}}_n$ allows $\mathbb{S}_n$ if and only if $\hat{\mathcal{A}}_n$ allows inertia* $(0,n-1,1)$.

A sign pattern $\mathcal{A}_n$ is *combinatorially singular* if $\det(A_n)=0$ for all $A_n \in Q(\mathcal{A}_n)$, and it is *sign nonsingular* if $\det(A_n) \neq 0$ for all $A_n \in Q(\mathcal{A}_n)$. A zero-nonzero pattern is *combinatorially nonsingular* if the determinant is nonzero for every matrix realization of the pattern.

Observation 2.2. *If sign pattern $\mathcal{A}_n$ allows $\mathbb{S}_n$, then $\mathcal{A}_n$ is not combinatorially singular and not sign nonsingular.*

For sign patterns this observation means that there are at least two terms in the determinant of $A_n \in Q(\mathcal{A}_n)$ and they are of opposite signs. The next observation follows as in Lemma 3.1 of [2].

Observation 2.3. *If sign pattern $\mathcal{A}_n$ has no zero entry on its main diagonal, then it allows $\mathbb{S}_n$ if and only if it allows inertia* $(0,n-1,1)$.

Note that Observations 2.1, 2.2, and 2.3 with sign nonsingular replaced with combinatorially nonsingular hold for zero-nonzero patterns and $\mathbb{S}_n^*$.

We next consider a sign pattern that has all of its diagonal entries negative, and the other nonzero entries correspond to a positive n-cycle in the digraph associated with the sign pattern.

Theorem 2.1. *Let $\mathcal{C}_n$ be the sign pattern of a positive n-cycle, i.e., $\mathcal{C}_n = [\gamma_{ij}]$ has $\gamma_{12} = \gamma_{23} = \cdots = \gamma_{n-1,n} = \gamma_{n1} = +$ and all other entries zero, and let $\mathcal{I}_n$ be the sign pattern with each diagonal entry $+$ and all other entries zero. Sign pattern $\mathcal{C}_n - \mathcal{I}_n$ allows $\mathbb{S}_n$ for all $n \geq 3$, but requires $\mathbb{S}_n$ only for $n=3$ and* 4.

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Proof. The eigenvalues of matrix $C_n \in Q(\mathcal{C}_n)$ are the nth roots of a positive scalar multiple of 1, thus C_n has a simple positive eigenvalue a. With I_n denoting the identity matrix of order n, the matrix $C_n - aI_n$ has inertia $(0, n-1, 1)$, and Observation 2.3 gives the allow result. The other seven (twelve) inertias for $n = 3$ ($n = 4$) can be eliminated by using the facts that any matrix realization of $\mathcal{C}_n - \mathcal{I}_n$ has negative trace, is essentially nonnegative, and has a simple zero as the only eigenvalue on the imaginary axis, together with continuity and the Routh-Hurwitz conditions. It follows that $\mathcal{C}_3 - \mathcal{I}_3$ ($\mathcal{C}_4 - \mathcal{I}_4$) requires $\mathbb{S}_3$ ($\mathbb{S}_4$). For $n \geq 5$, matrix $C_n - I_n$ may have more than one eigenvalue in the right half plane, and thus, $\mathcal{C}_n - \mathcal{I}_n$ does not require $\mathbb{S}_n$. ∎

Now we give sufficient conditions for a sign pattern and its superpatterns to allow $\mathbb{S}_n$. Let $A = [a_{ij}]$ be a real matrix of order n having $m \geq n$ nonzero entries and $i(A) = (0, n-1, 1)$. Let matrix X be the same as A, except that these m nonzero entries of A, namely $a_{i_1,j_1}, \ldots, a_{i_m,j_m}$, are replaced by variables $x_1, \ldots, x_m$, respectively. The characteristic polynomial of X is

$$c_X(z) = z^n + p_1 z^{n-1} + \cdots + p_{n-1}z + p_n,$$

with coefficients $p_1, \ldots, p_n$ depending on $x_1, \ldots, x_m$. The $n \times m$ Jacobian matrix J of A has (i, j)-entry equal to $\frac{\partial p_i(x_1,\ldots,x_m)}{\partial x_j}$ evaluated at $(x_1, \ldots, x_m) = (a_{i_1,j_1}, \ldots, a_{i_m,j_m})$. If J has rank n, then A *allows a Jacobian matrix of full rank.* This definition, which uses a rectangular Jacobian matrix as in [12], is equivalent to the determinantal property that A "allows a nonzero Jacobian" as defined in [4].

THEOREM 2.2. *Let $\mathcal{A}$ be an $n \times n$ sign pattern that allows inertia $(0, n-1, 1)$ and let $A \in \mathcal{Q}(\mathcal{A})$ with $i(A) = (0, n-1, 1)$. If A allows a Jacobian matrix of full rank, then every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ (including $\mathcal{A}$) allows $\mathbb{S}_n$.*

Proof. If A allows a nonzero Jacobian and $i(A) = (0, n-1, 1)$, then it follows that $\{(1, n-1, 0), (0, n, 0)\} \subseteq i(\mathcal{A})$ by Lemma 3.4(i) of [3]. Therefore, $\mathcal{A}$ allows $\mathbb{S}_n$. By Theorem 3.2 of [3], it follows that there exists $\hat{A} \in \mathcal{Q}(\hat{\mathcal{A}})$ for which $i(\hat{A}) = (0, n-1, 1)$. Thus, by Observation 2.1, $\hat{\mathcal{A}}$ allows $\mathbb{S}_n$. ∎

3. A family of sign patterns that allows $\mathbb{S}_n$. We employ the following bordering technique [13] to construct a family of irreducible sign patterns that allows $\mathbb{S}_n$.

For $n \geq 3$, consider the family of sign patterns $\mathcal{A}_n$ associated with the family of matrices A_n that are obtained from successive borderings of the form

$$A_{n+1} = \begin{bmatrix} I_n & 0 \\ {e_n}^T & 1 \end{bmatrix} \begin{bmatrix} A_n & e_n \\ 0^T & -1 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -{e_n}^T & 1 \end{bmatrix} = \begin{bmatrix} A_n - e_n{e_n}^T & e_n \\ {e_n}^T A_n & 0 \end{bmatrix},$$

where $e_n = (0, 0, \ldots, 0, 1)^T$ and ${e_n}^T A_n$ is the nth row of A_n. Note that if $i(A_n) = (n_+, n_-, n_0)$, then $i(A_{n+1}) = (n_+, n_- + 1, n_0)$.

EXAMPLE 3.1. To illustrate the above bordering, let $n = 3$ and A_3 be the matrix B_3 in Example 1.1 with $a = b = c = 1$. Then $i(B_3) = (0, 2, 1)$ and the bordering gives

$$B_4 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It can be shown (see the proof of Theorem 3.1) that B_4 allows a Jacobian matrix of rank 4. So, by Theorem 2.2, $\text{sgn}(B_4)$ allows $\mathbb{S}_4$. However, $\text{sgn}(B_4)$ does not require $\mathbb{S}_4$ since it allows $\mathbb{H}_4$ (see G9/23 in Table 3 of

Electronic Journal of Linear Algebra, ISSN 1081-3810
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[1]). A second iteration of bordering gives

$$B_5 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and by similar arguments, $\text{sgn}(B_5)$ allows $\mathbb{S}_5$.

Continued iterations of bordering, as in Example 3.1, lead to irreducible matrices $B_n = [b_{ij}]$ of order $n \geq 3$ defined as follows:

- $b_{13} = b_{21} = b_{n2} = 1$ and $b_{11} = b_{23} = -1$.
- For $3 \leq i \leq n-1$, $b_{ii} = -1$, $b_{i,i+1} = 1$, and $b_{i2} = 1$.
- $b_{ij} = 0$ otherwise.

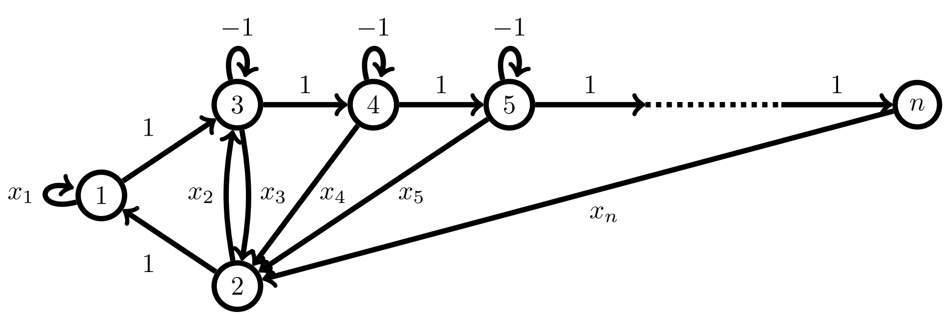
THEOREM 3.1. *With matrix B_n defined as above for $n \geq 3$, $\mathcal{B}_n = \text{sgn}(B_n)$ and every superpattern of $\text{sgn}(B_n)$ allow $\mathbb{S}_n$.*

Proof. Since $i(B_3) = (0, 2, 1)$, it follows that $i(B_4) = (0, 3, 1)$ and by induction, $i(B_n) = (0, n-1, 1)$.

We define $X_{B_n} = [w_{ij}]$ to be equal to B_n except that n nonzero entries of B_n have been replaced by variables $x_1, \ldots, x_n$ in the following way:

- $w_{13} = w_{21} = 1$.
- For $3 \leq i \leq n-1$, $w_{ii} = -1$ and $w_{i,i+1} = 1$.
- $w_{11} = x_1$, $w_{23} = x_2$, and for $3 \leq i \leq n$, $w_{i2} = x_i$.
- $w_{ij} = 0$ otherwise.

The labeled signed digraph of X_{B_n} is as follows:



It is well-known (see, e.g., [16, page 29-6, Fact 4]) that if

$$p_{X_{B_n}} = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n$$

is the characteristic polynomial of X_{B_n}, then for $1 \leq k \leq n$, p_k is the sum of all (disjoint) signed generalized cycle products of order k in the labeled digraph of X_{B_n}. Note that a *generalized k-cycle* in a digraph is a set of disjoint cycles that together cover k vertices, and a *generalized cycle product of order k* is the product

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A publication of the International Linear Algebra Society
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of the entries on these cycles. The sign of each generalized cycle product is $(-1)^q$, where q is the number of cycles in the product (see [6, Theorem 1.2]). In order to use Theorem 2.2, we need to determine the entries of the Jacobian matrix of X_{B_n} as functions of $x_1, \ldots, x_n$. The (i,j)-entry of the $n \times n$ Jacobian matrix $J_{X_{B_n}}$ is equal to $\dfrac{\partial p_i}{\partial x_j}$.

From the digraph of X_{B_n}, it can be seen that $p_1 = n - 3 - x_1$, so that the first row of $J_{X_{B_n}}$ is

$$[-1, 0, 0, \ldots, 0]\,.$$

Again using the digraph of X_{B_n}, it follows that

$$p_n = \det X_{B_n} = (-1)^2 x_1 x_2 x_n + (-1)^1 x_n$$

since there are only two generalized n-cycles. Thus, row n of $J_{X_{B_n}}$ is equal to

$$[x_2 x_n, x_1 x_n, 0, \ldots, 0, x_1 x_2 - 1]\,.$$

Furthermore, for $2 \le k \le n-1$, there exists a generalized k-cycle in the digraph of X_{B_n} with cycle product $(-1)^1 x_2 x_{k+1}$. This is the only generalized k-cycle involving x_{k+1}, and x_i is not on a generalized k-cycle for any $i \ge k+2$. Thus, for some possibly nonzero values $*$,

$$J_{X_{B_n}} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & -x_2 & 0 & \cdots & 0 & 0 \\ * & * & * & -x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & -x_2 & 0 \\ * & * & * & * & \cdots & * & -x_2 \\ x_2 x_n & x_1 x_n & 0 & 0 & \cdots & 0 & x_1 x_2 - 1 \end{bmatrix}.$$

Since $x_1 = x_2 = -1$ and $x_n = 1$ achieves inertia $(0, n-1, 1)$, it follows that for these values, the (n,n)-entry of $J_{X_{B_n}}$ is 0 and so $\det J_{X_{B_n}} = (-1)^n$. Thus, $\mathcal{B}_n$ allows a Jacobian matrix of full rank, and by Theorem 2.2, $\mathcal{B}_n$ and every superpattern of $\mathcal{B}_n$ allow $\mathbb{S}_n$. □

Note that $\mathcal{B}_n$ does not require $\mathbb{S}_n$ for $n \ge 4$ since it allows refined inertia $(0, n-2, 0, 2)$, and is not spectrally arbitrary since any $B_n \in \mathcal{Q}(\mathcal{B}_n)$ has negative trace.

4. Zero-nonzero patterns that allow $\mathbb{S}_3^*$. For $n \ge 3$, the sign pattern $\mathcal{B}_n$ in Theorem 3.1 gives an associated zero-nonzero pattern than allows $\mathbb{S}_n^*$. In this section, we describe all irreducible nonequivalent zero-nonzero patterns of order 3 that allow $\mathbb{S}_3^*$. With digraphs D1–D3 in Appendix A, Table 1 gives realizations $A = [a_{ij}]$ of three different sign (hence, zero-nonzero) patterns that allow $\mathbb{S}_3$ (hence, $\mathbb{S}_3^*$). Entries not mentioned are taken to be 0. Each of these realizations has inertia $(0, 2, 1)$ and allows a Jacobian matrix of full rank (we will illustrate the use of the Jacobian in the next section, which discusses patterns of order 4). By Theorem 2.2, these three zero-nonzero patterns and any superpattern of them allow $\mathbb{S}_3^*$. In fact, any irreducible zero-nonzero pattern of order 3 that is not equivalent to a superpattern of one of these three does not allow $\mathbb{S}_3^*$. The zero-nonzero pattern associated with digraph D2 with a loop at vertex 2 is combinatorially singular, and all others are combinatorially nonsingular, so do not allow $\mathbb{S}_3^*$ (see Observation 2.2).

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Digraph	Loops at	Nonzero Entries in Realization with Inertia $(0, 2, 1)$
D1	123	$a_{11} = a_{22} = -1, a_{12} = a_{23} = 1, a_{31} = 2, a_{33} = -2$
D2	13	$a_{11} = -1, a_{12} = a_{23} = a_{21} = 1, a_{32} = a_{33} = -2$
D3	3	$a_{33} = -1, a_{12} = a_{23} = 1, a_{21} = -1/4, a_{31} = 1/4$

TABLE 1
Order 3 realizations with inertia $(0, 2, 1)$.

Digraph	Loops at	Nonzero Entries in Realization with Inertia $(0, 3, 1)$
G1	1234	$a_{11} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{33} = -2, a_{44} = -3, a_{41} = 6$
G2	134	$a_{11} = a_{21} = a_{33} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1$
G3	12	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{41} = 1, a_{13} = -6/5, a_{22} = -5/6$
G3	24	$a_{22} = a_{13} = -1, a_{12} = a_{23} = a_{34} = a_{41} = 1, a_{44} = -2$
G4	24	$a_{22} = -1, a_{12} = a_{23} = a_{14} = a_{31} = 1, a_{43} = a_{44} = -2$
G5	34	$a_{21} = a_{33} = -1, a_{12} = a_{23} = a_{34} = 1, a_{41} = 2, a_{44} = -2$
G6	2	$a_{13} = a_{22} = a_{41} = -1, a_{12} = a_{23} = a_{34} = 1, a_{14} = 2$
G7	2	$a_{13} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{21} = -1/3, a_{41} = 1/27$
G8	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1, a_{21} = -1/2, a_{43} = -2$
G8	34	$a_{21} = a_{33} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{43} = 1, a_{42} = 3/2$
G9	2	$a_{13} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{31} = 1/3, a_{41} = 1/27$
G11	34	$a_{21} = a_{33} = a_{43} = a_{44} = -1, a_{12} = a_{14} = a_{23} = a_{31} = 1$
G13	14	$a_{11} = a_{32} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1, a_{21} = -2$
G15	12	$a_{11} = a_{22} = a_{32} = a_{43} = -1, a_{12} = a_{23} = a_{34} = a_{21} = 1$
G15	14	$a_{11} = a_{21} = a_{32} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{43} = 1$
G16	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = 1, a_{14} = -27, a_{32} = -1/27, a_{41} = 8/729$
G17	234	$a_{22} = a_{44} = -1, a_{12} = a_{13} = a_{14} = 1,$ $a_{33} = -2, a_{21} = -4, a_{31} = -3, a_{41} = 5.5$
G22	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{24} = 1, a_{13} = 3, a_{32} = -1/3, a_{41} = 2/27$
G29	2	$a_{22} = a_{34} = -1, a_{13} = a_{24} = a_{32} = a_{21} = 1, a_{12} = -1/3, a_{41} = 7/9$

TABLE 2
Order 4 realizations with inertia $(0, 3, 1)$.

5. Zero-nonzero patterns that allow $\mathbb{S}_4^*$. In this section, we describe all irreducible nonequivalent zero-nonzero patterns of order 4 that allow $\mathbb{S}_4^*$. In Appendix B, we list 38 of the 61 digraphs G1–G61 given in [1]; the remaining 23 digraphs are not needed here since their associated zero-nonzero patterns are superpatterns of patterns associated with the listed 38 digraphs. With the digraphs in Appendix B, Table 2 gives realizations $A = [a_{ij}]$ of 19 different sign (hence, zero-nonzero) patterns. Entries not mentioned are taken to be 0. Each of these realizations has inertia $(0, 3, 1)$ and allows a Jacobian matrix of full rank. By Theorem 2.2, any of these 19 zero-nonzero patterns and any superpattern of any of these 19 zero-nonzero patterns allow $\mathbb{S}_4^*$.

EXAMPLE 5.1. To illustrate the use of the Jacobian, we give justification for one of the realizations. Consider the zero-nonzero pattern $\mathcal{A}$ associated with digraph G3 with loops at vertices 2 and 4:

$$\mathcal{A} = \begin{bmatrix} 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & * \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
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A matrix with an associated sign pattern and inertia $(0,3,1)$ is

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}.$$

In order to apply Theorem 2.2, we let

$$X = \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & x_3 & x_4 & 0 \\ 0 & 0 & 0 & x_5 \\ x_6 & 0 & 0 & x_7 \end{bmatrix}.$$

The characteristic polynomial of X is

$$c_X(z) = z^4 + (-x_3 - x_7)z^3 + x_3x_7z^2 - x_2x_5x_6z + (x_2x_3x_5x_6 - x_1x_4x_5x_6)$$

and the Jacobian matrix J_{X_A} is

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & x_7 & 0 & 0 & 0 & x_3 \\ 0 & -x_5x_6 & 0 & 0 & -x_2x_6 & -x_2x_5 & 0 \\ -x_4x_5x_6 & x_3x_5x_6 & x_2x_5x_6 & -x_1x_5x_6 & x_2x_3x_6 - x_1x_4x_6 & x_2x_3x_5 - x_1x_4x_5 & 0 \end{bmatrix}.$$

Matrix J_{X_A} evaluated at $x_1 = x_4 = x_5 = x_6 = 1, x_2 = x_3 = -1, x_7 = -2$, gives the full rank matrix

$$J_{X_A} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, by Theorem 2.2, $\mathcal{A}$ and any superpattern of $\mathcal{A}$ allows $\mathbb{S}_4^*$.

In fact, any irreducible zero-nonzero pattern of order 4 that is not equivalent to a superpattern of one of these 19 in Table 2 does not allow $\mathbb{S}_4^*$. Tables 3–5 give justification for why each of the these zero-nonzero patterns do not allow $\mathbb{S}_4^*$. Some zero-nonzero patterns are combinatorially singular while others are combinatorially nonsingular (see Tables 3 and 4). While this covers most patterns, there are three patterns that are neither combinatorially singular nor combinatorially nonsingular yet still do not allow $\mathbb{S}_4^*$ (see Table 5). These patterns require a characteristic polynomial that has no quadratic term, and therefore, inertias $(0,3,0)$ and $(0,2,1)$ are not allowed.

6. Concluding remarks. As discussed in Section 1, there is no zero-nonzero pattern that requires $\mathbb{S}_2^*$. All zero-nonzero patterns that allow $\mathbb{S}_3^*$ (from Section 4) and $\mathbb{S}_4^*$ (from Section 5) have been checked, and each such pattern allows at least one additional inertia. Thus, no irreducible zero-nonzero patterns require $\mathbb{S}_3^*$ or $\mathbb{S}_4^*$. We do not know if this remains true for $n \geq 5$.

However, note that the sign pattern $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$ requires $\mathbb{S}_2$ and $\begin{bmatrix} - & - & 0 \\ - & - & - \\ 0 & - & - \end{bmatrix}$ requires $\mathbb{S}_3$ (see [15]). These sign patterns can be used as follows to construct higher order reducible sign patterns that require $\mathbb{S}_n$.

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Digraph(s)	Loop location(s) with combinatorially nonsingular pattern
G1	$1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234$
G2	$1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 234$
G3, G6	$1, 3, 4, 13, 14, 34, 134$
G4, G10	$2, 4, 12, 14, 23, 34, 123, 134$
G5	$1, 2, 3, 4, 12, 13, 14, 23, 24, 123, 124$
G7	$1, 3, 4, 13, 14$
G8, G28, G45	$2, 3, 4, 23, 24$
G9, G19, G24, G27, G35, G38	$1, 3, 13$
G11	$2, 4, 14, 23, 123$
G12	$1, 2, 3, 4, 12, 13, 23, 24, 123$
G13	$1, 12, 13, 34, 123, 234$
G14, G18, G21	$1, 4, 14$
G15	$1, 2, 3, 4, 13, 23, 24$
G17	$23, 24, 34, 123, 124, 134$
G23	$1, 2, 3, 12, 13, 23, 123$
G26	$1, 2, 3, 12, 13$
G29, G33	1
G30	$2, 4, 14, 23, 34, 134$
G31	$2, 4, 12, 14$
G32	$1, 12, 34, 234$
G42, G44	$2, 4, 24$
G47	$1, 4, 12, 24$
G51	2

Table 3
Digraphs of order 4 with combinatorially nonsingular associated patterns.

Digraph(s)	Loop location(s) with combinatorially singular pattern
G4, G10, G30, G31	$1, 3, 13$
G11	$1, 3, 12, 13$
G13, G32	$2, 3, 4, 23, 24$
G17	$1, 2, 3, 4, 12, 13, 14$
G47	$2, 3, 23$

Table 4
Digraphs of order 4 with combinatorially singular associated patterns.

Observation 6.1. *Consider the reducible sign pattern*

$$\mathcal{G}_{m+n} = \begin{bmatrix} \mathcal{A}_n & \# \\ 0 & \mathcal{P}_m \end{bmatrix},$$

with # an arbitrary $n \times m$ sign pattern.

(a) If $\mathcal{A}_n$ allows $\mathbb{S}_n$ and $\mathcal{P}_m$ allows inertia $(0, m, 0)$, then $\mathcal{G}_{m+n}$ allows $\mathbb{S}_{m+n}$.

(b) If $\mathcal{A}_n$ requires $\mathbb{S}_n$ and $\mathcal{P}_m$ is sign stable (i.e., $i(P_m) = (0, m, 0)$ for all $P_m \in Q(\mathcal{P}_m)$), then $\mathcal{G}_{m+n}$ requires $\mathbb{S}_{m+n}$.

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Digraph(s)	**Loop location(s) that do not allow** $(0,4,0)$
G3	2
G14	$2,3$

TABLE 5
Other digraphs of order 4 with associated patterns that do not allow inertia $(0,4,0)$.

As mentioned in Section 1, motivation for $\mathbb{S}_n$ comes from considering the onset of instability in dynamical systems. Consider an ecological example given by Weisser et al. [17, equation (5)] for a host-parasitoid system with a type II functional response of the parasitoids. For this system, the Jacobian matrix at the positive equilibrium has the sign pattern

$$\mathcal{A} = \begin{bmatrix} + & - & 0 \\ + & - & + \\ 0 & + & - \end{bmatrix}.$$

As shown in [5, 15], $\mathcal{A}$ allows $\mathbb{H}_3$. In addition, the following three realizations show that $\mathcal{A}$ allows $\mathbb{S}_3$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

Thus, for this sign pattern, instability of the positive equilibrium can occur through either a pair of nonzero complex eigenvalues or a zero eigenvalue, depending on the numerical values of a matrix realization.

Acknowledgment. The research of D.D. Olesky and P. van den Driessche was partially supported by NSERC Discovery Grants.

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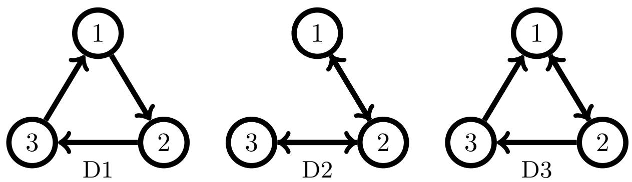
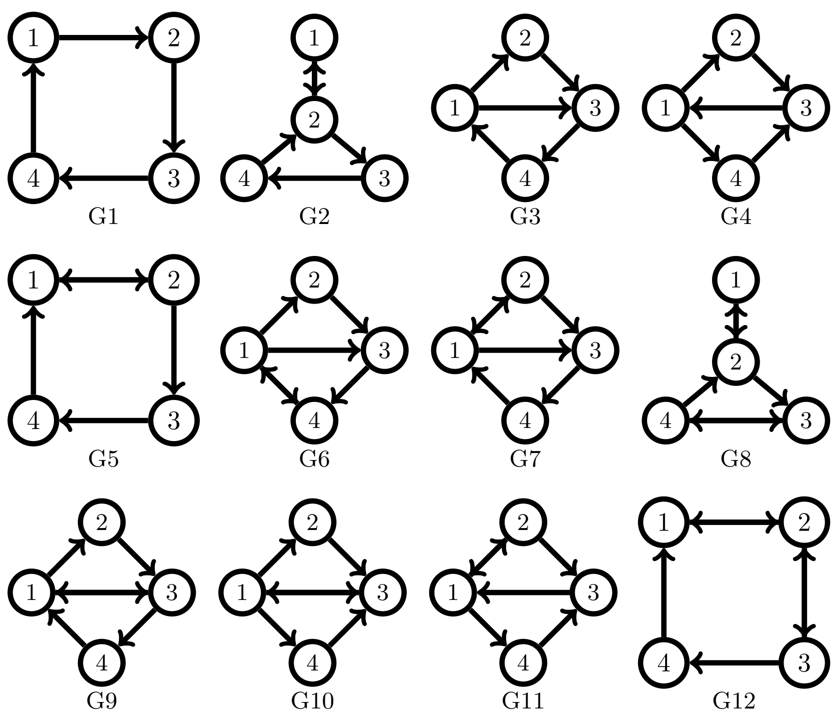
Appendix A. Non-equivalent strongly connected digraphs on 3 vertices used in Section 4 (retaining the labels from [1]).



Appendix B. Non-equivalent strongly connected digraphs on 4 vertices used in Section 5 (retaining the labels from [1]).



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A publication of the International Linear Algebra Society
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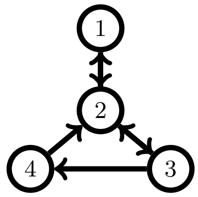
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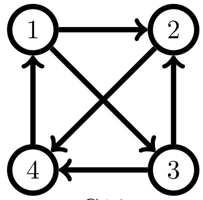
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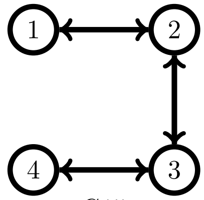
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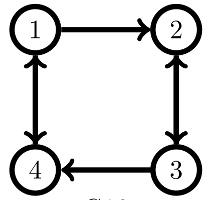
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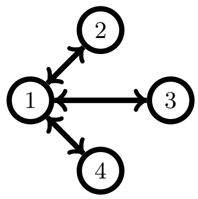
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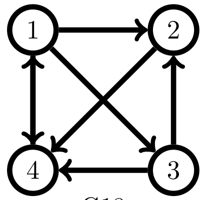
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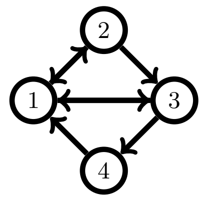
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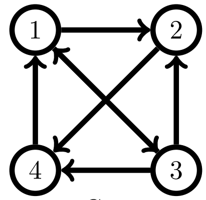
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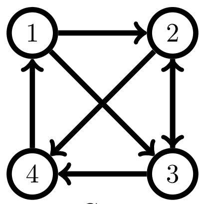
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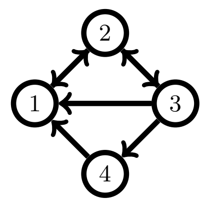
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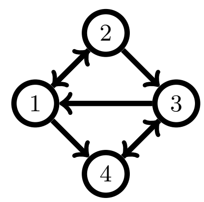
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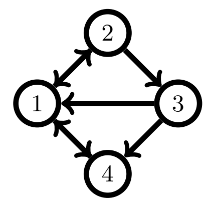
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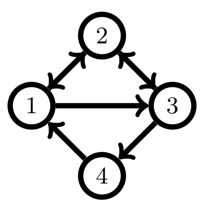
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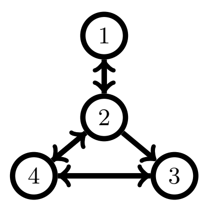
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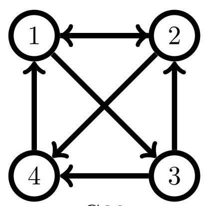
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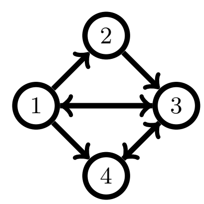
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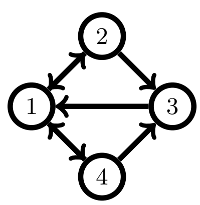
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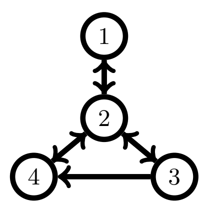
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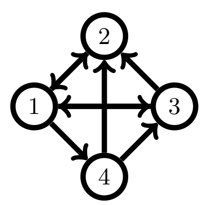
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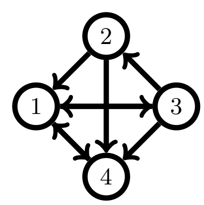
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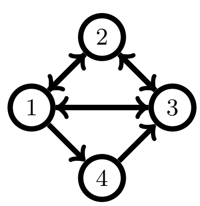
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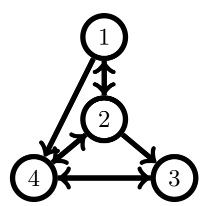
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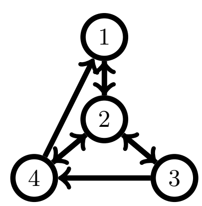
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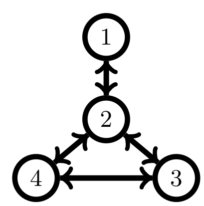
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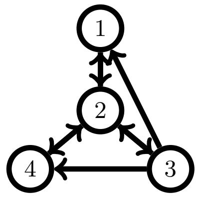
A publication of the International Linear Algebra Society
Volume 34, pp. 343-355, July 2018.



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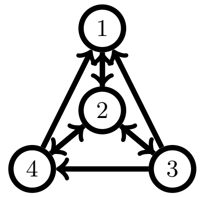

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