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RANKS OF QUANTUM STATES WITH PRESCRIBED REDUCED STATES*

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Abstract. Let $\mathcal{M}_n$ be the set of $n \times n$ complex matrices. In this note, all the possible ranks of a bipartite state in $\mathcal{M}_m \otimes \mathcal{M}_n$ with prescribed reduced states in the two subsystems, are determined. The results are used to determine the Choi rank of quantum channels $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ sending I/m to a specific state $\sigma_2 \in \mathcal{M}_n$.

Key words. Quantum state, Reduced state, Partial trace, Quantum channel, Rank, Eigenvalue.

AMS subject classifications. 15A18, 15A60, 15A42, 15B48, 46N50.

1. Introduction. In quantum information science, quantum states are used to store, process, and transmit information. Mathematically, quantum states are represented by density matrices, i.e., positive semidefinite matrices of trace 1.

Let $\mathcal{M}_n$ ($\mathcal{H}_n$) be the set of $n \times n$ complex (Hermitian) matrices. Let $\mathcal{D}_n$ be the set of density matrices in $\mathcal{H}_n$. Suppose $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$ are two quantum states. Their product state is $\sigma_1 \otimes \sigma_2 \in \mathcal{D}_{mn}$. The combined system is known as the bipartite system, and a general quantum state is represented by a density matrix $\rho \in \mathcal{D}_{mn}$.

Two important quantum operations used to extract information of the subsystems from a quantum state of the bipartite system are the partial traces defined as

$$\mathrm{tr}_1(\sigma_1 \otimes \sigma_2) = \sigma_2 \quad \text{and} \quad \mathrm{tr}_2(\sigma_1 \otimes \sigma_2) = \sigma_1$$

on tensor states $\sigma_1 \otimes \sigma_2 \in \mathcal{D}_{mn}$, and extended by linearity for general states in $\mathcal{D}_{mn}$. In particular, suppose $\rho = (\rho_{ij})_{1 \le i,j \le m} \in \mathcal{D}_{mn}$ such that $\rho_{ij} \in \mathcal{M}_n$. Then

$$\mathrm{tr}_1(\rho) = \rho_{11} + \cdots + \rho_{mm} \in \mathcal{M}_n \quad \text{and} \quad \mathrm{tr}_2(\rho) = (\mathrm{tr}\rho_{ij})_{1 \le i,j \le m} \in \mathcal{M}_m.$$

Let $\mathrm{rank}\,(\sigma)$ be the rank of a matrix σ. The purpose of this note is to give a complete answer to the following.

Problem. Determine all the possible values of $\mathrm{rank}\,(\rho)$ for $\rho \in \mathcal{D}_{mn}$ in the set

$$\mathcal{S}(\sigma_1, \sigma_2) = \{\rho \in \mathcal{D}_{mn} : \mathrm{tr}_1(\rho) = \sigma_2,\, \mathrm{tr}_2(\rho) = \sigma_1\}$$

for given $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$.

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It is well known and easy to verify that the maximum rank of $\rho \in \mathcal{S}(\sigma_1,\sigma_2)$ equals $R = \mathrm{rank}\,(\sigma_1)\mathrm{rank}\,(\sigma_2)$. In Section 2, we will present a finite algorithm for determining the minimum rank value r of $\rho \in \mathcal{S}(\sigma_1,\sigma_2)$ in terms of the eigenvalues of σ_1 and σ_2. Moreover, we will show that there is $\rho \in \mathcal{S}(\sigma_1,\sigma_2)$ with $\mathrm{rank}\,(\rho) = k$ for every $k \in \{r, r+1, \ldots, R\}$.

In Section 3, we will describe some implications of the results in Section 2 to the study of quantum channels. In particular, the results allow us to determine all the possible Choi ranks of a quantum channel $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ having a prescribed state $\Phi(I_m/m) \in \mathcal{D}_n$.

In our discussion, let $\mathcal{M}_{m,n}$ be the set of $m \times n$ complex matrices so that $\mathcal{M}_n = \mathcal{M}_{n,n}$. Denote by $\mathcal{U}_n$ and $\mathcal{H}_n$ the set of unitary matrices and the set of Hermitian matrices in $\mathcal{M}_n$, respectively. Let $\mathbb{R}^n_\downarrow$ denote the set of n-tuples in $\mathbb{R}^n$ with decreasing coordinates. Given $\sigma \in \mathcal{H}_n$, $\lambda(\sigma) = (\lambda_1(\sigma), \ldots, \lambda_n(\sigma)) \in \mathbb{R}^n_\downarrow$ will denote the eigenvalues of σ arranged in decreasing order.

Let $\{e_1^{(m)}, \ldots, e_m^{(m)}\}$ and $\{e_1^{(n)}, \ldots, e_n^{(n)}\}$ be the standard bases for $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then, clearly, $\{e_1^{(m)} \otimes e_1^{(n)}, e_1^{(m)} \otimes e_2^{(n)}, \ldots, e_m^{(m)} \otimes e_n^{(n)}\}$ is the standard basis for $\mathbb{C}^m \otimes \mathbb{C}^n \equiv \mathbb{C}^{mn}$. For simplicity, we use the notation e_i for $e_i^{(m)}$ or $e_i^{(n)}$ if the dimension of the vector is clear in the context. Also, we use $e_i \otimes e_j$ instead of $e_i^{(m)} \otimes e_j^{(n)}$, and $e_i e_j^t = E_{ij}$ to denote the basic complex unit of appropriate size.

The following observation is useful in our discussion.

Lemma 1.1. *Let $\sigma_1 \in \mathcal{D}_m$, $\sigma_2 \in \mathcal{D}_n$, $U \in \mathcal{M}_m$ and $V \in \mathcal{M}_n$. Then*

$$\mathcal{S}(U\sigma_1U^*, V\sigma_2V^*) = (U \otimes V)\mathcal{S}(\sigma_1,\sigma_2)(U \otimes V)^* = \{(U \otimes V)\rho(U \otimes V)^* : \rho \in \mathcal{S}(\sigma_1,\sigma_2)\}.$$

2. Ranks of matrices in $\mathcal{S}(\sigma_1,\sigma_2)$. In this section, we present results concerning the ranks of states in $\mathcal{S}(\sigma_1,\sigma_2)$. We will use the fact that $\mathcal{S}(\sigma_1,\sigma_2)$ contains a rank one matrix if and only if σ_1 and σ_2 have the same nonzero eigenvalues counting multiplicities; see [2, 5].

For $w = (w_1, \ldots, w_{mn})^t \in \mathbb{C}^{mn}$, let $W = [w]$ be the $m \times n$ matrix such that the ith row equals $(w_{(i-1)n+1}, \ldots, w_{in})$ for $i = 1, \ldots, m$. Suppose $W = [w]$ has singular value decomposition XSY^t such that $X \in \mathcal{M}_m$ is unitary with columns $x_1, \ldots, x_m$, $Y \in \mathcal{M}_n$ is unitary with columns $y_1, \ldots, y_n$ and $S = s_1e_1e_1^t + \cdots + s_ke_ke_k^t$, where k is the rank of W. It follows that $W = [w] = \sum_{j=1}^k s_jx_jy_j^t$, and $w = \sum_{j=1}^k s_jx_j \otimes y_j$, which is known as the Schmidt decomposition of w.

Theorem 2.1. *Let $(\sigma_1,\sigma_2) \in \mathcal{D}_m \times \mathcal{D}_n$ such that* $\mathrm{rank}\,(\sigma_1) = r_1 \geq r_2 = \mathrm{rank}\,(\sigma_2)$. *There is an element in $\mathcal{S}(\sigma_1,\sigma_2)$ attaining the lowest rank r with $r \leq r_1$. Moreover, there exists $\rho \in \mathcal{S}(\sigma_1,\sigma_2)$ of rank k if and only if $r \leq k \leq r_1r_2$.*

Proof. By Lemma 1.1, we may assume that

$$\sigma_1 = \mathrm{diag}(\mu_1, \ldots, \mu_{r_1}, 0, \ldots, 0) \quad \text{and} \quad \sigma_2 = \mathrm{diag}(\hat{\mu}_1, \ldots, \hat{\mu}_{r_2}, 0, \ldots, 0).$$

If $\rho = \sum_{j=1}^k z_jz_j^* \in \mathcal{S}(\sigma_1,\sigma_2)$, then $\mathrm{tr}_1(\rho) = \sigma_2$ and $\mathrm{tr}_2(\rho) = \sigma_1$.

For $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$, let $e_i \otimes e_j$ denote $e_i^{(m)} \otimes e_j^{(n)}$, and let $v_{ij} = \sqrt{\mu_i\hat{\mu}_j}e_i \otimes e_j$. Now, for $\ell = 1, \ldots, r_1$, let

$$z_\ell = \begin{cases} \sum_{i=1}^{r_2} v_{i+\ell-1,i} & \text{if } 1 \leq \ell \leq r_1 - r_2 + 1, \\ \\ \sum_{i=1}^{r_1+1-\ell} v_{i+\ell-1,i} + \sum_{i=r_1+2-\ell}^{r_2} v_{i+\ell-1-r_1,i} & \text{if } r_1 - r_2 + 1 < \ell \leq r_1\,. \end{cases}$$

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For example, $z_1 = v_{11} + \cdots + v_{r_2,r_2}$, $z_2 = v_{21} + v_{32} + \cdots + v_{r_2+1,r_2}, \ldots, z_{r_1} = v_{r_1,1} + v_{12} + \cdots + v_{r_2-1,r_2}$. Then

$$\operatorname{tr}_1(z_\ell z_\ell^*) = \begin{cases} \sum_{i=1}^{r_2} \mu_{i+\ell-1}\hat{\mu}_i E_{ii} & \text{if } 1 \le \ell \le r_1 - r_2 + 1, \\ \sum_{i=1}^{r_1+1-\ell} \mu_{i+\ell-1}\hat{\mu}_i E_{ii} + \sum_{i=r_1+2-\ell}^{r_2} \mu_{i+\ell-1-r_1}\hat{\mu}_i E_{ii} & \text{if } r_1 - r_2 + 1 < \ell \le r_1, \end{cases} \tag{2.1}$$

and

$$\operatorname{tr}_2(z_\ell z_\ell^*) = \begin{cases} \sum_{i=1}^{r_2} \mu_{i+\ell-1}\hat{\mu}_i E_{i+\ell-1\ i+\ell-1} & \text{if } 1 \le \ell \le r_1 - r_2 + 1, \\ \sum_{i=1}^{r_1+1-\ell} \mu_{i+\ell-1}\hat{\mu}_i E_{i+\ell-1\ i+\ell-1} + \sum_{i=r_1+2-\ell}^{r_2} \mu_{i+\ell-1-r_1}\hat{\mu}_i E_{i+\ell-1-r_1\ i+\ell-1-r_1} & \text{if } r_1 - r_2 + 1 < \ell \le r_1\,. \end{cases} \tag{2.2}$$

Let

$$\rho = \sum_{\ell=1}^{r_1} z_\ell z_\ell^* \in \mathcal{S}(\sigma_1, \sigma_2)\,. \tag{2.3}$$

Then ρ has rank r_1. It follows from (2.1) and (2.2) that $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$. In (2.3) if we replace $z_1 z_1^*$ by

$$v_{11}v_{11}^* + \cdots + v_{pp}v_{pp}^* + \left(\sum_{j>p} v_{jj}\right)\left(\sum_{j>p} v_{jj}\right)^*, \quad p = 1, \ldots, r_2 - 1,$$

the resulting state is in $\mathcal{S}(\sigma_1, \sigma_2)$ with rank $p + r_1$ for $p = 1, \ldots, r_2 - 1$. Similarly, we can replace each $z_j z_j^*$ by a rank $p + 1$ matrix for $p = 1, \ldots, r_2 - 1$, in such a way that the resulting state still lies in $\mathcal{S}(\sigma_1, \sigma_2)$. Hence, $\mathcal{S}(\sigma_1, \sigma_2)$ contains matrices of rank k for every $k \in \{r_1, \ldots, r_1 r_2\}$.

Now, suppose $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank $r < r_1$. We will show that there exists $\tilde{\rho} \in \mathcal{S}(\sigma_1, \sigma_2)$ with rank k for any $r < k < r_1$. We prove the result by induction on k. To this end, let

$$\rho = \sum_{j=1}^{r} z_j z_j^* \in \mathcal{S}(\sigma_1, \sigma_2). \tag{2.4}$$

By the Schmidt decomposition, $z_j = \sum_{\ell=1}^{t_j} s_{j\ell} x_{j\ell} \otimes y_{j\ell}$ for each j, where $t_j = \operatorname{rank}([z_j])$. Let $w_{j\ell} = s_{j\ell} x_{j\ell} \otimes y_{j\ell}$. Similar to the previous case, we can replace $z_j z_j^*$ by

$$\sum_{\ell \le p} w_{j\ell} w_{j\ell}^* + \left(\sum_{\ell>p} w_{j\ell}\right)\left(\sum_{\ell>p} w_{j\ell}\right)^*$$

so that the resulting state still lies in $\mathcal{S}(\sigma_1, \sigma_2)$. We may increase the number of summands in (2.4) by one at each time until we get $\hat{\rho} = \sum_{j,\ell} w_{j\ell} w_{j\ell}^*$. Note that in each step, the rank of the state will either stay the same or increase by 1, and $\operatorname{rank}(\hat{\rho}) \ge \operatorname{rank}(\operatorname{tr}_2(\hat{\rho})) = \operatorname{rank}(\sigma_1) = r_1$. As a result, the set $\mathcal{S}(\sigma_1, \sigma_2)$ contains matrices of ranks $r, \ldots, r_1$. $\square$

We illustrate the above theorem by the following example.

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Example 2.2. Suppose $\sigma_1 = \mathrm{diag}(a_1, a_2, a_3)$ and $\sigma_2 = \mathrm{diag}(b_1, b_2)$ with $a_1 \geq a_2 \geq a_3 > 0, b_1 \geq b_2 > 0$. Let

$$z_1 = (\sqrt{a_1 b_1}, 0, 0, \sqrt{a_2 b_2}, 0, 0)^t, \quad z_2 = (0, 0, \sqrt{a_2 b_1}, 0, 0, \sqrt{a_3 b_2})^t,$$
$$z_3 = (0, \sqrt{a_1 b_2}, 0, 0, \sqrt{a_3 b_1}, 0)^t.$$

Then $\rho = z_1 z_1^* + z_2 z_2^* + z_3 z_3^* \in \mathcal{S}(\sigma_1, \sigma_2)$ is of rank 3. One can replace $z_1 z_1^*$ by $v_1 v_1^* + v_2 v_2^*$ with $v_1 = (\sqrt{a_1 b_1}, 0, 0, 0, 0, 0)^t$ and $v_2 = (0, 0, 0, \sqrt{a_2 b_2}, 0, 0)^t$ to get a rank 4 matrix in $\mathcal{S}(\sigma_1, \sigma_2)$. Similarly, we can further replace $z_2 z_2^*$ by $v_3 v_3^* + v_4 v_4^*$, and $z_3 z_3^*$ by $v_5 v_5^* + v_6 v_6^*$, etc. to get matrices in $\mathcal{S}(\sigma_1, \sigma_2)$ of rank 5 and 6.

Corollary 2.3. *Let $(\sigma_1, \sigma_2) \in \mathcal{D}_m \times \mathcal{D}_n$. There is a rank one $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ if and only if σ_1 and σ_2 have the same (multi-)set of non-zero eigenvalues, say, $\lambda_1 \geq \cdots \geq \lambda_r$.*

In such a case, there exists $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ of rank k if and only if $1 \leq k \leq r^2$.

Proof. The first part follows from the fact that $\rho = vv^* \in \mathcal{S}(\sigma_1, \sigma_2)$, where v has Schmidt decomposition $\sum_{j=1}^r s_j x_j \otimes y_j$, if and only if $\sigma_1 = \sum_{j=1}^r s_j^2 x_j x_j^*$ and $\sigma_2 = \sum_{j=1}^r s_j^2 y_j y_j^*$.

The second part follows from Theorem 2.1. ◻

We illustrate the above Corollary by the following example.

Example 2.4. Suppose $\sigma_1 = \mathrm{diag}(\lambda_1, \lambda_2, 0)$ and $\sigma_2 = \mathrm{diag}(\lambda_1, \lambda_2)$ such that $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. Let $f_1 = (\sqrt{\lambda_1}, 0, 0, 0, 0, 0)^t$, $f_2 = (0, 0, 0, \sqrt{\lambda_2}, 0, 0)^t$. Then $(f_1 + f_2)(f_1 + f_2)^* \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank 1, and $f_1 f_1^* + f_2 f_2^* \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank 2. Let

$$v_{11} = (\lambda_1, 0, 0, 0, 0, 0)^t, \quad v_{12} = (0, 0, 0, \lambda_2, 0, 0)^t,$$

$$v_{21} = (0, 0, \sqrt{\lambda_2 \lambda_1}, 0, 0, 0)^t, \quad v_{31} = (0, \sqrt{\lambda_1 \lambda_2}, 0, 0, 0, 0)^t.$$

Then $(v_{11} + v_{12})(v_{11} + v_{12})^* + v_{21} v_{21}^* + v_{31} v_{31}^* \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank 3 and $v_{11} v_{11}^* + v_{12} v_{12}^* + + v_{21} v_{21}^* + v_{31} v_{31}^* \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank 4.

Next, we determine the minimal rank of ρ in $\mathcal{S}(\sigma_1, \sigma_2)$. For $w = (w_1, \ldots, w_{mn})^t \in \mathbb{C}^{mn}$, we continue to let $W = [w]$ be the $m \times n$ matrix such that the ith row equals $(w_{(i-1)n+1}, \ldots, w_{in})$ for $i = 1, \ldots, m$. One can easily construct the inverse map which converts an $m \times n$ matrix W to $w = \mathrm{vec}(W) \in \mathbb{C}^{mn}$ so that $W = [w]$. Note that $\mathrm{tr}_1(ww^*) = W^t (W^t)^*$ and $\mathrm{tr}_2(ww^*) = WW^*$. Suppose $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ has rank $\leq r$. Then there exists an $mn \times r$ matrix V such that $\mathrm{tr}_1(VV^*) = \sigma_2$ and $\mathrm{tr}_2(VV^*) = \sigma_1$. For $1 \leq j \leq r$, let $W_j = [v_j]$, where v_j is the j$^{\text{th}}$ column of V. Then we have $\sigma_1 = W_1(W_1)^* + \cdots + W_r(W_r)^*$ and $\sigma_2 = W_1^t (W_1^t)^* + \cdots + W_r^t (W_r^t)^*$.

Given $C \in H_m$, let $\lambda(C) = (c_1, \ldots, c_m)$ denote the eigenvalues of C with $c_1 \geq \cdots \geq c_m$. Let $\mathbb{R}^m_\downarrow = \{(x_i) \in \mathbb{R}^m : x_1 \geq \cdots \geq x_m\}$. By a result of Klyachko [4, 2], the eigenvalues of a sum of Hermitian matrices can be characterized by a set $LR(m, r)$ of $r+1$ tuples $(J_0, J_1, \ldots, J_r)$, where $J_0, \ldots, J_r$ are subsets of $\{1, \ldots, m\}$. More specifically, given $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{c}_j = (c_{1j}, \ldots, c_{mj}) \in \mathbb{R}^m$, $1 \leq j \leq r$, there exist $A, C_1, \ldots, C_r \in \mathcal{H}_n$ such that

$$A = \sum_{j=1}^r C_j \text{ with } \lambda(A) = \mathbf{a}, \quad \text{and} \quad \lambda(C_j) = \mathbf{c}_j, \ 1 \leq j \leq r$$

if and only if $\mathrm{tr} A = \sum_{j=1}^r \mathrm{tr}\,(C_j)$ and for every $(J_0, J_1, \ldots, J_r) \in LR(m, r)$, the inequality

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$$\sum_{i\in J_0} a_i \le \sum_{j=1}^{r}\sum_{i\in J_j} c_{ij} \tag{2.5}$$

holds. We have the following.

THEOREM 2.5. *Suppose $m \le n$, $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$. The following conditions are equivalent:*

(1) *There exists $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ with rank $\le r$.*
(2) *There exist $C_1, \dots, C_r \in \mathcal{H}_m$ and $\tilde{C}_1, \dots, \tilde{C}_r \in \mathcal{H}_n$ such that*

$$\text{(i) } \lambda(\tilde{C}_j) = \lambda(C_j \oplus O_{n-m}), \quad \text{(ii) } \sigma_1 = \sum_{j=1}^{r} C_j, \quad \textit{and} \quad \text{(iii) } \sigma_2 = \sum_{j=1}^{r} \tilde{C}_j.$$

(3) *There exists $C \in \mathcal{H}_m$ such that (2) holds with $\lambda(C_j) = \lambda(C)$ and $\lambda(\tilde{C}_j) = \lambda(C \oplus O_{n-m})$ for all $1 \le j \le r$.*

Proof. "(1) $\Rightarrow$ (2)": Suppose there exists $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ with rank r. Then there exist $W_1, \dots, W_r \in \mathcal{M}_{m,n}$ such that

$$\sigma_1 = W_1(W_1)^* + \cdots + W_r(W_r)^* \quad \text{and} \quad \sigma_2 = W_1^t \left(W_1^t\right)^* + \cdots + W_r^t \left(W_r^t\right)^*.$$

Then (2) holds with $C_j = W_j W_j^*$ and $\tilde{C}_j = W_j^t (W_j^t)^*$.

"(2) $\Rightarrow$ (1)" can be proven by reversing the above argument.

"(2) $\Rightarrow$ (3)": Suppose $\lambda(\sigma_1) = (a_1, \dots, a_m)$, and $\lambda(C_j) = (c_{1j}, \dots, c_{mj})$ for $j = 1, \dots, r$. Let $(J_0, J_1, \dots, J_r) \in LR(m, r)$, and $\pi = (1, 2, \dots, r)$ be the cyclic permutation of $\{1, \dots, r\}$, i.e., $\pi(j) = j + 1$ for $1 \le j < r$ and $\pi(r) = 1$. Then $(J_0, J_{\pi^k(1)}, \dots, J_{\pi^k(r)}) \in LR(m, r)$. Since $\sigma_1 = C_1 + \cdots + C_r$, by (2.5) we have

$$\begin{aligned}
& \textstyle\sum_{i\in J_0} a_i \le \sum_{j=1}^{r}\sum_{i\in J_j} c_{i\sigma^k(j)} \quad \text{for every } 0 \le k \le r-1 \\
\Rightarrow\quad & \textstyle\sum_{i\in J_0} a_i \le \sum_{j=1}^{r}\sum_{i\in J_j} c_i, \ \text{ where } c_i = (c_{i1} + \cdots + c_{ir})/r \\
\Rightarrow\quad & \sigma_1 = \hat{C}_1 + \cdots + \hat{C}_r
\end{aligned}$$

for some $\hat{C}_1, \dots, \hat{C}_r \in \mathcal{H}_m$, where $\lambda(\hat{C}_j) = (c_1, \dots, c_m)$ for $1 \le j \le r$. Similarly, we can choose $\tilde{C}_j$ such that $\lambda(\tilde{C}_j) = (c_1, \dots, c_m, 0, \dots, 0)$.

Clearly, (3) $\Rightarrow$ (2). ∎

REMARK 2.6. For every $m, n \ge 1$, there exists a permutation matrix $P \in \mathcal{M}_{mn}$ such that $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ if and only if $P\rho P^T \in \mathcal{S}(\sigma_2, \sigma_1)$. For example, let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then for every $\sigma_1 \in \mathcal{D}_2$ and $\sigma_1 \in \mathcal{D}_3$, $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ if and only if $P\rho P^T \in \mathcal{S}(\sigma_2, \sigma_1)$. Thus, there is no loss of generality in the restriction of $m \le n$ in Theorem 2.5.

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EXAMPLE 2.7. Suppose $m = 2,\ n = 3,\ \rho = \frac{1}{12}\begin{bmatrix} 3 & -2 & 1 & 1 & 2 & -1 \\ -2 & 2 & 0 & 0 & -2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & -2 & 0 & 0 & 2 & -2 \\ -1 & 2 & 1 & 1 & -2 & 3 \end{bmatrix}$. Then $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ with $\sigma_1 = \frac{1}{12}\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix},\ \sigma_2 = \frac{1}{12}\begin{bmatrix} 5 & 1 & 4 \\ 1 & 2 & 1 \\ 4 & 1 & 5 \end{bmatrix}$. Let

$$C_1 = \frac{1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\quad C_2 = \frac{1}{12}\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix},$$

$$\tilde{C}_1 = \frac{1}{12}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},\quad \tilde{C}_2 = \frac{1}{12}\begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}.$$

We have:

(i) $\lambda(C_1) = \frac{1}{12}(3,1),\ \lambda(\tilde{C}_1) = \frac{1}{12}(3,1,0), \lambda(C_2) = \frac{1}{12}(8,0),\ \lambda(\tilde{C}_2) = \frac{1}{12}(8,0,0)$.
(ii) $\sigma_1 = C_1 + C_2$.
(iii) $\sigma_2 = \tilde{C}_1 + \tilde{C}_2$.

Let $c_1 = \frac{1}{2}\left(\frac{3}{12} + \frac{8}{12}\right) = \frac{11}{24}$ and $c_2 = \frac{1}{2}\left(\frac{1}{12} + 0\right) = \frac{1}{24}$. Then we can choose

$$C_1' = \frac{1}{24}\begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix},\ C_2 = \frac{1}{24}\begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix},$$

$$\hat{C}_1 = \frac{1}{24}\begin{bmatrix} 5+4\sqrt{\frac{2}{57}} & 1-14\sqrt{\frac{2}{57}} & 4+4\sqrt{\frac{2}{57}} \\ 1-14\sqrt{\frac{2}{57}} & 2-8\sqrt{\frac{2}{57}} & 1-14\sqrt{\frac{2}{57}} \\ 4+4\sqrt{\frac{2}{57}} & 1-14\sqrt{\frac{2}{57}} & 5+4\sqrt{\frac{2}{57}} \end{bmatrix},$$

and

$$\hat{C}_2 = \frac{1}{24}\begin{bmatrix} 5-4\sqrt{\frac{2}{57}} & 1+14\sqrt{\frac{2}{57}} & 4-4\sqrt{\frac{2}{57}} \\ 1+14\sqrt{\frac{2}{57}} & 2+8\sqrt{\frac{2}{57}} & 1+14\sqrt{\frac{2}{57}} \\ 4-4\sqrt{\frac{2}{57}} & 1+14\sqrt{\frac{2}{57}} & 5-4\sqrt{\frac{2}{57}} \end{bmatrix}.$$

Then we have $\sigma_1 = C_1' + C_2'$, $\sigma_2 = \hat{C}_1 + \hat{C}_2$ with $\lambda(C_1') = \lambda(C_2') = (c_1, c_2)$ and $\lambda(\hat{C}_1) = \lambda(\hat{C}_2) = (c_1, c_2, 0)$.

When $n = rm$, we have the following corollary of Theorem 2.5.

COROLLARY 2.8. *Suppose $n = rm$, $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$. The following conditions are equivalent:*

(a) *There exists $\rho \in \mathcal{S}(\sigma_1, \sigma_2)$ with rank $\leq r$.*
(b) *There exists $C = (C_{ij}) \in H_n$ with $C_{ij} \in M_m$ such that $\lambda(C) = \lambda(\sigma_2)$ and $\lambda(C_{11} + \cdots + C_{rr}) = \lambda(\sigma_1)$.*
(c) *Condition* (b) *holds with $\lambda(C_{11}) = \cdots = \lambda(C_{rr})$.*

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Proof. "(a) $\Rightarrow$ (b)": Suppose (a) holds. Let C_j and $\tilde{C}_j$, $1 \le j \le r$, be as given by condition (2) in Theorem 2.5. For each $1 \le j \le r$, there exists an $n \times n$ unitary matrix U_j such that $\tilde{C}_j = U_j \left(C_j \oplus O_{(r-1)m} \right) U_j^*$. Let $V_j \in \mathcal{M}_{n,m}$ be formed by the first m columns of U_j. Let $R_j = V_j C_j^{1/2}$ and $R = [R_1 | \cdots | R_r]$. Set $C = R^*R$. Then

$$\lambda(C) = \lambda(R^*R) = \lambda(RR^*) = \lambda\left(\sum_{j=1}^r \tilde{C}_j\right) = \lambda(\sigma_2)$$

and

$$\lambda\left(C_{11} + \cdots + C_{rr}\right) = \lambda\left(\sum_{j=1}^r R_j^* R_j\right) = \lambda\left(\sum_{j=1}^r C_j\right) = \lambda(\sigma_1)\,.$$

Therefore, $C = R^*R$ will satisfy condition (b).

"(b) $\Rightarrow$ (a)": Suppose (b) holds. Let $C = R^*R$ where $R = [R_1|\cdots|R_r]$, with $R_j \in \mathcal{M}_{n,m}$. Then $C_{ii} = R_i^* R_i$ and $\lambda(C) = \lambda(RR^*)$. Hence, we have

$$\lambda(\sigma_1) = \lambda\left(R_1^*R_1 + \cdots + R_r^*R_r\right) \quad \text{and} \quad \lambda(\sigma_2) = \lambda\left(R_1^*R_1 + \cdots + R_r^*R_r\right).$$

Since $\lambda(R_i^*R_i) = \lambda\left(R_iR_i^* \oplus O_{(r-1)m}\right)$, the result follows from condition (2) in Theorem 2.5.

"(b) $\Leftrightarrow$ (c)" follows from "(2) $\Leftrightarrow$ (3)" in Theorem 2.5. ∎

Applying Theorem 2.5, we have the following algorithm for finding the minimal rank r of matrices in $\mathcal{S}(\sigma_1, \sigma_2)$.

ALGORITHM 2.9. Suppose $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$, with $n \ge m$. Let

$$\lambda(\sigma_1) = (a_1, \ldots, a_m) \quad \text{and} \quad \lambda(\sigma_2) = (b_1, \ldots, b_n).$$

Step 1. If $b_i = a_i$ for $1 \le i \le m$, then $r = 1$. If not, $r > 1$, then go to step 2.

Step 2. For $r > 1$, suppose the previous steps shows that the minimal rank is $\ge r$. Let

$$P_r(a_1, \ldots, a_m) = \left\{ \mathbf{c} \in \mathbb{R}^m_\downarrow : c_1, \ldots, c_m \ge 0,\ \sum_{j=1}^m c_j = 1/r, \right.$$
$$\left. \sum_{i \in J_0} a_i \le \sum_{j=1}^r \sum_{i \in J_j} c_i \ \text{ for all } (J_0, J_1, \ldots, J_r) \in LR(m, r) \right\},$$

$$Q_r(b_1, \ldots, b_n) = \left\{ \mathbf{c} \in \mathbb{R}^m_\downarrow : c_1, \ldots, c_m \ge 0,\ \sum_{j=1}^m c_j = 1/r, \right.$$
$$\left. \sum_{i \in J_0} b_i \le \sum_{j=1}^r \sum_{i \in J_j} \hat{c}_i \ \text{ for all } (J_0, J_1, \ldots, J_r) \in LR(n, r) \right\},$$

where in $Q_r(b_1, \ldots, b_n)$, $\hat{c}_i = c_i$ for $1 \le i \le m$ and $\hat{c}_i = 0$ for $m+1 \le i \le n$.

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If $S = P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n)$ is non-empty, then the minimal rank of ρ in $\mathcal{S}(\sigma_1, \sigma_2)$ is r.

Otherwise, the minimal rank is larger than r and we have to repeat Step 2 with r increased by 1.

By Theorem 2.1, Algorithm 2.9 will terminate for some

$$r \leq \max\{\mathrm{rank}\,(\sigma_1), \mathrm{rank}\,(\sigma_2)\}.$$

In fact, we can focus on $\sigma_1 \in \mathcal{D}_m$ and $\sigma_2 \in \mathcal{D}_n$ with rank m and n, respectively.

The set $P_r(a_1, \ldots, a_m)$ and $Q_r(b_1, \ldots, b_n)$ are polyhedral. There are standard linear programming packages for checking whether $S = P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n)$ is empty. Actually, we have the following.

Proposition 2.10. *Let $S = P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n)$ be defined as in Algorithm 2.9. Then the set S is non-empty if and only if $a_1, \ldots, a_m, b_1, \ldots, b_n$ satisfy a finite set of linear inequalities.*

Proof. Because $(a_1/r, \ldots, a_m/r) \in P_r(a_1, \ldots, a_m)$, and $P_r(a_1, \ldots, a_m)$ is governed by a finite set of inequalities, it is a non-empty polyhedron. Thus, there are finitely many extreme points expressed as linear combinations of $a_1, \ldots, a_m$. Now, $Q_r(b_1, \ldots, b_n)$ is determined a finite set of inequalities, say, $v_j^t x \leq \beta_j$ for $j = 1, \ldots, N$, where $v_1, \ldots, v_N \in \mathbb{R}^n$ with entries in $\{b_1, \ldots, b_n\}$ and $\beta_1, \ldots, \beta_N \in \mathbb{R}$. If all the extreme points of $P_r(a_1, \ldots, a_m)$ lies in the complement of the half space defined by $v_1^t x \leq \beta_1$, then S is empty. Otherwise, the intersection of $P_r(a_1, \ldots, a_m)$ and the half space defined by $v_1^t x \leq \beta_1$ is a non-empty polytope, and has a finite number of extreme points expressed as linear combinations of $a_1, \ldots, a_m, b_1, \ldots, b_n$. We can repeat the argument to this new polytope and the half space $v_2^t x \leq \beta_2$. We may conclude either the set and the half space has empty intersection or non-empty intersection. Repeating the process, we get a finite set of inequalities involving $a_1, \ldots, a_m, b_1, \ldots, b_n$, such that any one of them being violated will imply that $S = \emptyset$, and $S \neq \emptyset$ if all the inequalities are satisfied. ∎

By the above proposition, one can determine whether

$$S = P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n) \neq \emptyset$$

by checking a finite set of inequalities in terms of $a_1, \ldots, a_m, b_1, \ldots, b_n$. Using this result, one may determine the set

$$\mathcal{S}_r(m : \sigma_2) = \{\sigma \in \mathcal{D}_m : \text{ there is } \rho \in \mathcal{S}(\sigma, \sigma_2) \text{ with rank at most } r\}$$

for a given $\sigma_2 \in \mathcal{D}_n$; and

$$\mathcal{S}_r(\sigma_1 : n) = \{\sigma \in \mathcal{D}_n : \text{ there is } \rho \in \mathcal{S}(\sigma_1, \sigma) \text{ with rank at most } r\}$$

for a given $\sigma_1 \in \mathcal{D}_m$. We have the following.

Proposition 2.11. *Suppose $\sigma_2 \in \mathcal{D}_n$ has eigenvalues $b_1 \geq \cdots \geq b_n$. Then $\sigma \in S_r(m : \sigma_2)$ if and only if σ has eigenvalues $a_1 \geq \cdots \geq a_m$ such that $P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n) \neq \emptyset$.*

Suppose $\sigma_1 \in \mathcal{D}_m$ has eigenvalues $a_1 \geq \cdots \geq a_m$. Then $\sigma \in S_r(\sigma_1 : n)$ if and only if σ has eigenvalues $b_1 \geq \cdots \geq b_m$ such that $P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n) \neq \emptyset$.

Although one can determine whether the set $S = P_r(a_1, \ldots, a_m) \cap Q_r(b_1, \ldots, b_n)$ is non-empty by checking a finite set of inequalities, the number of inequalities involved may be very large. For low dimension

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case, the inequalities may be reduced to a smaller set after the redundant inequalities are removed. We illustrate this in the following proposition with a direct proof. It would be nice if one can give a description of non-redundant inequalities governing the eigenvalues of the reduced states of a bipartite state with prescribed rank.

Given $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we say that $\mathbf{a}$ is majorized by $\mathbf{b}$, denoted by $\mathbf{a} \prec \mathbf{b}$, if for $1 \leq k \leq n-1$ the sum of the k largest components of $\mathbf{a}$ is less than or equal to that of $\mathbf{b}$. By Horn's result [3], $\mathbf{a}$ is the diagonal of some $B \in H_n$ with eigenvalues $b_1, \ldots, b_n$ if and only if $\mathbf{a} \prec \mathbf{b}$.

Proposition 2.12. *Suppose $\sigma_1 \in \mathcal{D}_3$ has eigenvalues $a_1 \geq a_2 \geq a_3$ and $\sigma_2 \in \mathcal{D}_6$ has eigenvalues $b_1 \geq \cdots \geq b_6$. Then $\sigma_1 \in \mathcal{S}_2(3 : \sigma_2)$ if and only if a_1, a_2, a_3 satisfying $\sum_{i=1}^3 a_i = \sum_{j=1}^6 b_j$ and the following inequalities:*

$$b_3 + b_6,\ b_4 + b_5 \ \leq\ a_1 \ \leq\ b_1 + b_2,$$

$$\frac{b_3 + b_4 + b_5 + b_6}{2} \ \leq\ a_2 \ \leq\ \frac{b_1 + b_2 + b_3 + b_4}{2}\,, \tag{2.6}$$

$$b_5 + b_6 \ \leq\ a_3 \ \leq\ b_1 + b_4,\ b_2 + b_3.$$

Proof. Suppose $\sigma_2 \in \mathcal{D}_6$ has eigenvalues $b_1 \geq \cdots \geq b_6$ and $\sigma_1 \in \mathcal{S}_2(3 : \sigma_2)$ has eigenvalues $a_1 \geq a_2 \geq a_3$. Then by Corollary 2.8, there exists a unitary matrix $U \in M_6$ such that $U^*\sigma_2 U = (C_{ij})_{1\leq i,j\leq 2}$ with

$$\lambda(C_{11}) = \lambda(C_{22}) = (c_1, c_2, c_3) \quad \text{and} \quad \lambda(C_{11} + C_{22}) = (a_1, a_2, a_3).$$

Then there exist $c_1 \geq c_2 \geq c_3$ and 3×3 unitary matrices V_1 and V_2 such that $\operatorname{diag}(V_i^* C_{ii} V_i) = (c_1, c_2, c_3)$ for $i = 1,\ 2$. Hence, $(c_1, c_1, c_2, c_2, c_3, c_3) \prec (b_1, \ldots, b_6)$ [3], and we have:

1. $a_1 \leq 2c_1 \leq b_1 + b_2$.
2. $a_2 \leq c_1 + c_2 \leq \dfrac{b_1 + b_2 + b_3 + b_4}{2}$.
3. $a_3 \leq 2c_2 = (b_1+b_2+b_3+b_4+b_5+b_6) - 2(c_1+c_3) \leq (b_1+b_2+b_3+b_4+b_5+b_6) - (b_2+b_3+b_5+b_6) = b_1+b_4$, and $(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - (b_1 + b_4 + b_5 + b_6) = b_2 + b_3$.

Here, $2(c_1 + c_3) \geq b_2 + b_3 + b_5 + b_6,\ b_1 + b_4 + b_5 + b_6$ follows from the fact that [2]

$$(\{2,3,6,7\},\ \{1,3,4,5\},\ \{1,3,4,5\}),\ (\{1,4,5,6\},\ \{1,3,4,5\},\ \{1,3,4,5\}) \in LR(6,2)\,.$$

The other inequalities can be deduced by looking at $2\mu I_3 - \operatorname{diag}(a_1, a_2, a_3)$ and $\mu I_6 - \operatorname{diag}(b_1, \ldots, b_6)$, where $\mu = (b_1 + \cdots + b_6)/6$.

Given $b_1 \geq \cdots \geq b_6$, let S be the set of (a_1, a_2, a_3) satisfying $a_1 \geq a_2 \geq a_3$, $\sum_{i=1}^3 a_i = \sum_{i=1}^6 b_i = 6\mu$ and (2.6). Then S is a convex polyhedron in $\mathbb{R}^3$. If S is non-empty, we can choose an extreme point (a_1, a_2, a_3) of S. Therefore, among the inequalities $a_1 \geq a_2 \geq a_3$ and (2.6), at least two equalities hold. If $a_1 = a_2 = a_3 = 2\mu$, then from (2.6) we have

$$b_3 + b_6,\ b_4 + b_5 \ \leq\ 2\mu \ \leq\ b_1 + b_4,\ b_2 + b_3,$$

$$b_5 \ \leq\ 2\mu - b_4 \ \leq\ b_1 \quad \text{and} \quad b_6 \ \leq\ 2\mu - b_3 \ \leq\ b_2.$$

Thus, $\operatorname{diag}(b_1, b_5)$ is unitarily similar to B_1 with diagonal entries $2\mu - b_4, b$ and $\operatorname{diag}(b_2, b_6)$ is unitarily similar to B_2 with diagonal entries $2\mu - b_3, c$; see [3]. Thus, $B = \operatorname{diag}(b_1, \ldots, b_6)$ is unitarily similar

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to $\mathrm{diag}(b_3, b_4) \oplus B_1 \oplus B_2$. There exists a permutation matrix P such that $P^*BP = (D_{ij})$ with $D_{11} = \mathrm{diag}(b_3, 2\mu - b_4, b)$ and $D_{22} = \mathrm{diag}(2\mu - b_3, b_4, c)$. By the trace condition, we see that $b + c = 2\mu$. The result follows from Corollary 2.8.

Suppose either $a_1 > a_2$ or $a_2 > a_3$. Then at least one of the equalities in (2.6) holds. Consider the following cases:

1. $a_1 = b_1 + b_2$. Then we have $\dfrac{b_3 + b_4 + b_5 + b_6}{2} \le a_2 = (b_3 + b_4 + b_5 + b_6) - a_3 \le b_3 + b_4$. Therefore, $(a_2, a_3) \prec (b_3 + b_4, b_5 + b_6)$. So there exists a 2×2 unitary matrix U_1 such that $U_1^*\mathrm{diag}(b_3 + b_4, b_5 + b_6)U_1 = (a_2, a_3)$. Let $U_2 = U_1\mathrm{diag}(1, -1)$. Let
$$B_1 = \frac{1}{2}\begin{bmatrix} b_3 + b_4 & b_3 - b_4 \\ b_3 - b_4 & b_3 + b_4 \end{bmatrix} \quad \text{and} \quad B_2 = \frac{1}{2}\begin{bmatrix} b_5 + b_6 & b_5 - b_6 \\ b_5 - b_6 & b_5 + b_6 \end{bmatrix}.$$
Then $B = \mathrm{diag}(b_1, b_2) \oplus B_1 \oplus B_2$ has eigenvalues $b_1, \ldots, b_6$. There exists a permutation matrix P such that $P^*BP = (D_{ij})$ with
$$D_{11} = \frac{1}{2}\mathrm{diag}(2b_1, b_3 + b_4, b_5 + b_6) \quad \text{and} \quad D_{22} = \frac{1}{2}\mathrm{diag}(2b_2, b_3 + b_4, b_5 + b_6).$$
Let $U = P\,([1] \oplus U_1 \oplus [1] \oplus U_2)$. Then U^*BU will satisfy condition (2) in Corollary 2.8.
2. $a_2 = \dfrac{b_1 + b_2 + b_3 + b_4}{2}$. Then $a_1 + a_2 \ge 2a_2 = b_1 + b_2 + b_3 + b_4 \Rightarrow a_3 \le b_5 + b_6$. Therefore, $a_3 = b_5 + b_6$ and $(a_1, a_2) \prec (b_1 + b_2, b_3 + b_4)$. Thus, the result follows as in the previous case.
3. $a_3 = b_1 + b_4$. Then
$$\begin{aligned} & (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) = a_1 + a_2 + a_3 \ge 3a_3 \\ & = 3(b_1 + b_4) \ge (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) \\ \Rightarrow\quad & a_1 = a_2 = a_3 = b_1 + b_4 = b_2 + b_5 = b_3 + b_6 \\ \Rightarrow\quad & C = \mathrm{diag}(b_1, \ldots, b_6) \text{ will satisfy (2) in Corollary 2.8} \end{aligned}.$$
4. $a_3 = b_2 + b_3$. Then $a_1 + a_2 = b_1 + b_4 + b_5 + b_6$ and $a_2 \ge a_3 \ge b_5 + b_6$. Therefore, $(a_1, a_2) \prec (b_1 + b_4, b_5 + b_6)$. Thus, the result follows as in the Case 2.

The proof for the other equalities are similar. □

Note that the same set of inequalities (2.6) will determine whether $\sigma \in \mathcal{D}_6$ with eigenvalues $b_1 \ge \cdots \ge b_6$ lying in $\mathcal{S}_2(\sigma_1 : 6)$ for a given $\sigma_1 \in \mathcal{D}_3$ with eigenvalues $a_1 \ge a_2 \ge a_3$.

In case $a_3 = 0$, then $b_5 = b_6 = 0$, and the set of inequalities reduce to:
$$(b_3 + b_4)/2 \le a_2 \quad \text{and} \quad a_1 \le b_1 + b_2.$$
These inequalities will determine $\sigma \in \mathcal{S}_2(\sigma_1 : 4)$ with eigenvalues $b_1 \ge \cdots \ge b_4$ for a given $\sigma_1 \in \mathcal{D}_2$ with eigenvalues $a_1 \ge a_2$. The same set of inequalities will also determine $\sigma \in \mathcal{S}_2(2 : \sigma_2)$ with eigenvalues $a_1 \ge a_2$ for a given $\sigma_2 \in \mathcal{D}_4$ with eigenvalues $b_1 \ge \cdots \ge b_4$.

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Note that $\mathbf{a}$ satisfies (2.6) if and only if $(c_1, c_2, c_3) \prec \mathbf{a} \prec (b_1 + b_2, b_3 + b_4, b_5 + b_6)$, where

$$\mathbf{c} = \begin{cases} \left(b_4 + b_5, \dfrac{b_1 + b_2 + b_3 + b_6}{2}, \dfrac{b_1 + b_2 + b_3 + b_6}{2}\right) & \text{if } \dfrac{1}{3} \le b_4 + b_5, \\[2ex] \left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) & \text{if } b_4 + b_5 \le \dfrac{1}{3} \le b_2 + b_3, \\[2ex] \left(\dfrac{b_1 + b_4 + b_5 + b_6}{2}, \dfrac{b_1 + b_4 + b_5 + b_6}{2}, b_2 + b_3\right) & \text{if } b_2 + b_3 \le \dfrac{1}{3}\,. \end{cases}$$

3. Quantum channels. Recall that quantum channels are completely positive linear maps $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ that admit the operators sum representation

$$\phi(A) = \sum_{j=1}^{r} F_j A F_j^*, \tag{3.7}$$

for some $n \times m$ matrices $F_1, \ldots, F_j$ such that $\sum_{j=1}^{r} F_j^* F_j = I_m$; see [1, 6]. By the result in [1], Φ is a quantum channel if and only if the Choi matrix $C(\Phi) = (\Phi(E_{ij})) \in \mathcal{M}_m(\mathcal{M}_n)$ is positive semidefinite and $\operatorname{tr} \Phi(E_{ij}) = \delta_{ij}$. Thus, the set of quantum channels can be identified with the set

$$\begin{aligned} QC(m,n) &= \{P = (P_{ij}) \in \mathcal{M}_m(\mathcal{M}_n) : P \text{ is positive semidefinite}, (\operatorname{tr}(P_{ij})) = I_m\} \\ &= \{m\rho \in \mathcal{D}_{mn} : \operatorname{tr}_2(\rho) = I_m/m\}\,. \end{aligned}$$

Consequently, the set of quantum channels $\Phi : M_m \to M_n$ satisfying $\Phi(I_m/m) = \rho_2 \in \mathcal{D}_n$ can be identified with $\mathcal{S}(I_m/m, \rho_2)$. In particular, $\mathcal{S}(I_n/n, I_n/n)$ can be identified with the set of unital quantum channels from M_n to M_n.

For a quantum channel Φ, its Choi rank is defined as the rank of its Choi matrix $C(\Phi)$. Moreover, it is known that Φ has Choi rank r if and only if r is the minimum number of matrices $F_1, \ldots, F_r$ required in the operator sum representation of Φ. By Theorem 2.3, we have the following.

Proposition 3.1. *There is $\rho \in \mathcal{S}(I_n/n, I_n/n)$ of rank k if and only if $1 \le k \le n^2$. Equivalently, there is a unital quantum channel with Choi rank k if and only if $1 \le k \le n^2$.*

By the result in the previous section, we have the following.

Proposition 3.2. *Let $\rho_2 \in \mathcal{D}_n$. There is a quantum channel $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ with Choi rank k and $\Phi(I_m/m) = \rho_2$ if and only if there is a rank k element in $\mathcal{S}(I_m/m, \rho_2)$. As a result, the value k can be any value between the minimum value r determined by Algorithm 2.9 and the maximum value* $\operatorname{rank}(\rho) \cdot m$.

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