Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

ON THE MAXIMAL NUMERICAL RANGE OF SOME MATRICES*

ALI N. HAMED† AND ILYA M. SPITKOVSKY‡

Abstract. The maximal numerical range $W_0(A)$ of a matrix A is the (regular) numerical range $W(B)$ of its compression B onto the eigenspace $\mathcal{L}$ of A^*A corresponding to its maximal eigenvalue. So, always $W_0(A) \subseteq W(A)$. Conditions under which $W_0(A)$ has a non-empty intersection with the boundary of $W(A)$ are established, in particular, when $W_0(A) = W(A)$. The set $W_0(A)$ is also described explicitly for matrices unitarily similar to direct sums of 2-by-2 blocks, and some insight into the behavior of $W_0(A)$ is provided when $\mathcal{L}$ has codimension one.

Key words. Numerical range, Maximal numerical range, Normaloid matrices.

AMS subject classifications. 15A60, 15A57.

1. Introduction. Let $\mathbb{C}^n$ stand for the standard n-dimensional inner product space over the complex field $\mathbb{C}$. Also, denote by $\mathbb{C}^{m\times n}$ the set (algebra, if $m = n$) of all m-by-n matrices with the entries in $\mathbb{C}$. We will think of $A \in \mathbb{C}^{n\times n}$ as a linear operator acting on $\mathbb{C}^n$.

The *numerical range* (also known as the *field of values*, or the *Hausdorff set*) of such A is defined as

$$W(A) := \{x^*Ax\colon x^*x = 1,\ x \in \mathbb{C}^n\}.$$

It is well known that $W(A)$ is a convex compact subset of $\mathbb{C}$ invariant under unitary similarities of A; see e.g. [6] for this and other properties of $W(A)$ needed in what follows.

The notion of a *maximal* numerical range $W_0(A)$ was introduced in [14] in a general setting of A being a bounded linear operator acting on a Hilbert space $\mathcal{H}$. In the case we are interested in, $W_0(A)$ can be defined simply as the (usual) numerical range of the compression B of A onto the eigenspace $\mathcal{L}$ of A^*A corresponding to its largest eigenvalue:

$$W_0(A) = \{x^*Ax\colon x^*x = 1,\ x \in \mathcal{L}\}. \tag{1.1}$$

Consequently, $W_0(A)$ is a convex closed subset of $W(A)$, invariant under unitary similarities of A. Moreover, for A unitarily similar to a direct sum of several blocks A_j:

$$W_0(A) = \operatorname{conv}\{W_0(A_j)\colon j \text{ such that } \|A_j\| = \|A\|\}; \tag{1.2}$$

here and below, we are using a standard notation conv X for the convex hull of the set X.

*Received by the editors on April 2, 2018. Accepted for publication on May 17, 2018. Handling Editor: Tin-Yau Tam. Corresponding Author: Ilya M. Spitkovsky.

†Division of Science and Mathematics, New York University Abu Dhabi (NYUAD), Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates (anh334@nyu.edu). The results are partially based on the Capstone project of the first named author under the supervision of the second named author.

‡Division of Science and Mathematics, New York University Abu Dhabi (NYUAD), Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates (ims2@nyu.edu, imspitkovsky@gmail.com). Supported in part by Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

In the finite dimensional setting, property (1.2) is rather trivial; the infinite dimensional version is in [7, Lemma 2].

Since $W_0(A) \subseteq W(A)$, two natural questions arise: (i) whether $W_0(A)$ intersects with the boundary $\partial W(A)$ of $W(A)$ or lies completely in its interior, and (ii) more specifically, for which A do the two sets coincide. We deal with these questions in Section 2. These results are illustrated in Section 3 by the case $n = 2$ in which a compete description of $W_0(A)$ is readily accessible. With the use of (1.2), we then (in Section 4) tackle the case of marices A unitarily reducible to 2-by-2 blocks. The last Section 5 is devoted to matrices with the norm attained on a hyperplane.

2. Position within the numerical range. In order to state the main result of this section, we need to introduce some additional notation and terminology. The *numerical radius* $w(A)$ of A is defined by the formula

$$w(A) = \max\{|z| : z \in W(A)\}.$$

The Cauchy-Schwarz inequality implies that $w(A) \le \|A\|$, and the equality $w(A) = \|A\|$ holds if and only if there is an eigenvalue λ of A with $|\lambda| = \|A\|$, i.e., the norm of A coincides with its spectral radius $\rho(A)$. If this is the case, A is called *normaloid* . In other words, A is normaloid if

$$\Lambda(A) := \{\lambda \in \sigma(A) \colon\ |\lambda| = \|A\|\}\ (= \{\lambda \in W(A) \colon\ |\lambda| = \|A\|\}) \neq \emptyset.$$

THEOREM 2.1. *Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

(i) *A is normaloid;*
(ii) *$W_0(A) \cap \partial W(A) \neq \emptyset$;*
(iii) *$\{\lambda \in W_0(A) \colon\ |\lambda| = \|A\|\} \neq \emptyset$.*

Proof. (i) $\Rightarrow$ (iii). As was shown in [5], $\rho(A) = \|A\|$ if and only if A is unitarily similar to a direct sum $cU \oplus B$, where U is unitary, c is a positive constant and the block B (which may or may not be actually present) is such that $\rho(B) < c$ and $\|B\| \le c$.

For such A we have $\|A\| = \rho(A) = c$, and according to (1.2):

$$W_0(A) = \begin{cases} W(cU) = \operatorname{conv}\Lambda(A) & \text{if } \|B\| < c, \\ \operatorname{conv}(\Lambda(A) \cup W_0(B)) & \text{otherwise.} \end{cases} \tag{2.1}$$

Either way, $W_0(A) \supset \Lambda(A)$.

(iii) $\Rightarrow$ (ii). Since $w(A) \le \|A\|$, the points of $W(A)$ (in particular, $W_0(A)$) having absolute value $\|A\|$ automatically belong to $\partial W(A)$.

(ii) $\Rightarrow$ (i). Considering $A/\|A\|$ in place of A itself, we may without loss of generality suppose that $\|A\| = 1$. Pick a point $a \in W_0(A) \cap \partial W(A)$. By definition of $W_0(A)$, there exists a unit vector $x \in \mathbb{C}^n$ for which $\|Ax\| = 1$ and $x^*Ax = a$. Choose also a unit vector y orthogonal to x, requiring in addition that $y \in \operatorname{Span}\{x, Ax\}$ if x is not an eigenvector of A. Let C be the compression of A onto the 2-dimensional subspace $\operatorname{Span}\{x, y\}$. The matrix $A_0 := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of C with respect to the orthonormal basis $\{x, y\}$ then satisfies $|a|^2 + |c|^2 = 1$. From here:

$$A_0^*A_0 = \begin{bmatrix} 1 & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & |b|^2 + |d|^2 \end{bmatrix}. \tag{2.2}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

But $\|A_0\| \leq \|A\| = 1$. Comparing this with (2.2), we conclude that

$$\overline{a}b + \overline{c}d = 0. \tag{2.3}$$

Moreover, $W(A_0) \subset W(A)$, and so $a \in \partial W(A_0)$. This implies $|b| = |c|$, as was stated explicitly in [16, Corollary 4] (see also [4, Proposition 4.3]). Therefore, (2.3) is only possible if $b = c = 0$ or $|a| = |d|$.

In the former case, $|a| = 1$, immediately implying $w(A) = 1 = \|A\|$.

In the latter case, some additional reasoning is needed. Namely, then $|b|^2 + |d|^2 = |c|^2 + |a|^2 = 1$ which in combination with (2.3) means that A_0 is unitary. Since $W(A) \supset \sigma(A_0)$, we see that $w(A) \geq 1$. On the other hand, $w(A) \leq \|A\| = 1$, and so again $w(A) = \|A\|$. □

Note that Theorem 2.1 actually holds in the infinite-dimensional setting. For this (more general) situation, the equivalence *(i)* ⇔ *(iii)* was established in [2, Corollary 1], while *(i)* ⇔ *(ii)* is from [13]. Moreover, the paper [2] prompted [13]. The proof in the finite-dimensional case is naturally somewhat simpler, and we provide it here for the sake of completeness.

If the matrix B introduced in the proof of Theorem 2.1 is itself normaloid, then $\|B\| < c$ and $W_0(A)$ is given by the first line of (2.1). This is true in particular for normal A, when B is also normal. On the other hand, if $\|B\| = c$, then Theorem 2.1 (applied to B) implies that $W_0(B)$ lies strictly in the interior of $W(B)$. In particular, there are no points in $W(B)$ with absolute value $c(= \|A\|)$. From here we immediately obtain

Corollary 2.2. *For any $A \in \mathbb{C}^{n\times n}$,*

$$\{\lambda \in W_0(A) \colon |\lambda| = \|A\|\} = \Lambda(A).$$

This is a slight refinement of condition (ii) in Theorem 2.1.

Theorem 2.3. *Given a matrix $A \in \mathbb{C}^{n\times n}$, its numerical range and maximal numerical range coincide if and only if A is unitarily similar to a direct sum $cU \oplus B$ where U is unitary, $c > 0$, and $W(B) \subseteq cW(U)$.*

Proof. Sufficiency. Under the condition imposed on B, $W(A) = cW(U)$. At the same time, $W_0(A) \supseteq W_0(cU) = cW(U)$.

Necessity. If $W(A) = W_0(A)$, then in particular $W_0(A)$ has to intersect $\partial W(A)$, and by Theorem 2.1 A is normaloid. As such, A is unitarily similar to $cU \oplus B$ with $\|B\| \leq c$, $\rho(B) < c$. It was observed in the proof of Theorem 2.1 that, if B itself is normaloid, then $W_0(A) = cW(U)$, and so we must have $W(A) = cW(U)$, implying $W(B) \subseteq cW(U)$.

Consider now the case when B is not normaloid. If $W(B) \subseteq cW(U)$ does not hold, draw a support line ℓ of $W(B)$ such that $cW(U)$ lies to the same side of it as $W(B)$ but at a positive distance from it. Since $W(A) = \operatorname{conv}(cW(U) \cup W(B))$, ℓ is also a support line of $W(A)$. Meanwhile $W_0(B)$ is contained in the interior of $W(B)$, making the distance between $W_0(B)$ and ℓ positive, and implying that ℓ is not a support line of $\operatorname{conv}(cW(U) \cup W_0(B))$. According to (2.1), the latter set is the same as $W_0(A)$. Thus, $W_0(A) \neq W(A)$, which is a contradiction. □

3. 2-by-2 matrices. A 2-by-2 matrix A is normaloid if and only if it is normal. The situation is then rather trivial: denoting $\sigma(A) = \{\lambda_1, \lambda_2\}$ with $|\lambda_1| \leq |\lambda_2|$, we have $W(A) = [\lambda_1, \lambda_2]$ (the line segment

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

connecting λ_1 with λ_2), and

$$W_0(A) = \begin{cases} \{\lambda_2\} & \neq W(A) \quad \text{if } |\lambda_1| < |\lambda_2|, \\ [\lambda_1, \lambda_2] & = W(A) \quad \text{otherwise.} \end{cases}$$

So, the only interesting case is that of a non-normal A. The eigenvalues of A^*A are then simple, and $W_0(A)$ is therefore a point. According to Theorem 2.1, this point lies inside the ellipse $W(A)$. Our next result is the formula for its exact location.

THEOREM 3.1. *Let $A \in \mathbb{C}^{2\times 2}$ be not normal but otherwise arbitrary. Then $W_0(A) = \{z_0\}$, where*

$$z_0 = \frac{(t_0 - |\lambda_2|^2)\lambda_1 + (t_0 - |\lambda_1|^2)\lambda_2}{2t_0 - \operatorname{trace}(A^*A)}, \tag{3.1}$$

λ_1, λ_2 are the eigenvalues of A, and

$$t_0 = \frac{1}{2}\left(\operatorname{trace}(A^*A) + \sqrt{(\operatorname{trace}(A^*A))^2 - 4\,|\det A|^2}\right). \tag{3.2}$$

Note that an alternative form of (3.1),

$$z_0 = \frac{t_0 \cdot \operatorname{trace} A - (\det A) \cdot \overline{\operatorname{trace} A}}{2t_0 - \operatorname{trace}(A^*A)}, \tag{3.3}$$

without λ_j explicitly present, is sometimes more convenient.

Proof. Since both the value of z_0 and the right-hand sides of formulas (3.1)–(3.3) are invariant under unitary similarities, it suffices to consider A in the upper triangular form

$$A = \begin{bmatrix} \lambda_1 & c \\ 0 & \lambda_2 \end{bmatrix}, \quad c > 0.$$

Then

$$A^*A - tI = \begin{bmatrix} |\lambda_1|^2 - t & c\overline{\lambda_1} \\ c\lambda_1 & c^2 + |\lambda_2|^2 - t \end{bmatrix},$$

so the maximal eigenvalue t_0 of A^*A satisfies

$$c^2\,|\lambda_1|^2 = (t_0 - |\lambda_1|^2)(t_0 - |\lambda_2|^2 - c^2) \tag{3.4}$$

and is thus given by formula (3.2). Choosing a respective eigenvector as

$$x = \left[c\overline{\lambda_1},\ t_0 - |\lambda_1|^2\right]^T,$$

we obtain successively

$$\|x\|^2 = c^2\,|\lambda_1|^2 + (t_0 - |\lambda_1|^2)^2,$$

$$Ax = \left[ct_0,\ (t_0 - |\lambda_1|^2)\lambda_2\right]^T,$$

$$x^*Ax = c^2 t_0 \lambda_1 + (t_0 - |\lambda_1|^2)^2 \lambda_2,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

and so finally,

$$z_0 = \frac{x^*Ax}{\|x\|^2} = \frac{c^2 t_0\lambda_1 + (t_0 - |\lambda_1|^2)^2\lambda_2}{c^2|\lambda_1|^2 + (t_0 - |\lambda_1|^2)^2}. \tag{3.5}$$

To put this expression for z_0 in form (3.1), we proceed as follows. Due to (3.4), the denominator in the right-hand side of (3.5) is nothing but

$$(t_0 - |\lambda_1|^2)\left((t_0 - |\lambda_2|^2 - c^2) + (t_0 - |\lambda_1|^2)\right) = (t_0 - |\lambda_1|^2)(2t_0 - \operatorname{trace}(A^*A)). \tag{3.6}$$

On the other hand, also from (3.4),

$$c^2 t_0 = (t_0 - |\lambda_1|^2)(t_0 - |\lambda_2|^2),$$

and the numerator in the right-hand side of (3.5) can be rewritten as

$$(t_0 - |\lambda_1|^2)(t_0 - |\lambda_2|^2)\lambda_1 + (t_0 - |\lambda_1|^2)\lambda_2 = (t_0 - |\lambda_1|^2)\left((t_0 - |\lambda_2|^2)\lambda_1 + (t_0 - |\lambda_1|^2)\lambda_2\right). \tag{3.7}$$

It remains to divide (3.7) by (3.6). ∎

To interpret formula (3.1) geometrically, let us rewrite it as

$$z_0 = t_1\lambda_1 + t_2\lambda_2,$$

where

$$t_1 = \frac{t_0 - |\lambda_2|^2}{2t_0 - \operatorname{trace}(A^*A)} \quad \text{and} \quad t_2 = \frac{t_0 - |\lambda_1|^2}{2t_0 - \operatorname{trace}(A^*A)}.$$

According to (3.2), the denominator of these formulas can be rewritten as

$$\begin{aligned} &\sqrt{(\operatorname{trace}(A^*A))^2 - 4|\det A|^2} = \sqrt{(|\lambda_1|^2 + |\lambda_2|^2 + c^2)^2 - 4|\lambda_1\lambda_2|^2} \\ &= \sqrt{(|\lambda_1|^2 - |\lambda_2|^2)^2 + 2c^2(|\lambda_1|^2 + |\lambda_2|^2) + c^4} \\ &> 0. \end{aligned}$$

Also, $t_0 \geq c^2 + \max\{|\lambda_1|^2, |\lambda_2|^2\}$, and

$$(t_0 - |\lambda_1|^2) + (t_0 - |\lambda_2|^2) = 2t_0 - \operatorname{trace}(A^*A) + c^2 > 2t_0 - \operatorname{trace}(A^*A).$$

Consequently, $t_1, t_2 > 0$ and $t_1 + t_2 > 1$, implying that in case of non-collinear λ_1, λ_2 (equivalently, $\lambda_1\overline{\lambda_2} \notin \mathbb{R}$) z_0 lies in the sector spanned by λ_1, λ_2 and is separated from the origin by the line passing through λ_1, λ_2.

If, on the other hand, λ_1 and λ_2 lie on the line passing through the origin, the point z_0 also lies on this line. More specifically, the following statement holds.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

Corollary 3.2. *Let A be a non-normal 2-by-2 matrix, with the maximal numerical range $W_0(A) = \{z_0\}$. Then the point z_0:*

(i) *is collinear with the spectrum $\sigma(A) = \{\lambda_1, \lambda_2\}$ if and only if $\lambda_1\overline{\lambda_2} \in \mathbb{R}$;*
(ii) *coincides with one of the eigenvalues of A if and only if the other one is zero;*
(iii) *lies in the open interval with the endpoints λ_1, λ_2 if and only if $\lambda_1\overline{\lambda_2} < 0$;*
(iv) *is the midpoint of the above interval if and only if* $\operatorname{trace} A = 0$*;*
(v) *lies on the line passing through λ_1 and λ_2 but outside of the interval $[\lambda_1, \lambda_2]$ if and only if $\lambda_1\overline{\lambda_2} > 0$.*

Proof. Part (i) follows from the discussion preceding the statement. When proving (ii)–(v) we may therefore suppose that $\lambda_1\overline{\lambda_2} \in \mathbb{R}$ holds. Since all the statements in question are invariant under rotations of A, without loss of generality even $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $z_0 \in \mathbb{R}$ as well. Using formula (3.5) for z_0:

$$z_0 - \lambda_2 = \frac{c^2\lambda_1(t_0 - \lambda_1\lambda_2)}{c^2\lambda_1^2 + (t_0 - \lambda_1^2)^2},$$

and so the signs of $z_0 - \lambda_2$ and λ_1 are the same. Relabeling the eigenvalues of A (which of course does not change z_0) we thus also have that the signs of $z_0 - \lambda_1$ and λ_2 are the same. Statements (ii)–(v) follow immediately. ∎

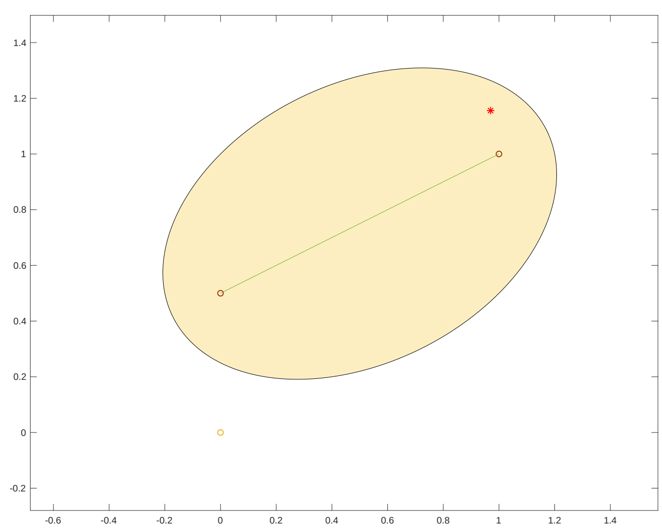

Figure 1. $A = \begin{bmatrix} 0.5i & -1 \\ 0 & 1+i \end{bmatrix}$; *$z_0$ is not collinear with the spectrum.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

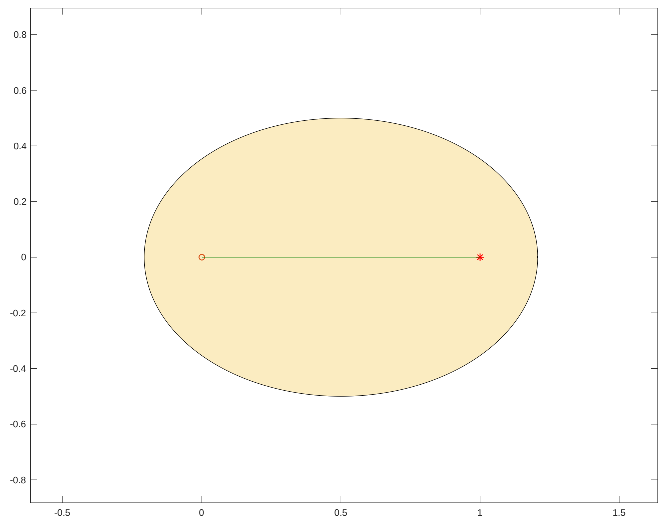

FIGURE 2. $A = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$; z_0 *coincides with one of the eigenvalues since the other is zero.*

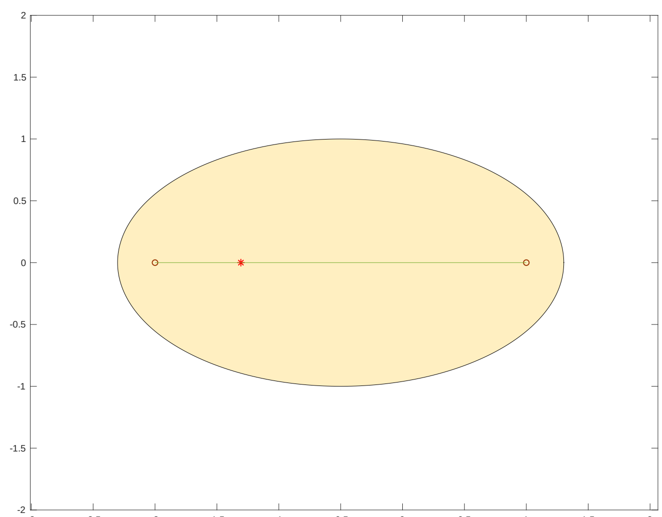

FIGURE 3. $A = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$; z_0 *is collinear with the spectrum and lies inside the interval connecting it.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

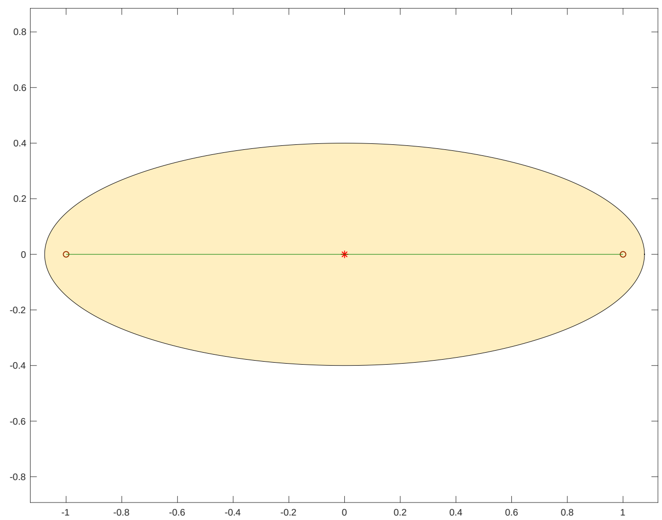

FIGURE 4. $A = \begin{bmatrix} 1 & 0.8 \\ 0 & -1 \end{bmatrix}$; z_0 *is the midpoint of the line connecting the eigenvalues.*

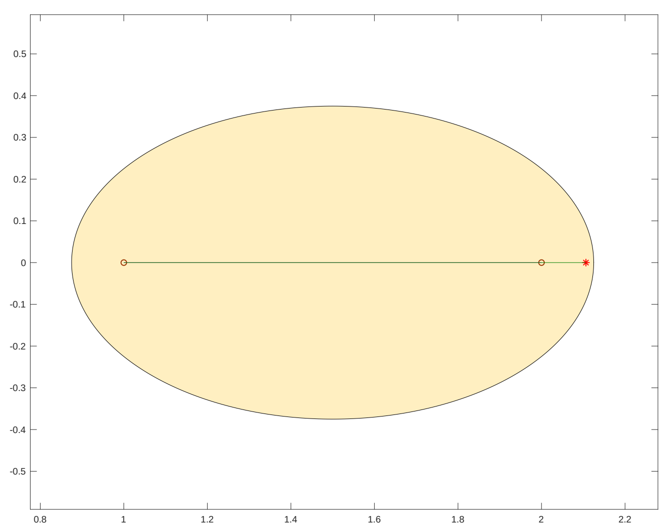

FIGURE 5. $A = \begin{bmatrix} 2 & 0.75 \\ 0 & 1 \end{bmatrix}$; z_0 *is collinear with the spectrum and lies outside the interval connecting it.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

4. Matrices decomposing into small blocks. A straightforward generalization of Theorem 3.1, based on property (1.2), is the description of $W_0(A)$ for matrices A unitarily similar to direct sums of 2-by-2 and 1-by-1 blocks.

THEOREM 4.1. *Let A be unitarily similar to*

$$\operatorname{diag}[\lambda_1, \dots, \lambda_k] \oplus A_1 \oplus \cdots \oplus A_m,$$

with $A_j \in \mathbb{C}^{2\times 2}$, $j = 1, \dots, m$. Denote

$$t_j = \frac{1}{2}\left(\operatorname{trace}(A_j^* A_j) + \sqrt{(\operatorname{trace}(A_j^* A_j))^2 - 4\left|\det A_j\right|^2}\right), \tag{4.1}$$

$$z_j = \frac{t_j \cdot \operatorname{trace} A_j - (\det A_j) \cdot \overline{\operatorname{trace} A_j}}{2t_j - \operatorname{trace}(A^* A)}, \quad j = 1, \dots, m, \tag{4.2}$$

$$t_0 = \max\{t_j, |\lambda_i|^2 : i = 1, \dots, k;\ j = 1, \dots, m\},$$

and let I (resp., J) stand for the set of all i (resp., j) for which $|\lambda_i|^2$ (resp., t_j) equals t_0. Then

$$W_0(A) = \operatorname{conv}\{\lambda_i, z_j : i \in I,\ j \in J\}. \tag{4.3}$$

According to (4.3), in the setting of Theorem 4.1, $W_0(A)$ is always a polygon.

Consider in particular A unitarily similar to

$$\begin{bmatrix} a_1 I_{n_1} & X \\ Y & a_2 I_{n_2} \end{bmatrix}, \tag{4.4}$$

with $X \in \mathbb{C}^{n_1\times n_2}$ and $Y \in \mathbb{C}^{n_2\times n_1}$ such that XY and YX are both normal. As was shown in the proof of [1, Theorem 2.1], yet another unitary similarity can be used to rewrite A as the direct sum of $\min\{n_1, n_2\}$ two-dimensional blocks

$$A_j = \begin{bmatrix} a_1 & \sigma_j \\ \delta_j & a_2 \end{bmatrix} \tag{4.5}$$

and $\max\{n_1, n_2\} - \min\{n_1, n_2\}$ one-dimensional blocks equal either a_1 or a_2.

Here σ_j are the s-numbers of X, read from the diagonal of the middle term in its singular value decomposition $X = U_1 \Sigma U_2^*$, while δ_j are the respective diagonal entries of the matrix $\Delta = U_2^* Y U_1$, which can also be made diagonal due to the conditions imposed on X, Y.

Since $\|A_j\| \geq \max\{|a_1|, |a_2|\}$, for matrices (4.4) (or unitarily similar to them) formula (4.3) implies that $W_0(A)$ is the convex hull of z_j given by (4.2) taken over those j only which deliver the maximal value of $\|A_j\|$.

Here are some particular cases in which all z_j, λ_i contributing to $W_0(A)$ happen to coincide. Then $W_0(A)$ is a singleton, as it was the case for 2-by-2 matrices A different from scalar multiples of a unitary matrix.

PROPOSITION 1. *Let, in (4.4), $a_1 = -a_2$. Then $W_0(A) = \{0\}$.*

Proof. Indeed, in this case $\operatorname{trace} A_j = 0$, and formula (4.2) implies that $z_j = 0$ for all j, in particular for those with maximal $\|A_j\|$ is attained. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

Recall that a *continuant* matrix is by definition a tridiagonal matrix $A \in \mathbb{C}^{n\times n}$ such that its off-diagonal entries satisfy the requirement

$$a_{k,k+1} = -\overline{a_{k+1,k}}, \quad k = 1, \dots, n-1.$$

Such a matrix can be written as

$$C = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ -\overline{b_1} & a_2 & b_2 & \ddots & \vdots \\ 0 & -\overline{b_2} & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & -\overline{b_{n-1}} & a_n \end{bmatrix}. \tag{4.6}$$

PROPOSITION 2. *Let C be the continuant matrix (4.6) with a 2-periodic main diagonal: $a_1 = a_3 = \dots$, $a_2 = a_4 = \dots$ Then $W_0(C)$ is a singleton.*

Proof. Let T be the matrix with the columns $e_1, e_3, \dots, e_2, e_4, \dots$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$. It is easy to see that a unitary similarity performed by T transforms the continuant matrix (4.6) with the 2-periodic main diagonal into the matrix (4.4) for which

$$X = \begin{bmatrix} b_1 & & & \\ b_2 & b_3 & & \\ & b_4 & b_5 & \\ & & \ddots & \ddots \end{bmatrix}, \quad Y = -X^*.$$

So, in (4.5) we have $\delta_j = -\sigma_j$, and thus, $\|A_j\|$ depends monotonically on σ_j. The block on which the maximal norm is attained is therefore uniquely defined (though might appear repeatedly), and the respective maximal value of σ_j is nothing but $\|X\|$. ∎

It is clear from the proof of Proposition 2 how to determine the location of $W_0(C)$: it is given by formulas (4.1), (4.2) with trace A_j, trace$(A_j^*A_j)$ and det A_j replaced by a_1+a_2, $|a_1|^2+|a_2|^2+2\,\|X\|^2$, and $a_1a_2+\|X\|^2$, respectively.

Finally, let A be *quadratic*, i.e., having the minimal polynomial of degree two. As is well known (and easy to show), A is then unitarily similar to a matrix

$$\begin{bmatrix} \lambda_1 I_{n_1} & X \\ 0 & \lambda_2 I_{n_2} \end{bmatrix}. \tag{4.7}$$

This fact was used e.g. in [15] to prove that for such matrices $W(A)$ is the same as $W(A_0)$, where $A_0 \in \mathbb{C}^{2\times 2}$ is defined as

$$A_0 = \begin{bmatrix} \lambda_1 & \|X\| \\ 0 & \lambda_2 \end{bmatrix},$$

and thus, $W(A)$ is an elliptical disk.

The next statement shows that the relation between A unitarily similar to (4.7) and A_0 persists when maximal numerical ranges are considered.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

Proposition 3. *Let $A \in \mathbb{C}^{n\times n}$ be quadratic and thus unitarily similar to (4.7). Then $W_0(A)$ is a singleton $\{z_0\}$, where*

$$z_0 = \frac{(\|A_0\|^2 - |\lambda_2|^2)\lambda_1 + (\|A_0\|^2 - |\lambda_1|^2)\lambda_2}{2\|A_0\|^2 - (|\lambda_1|^2 + |\lambda_2|^2 + \|X\|^2)}.$$

Proof. Observe that (4.7) is a particular case of (4.4) in which $Y = 0$ and $a_j = \lambda_j$, $j = 1, 2$. So, the normality of XY and YX holds in a trivial way and, moreover, $\delta_j = 0$ for all the blocks A_j appearing in the unitary reduction of A. Similarly to the situation in Proposition 2, the norms of A_j depend monotonically on σ_j, and thus, the maximum is attained on the blocks (of which there is at least one) coinciding with A_0. It remains only to invoke formula (3.1), keeping in mind that $t_0 = \|A_0\|^2$ and $\operatorname{trace}(A_0^*A_0) = |\lambda_1|^2 + |\lambda_2|^2 + \|X\|^2$. ∎

In general, however, there is no reason for the set (4.3) to be a singleton. To illustrate, let $A = A_1 \oplus A_2 \oplus A_3$, where

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1-i & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 1+i \\ 1 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} -1 & \sqrt{\frac{3+3\sqrt{6}}{5}} \\ 0 & 2 \end{bmatrix}. \tag{4.8}$$

Then $\|A_j\| = \sqrt{4+\sqrt{6}}$ for each $j = 1, 2, 3$, while $W_0(A_j) = \{z_j\}$, with

$$z_{1,2} \approx 1.93 \mp 0.20i, \quad z_3 \approx 1.45. \tag{4.9}$$

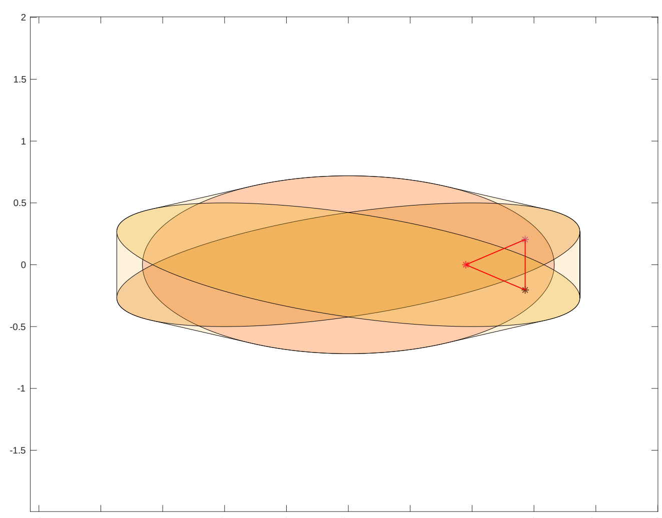

Figure 6. *A is the direct sum of A_j given by (4.8). The maximal numerical range is the triangle with the vertices z_j given by (4.9).*

5. Matrices with the norm attained on a hyperplane. Generically, the eigenvalues of A^*A are all distinct, and $W_0(A)$ is therefore a singleton. In more rigorous terms, the set of n-by-n matrices A with $W_0(A)$ being a point has the interior dense in $\mathbb{C}^{n\times n}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

An opposite extreme is the case when A^*A has just one eigenvalue. This happens if and only if A is a scalar multiple of a unitary matrix – a simple situation, covered by Theorem 2.3.

If $n = 2$, these are the only options, which is of course in agreement with the description of $W_0(A)$ provided for this case in Section 3. Starting with $n = 3$, however, the situation when the maximal eigenvalue of A^*A has multiplicity $n-1$ becomes non-trivial. We here provide some additional information about the shapes of $W(A), W_0(A)$ in this case.

The only way in which such matrices A can be unitarily reducible is if they are unitarily similar to $cU \oplus B$, with U unitary and $\|B\| = c$ attained on a subspace of codimension one. Therefore, it suffices to consider the case of unitarily irreducible A only.

To state the pertinent result, we need to recall one more notion. Namely, Γ is a *Poncelet curve* (of rank m with respect to a circle $\mathcal{C}$) if it is a closed convex curve lying inside $\mathcal{C}$ and such that for any point $z \in \mathcal{C}$ there is an m-gon inscribed in $\mathcal{C}$, circumscribed around Γ, and having z as one of its vertices.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ be unitarily irreducible, with the norm of A attained on an $(n-1)$-dimensional subspace. Then $\partial W(A)$ and $\partial W_0(A)$ both are Poncelet curves (of rank $n+1$ and n, respectively) with respect to the circle $\{z\colon\ |z| = \|A\|\}$.*

Proof. Considering $A/\|A\|$ in place of A, we may without loss of generality suppose that $\mathcal{C}$ is the unit circle $\mathbb{T}$, the matrix in question is a contraction with $\|A\| = 1$ and $\operatorname{rank}(I - A^*A) = 1$. Also, $\rho(A) < 1$ since otherwise A would be normaloid and thus unitarily reducible. In the notation of [3] (adopted in later publications), $A \in S_n$, and the result follows directly from [3, Theorem 2.1].

Moving to $W_0(A)$, consider the polar form UR of A. Since the statement in question is invariant under unitary similarities, we may suppose that the positive semi-definite factor R is diagonalized. Condition $\operatorname{rank}(I - A^*A) = 1$ then implies that $R = \operatorname{diag}[1, \dots, 1, c]$, where $0 \le c < 1$. In agreement with (1.1), $W_0(A) = W(U_0)$, where U_0 is the matrix obtained from U by deleting its last row and column. Note that U has no eigenvectors with the last coordinate equal to zero, since otherwise they would also be eigenvectors of R, implying unitary reducibility of A. In particular, the eigenvalues of U are distinct. The statement now follows by applying [11, Theorem 1] to $W(U_0)$. ◻

Note that the matrix U_0 constructed in the second part of the proof actually belongs to S_{n-1}. The properties of $W(T)$ for $T \in S_n$ stated in [3, Lemma 2.2] thus yield:

Corollary 5.2. *In the setting of Theorem 5.1, both $\partial W(A)$ and $\partial W_0(A)$ are smooth curves, with each point generated by exactly one (up to a unimodular scalar multiple) vector.*

The above mentioned uniqueness of the generating vectors implies in particular that $\partial W(A)$, $\partial W_0(A)$ contain no flat portions.

To illustrate, consider the Jordan block $J_n \in \mathbb{C}^{n\times n}$ corresponding to the zero eigenvalue. Then $J_n \in S_n$, with the norm of J_n attained on the span $\mathcal{L}$ of the elements $e_2, \dots, e_n$ of the standard basis of $\mathbb{C}^n$. Consequently, the compression of J_n onto $\mathcal{L}$ is J_{n-1}, and $W_0(J_n) = W(J_{n-1})$ is the circular disk $\{z\colon\ |z| \le \cos\frac{\pi}{n}\}$, while $W(J_n)$ is the (concentric but strictly larger) circular disk $\{z\colon\ |z| \le \cos\frac{\pi}{n+1}\}$.

Finally, let us concentrate on the smallest size for which the situation of this Section is non-trivial, namely $n = 3$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

PROPOSITION 4. *A matrix $A \in \mathbb{C}^{3\times 3}$ is unitarily irreducible with the norm attained on a 2-dimensional subspace if and only if it is unitarily similar to*

$$\omega \begin{bmatrix} \lambda_1 & \sqrt{(1-|\lambda_1|^2)(1-|\lambda_2|^2)} & -\lambda_2\sqrt{(1-|\lambda_1|^2)(1-|\lambda_3|^2)} \\ 0 & \lambda_2 & \sqrt{(1-|\lambda_2|^2)(1-|\lambda_3|^2)} \\ 0 & 0 & \lambda_3 \end{bmatrix}, \tag{5.1}$$

where $\omega \in \mathbb{C}\setminus\{0\}$, $-1 < \lambda_2 \le 0$, and $|\lambda_j| < 1$, $j = 1, 3$.

Proof. According to Schur's lemma, we can put any $A \in \mathbb{C}^{3\times 3}$ in an upper triangular form

$$A_0 = \begin{bmatrix} \lambda_1 & x & y \\ 0 & \lambda_2 & z \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{5.2}$$

Further multiplication by an appropriate non-zero complex number w allows us without loss of generality to suppose that $\|A\| = 1$ and $x\overline{y}z \ge 0$. An additional (diagonal) unitary similarity can then be used to arrange for x, y, z all to be non-negative. Being an irreducible contraction, the matrix (5.2) has to satisfy $|\lambda_j| < 1$ ($j = 1, 2, 3$) and $xz \ne 0$. Rewriting the rank-one condition for $I - A_0^*A_0$ as the collinearity of its columns and solving the resulting system of equations for x, y, z yields

$$\begin{aligned} x &= \sqrt{(1-|\lambda_1|^2)(1-|\lambda_2|^2)}, \\ y &= -\lambda_2\sqrt{(1-|\lambda_1|^2)(1-|\lambda_3|^2)}, \\ z &= \sqrt{(1-|\lambda_2|^2)(1-|\lambda_3|^2)}. \end{aligned} \tag{5.3}$$

In particular, λ_2 has to be non-positive, due to the non-negativity of y.

Setting $\omega = w^{-1}$, we arrive at representation (5.1).

A straightforward verification shows that the converse is also true, i.e., any matrix of the form (5.1) is unitarily irreducible with a norm attained on a 2-dimensional subspace. □

Note that the form (5.1) can also be established by invoking [11, Theorem 4], instead of solving for x, y, z in terms of λ_j straightforwardly.

In the setting of Proposition 4, the set $W_0(A)$ is the numerical range of a 2-by-2 matrix, and in agreement with Corollary 5.2 is an elliptical disk. By the same Corollary 5.2, $W(A)$ also cannot have flat portions on its boundary (this of course can also be established by applying the respective criteria for 3-by-3 matrices from [8, Section 3] or [12]). According to Kippenhahn's classification of the shapes of numerical ranges in the $n = 3$ case [9] (see also the English translation [10]), $W(A)$ can a priori be either an elliptical disk or an ovular figure bounded by a convex algebraic curve of degree 6. As it happens, both options materialize. The next result singles out the case in which $W(A)$ is elliptical; in all other cases it is therefore ovular.

THEOREM 5.3. *Let A be given by formula (5.1), with λ_j as described by Proposition 4. Then $W(A)$ is an elliptical disk if and only if*

$$\lambda_i = \lambda_j \frac{1-|\lambda_k|^2}{1-|\lambda_j\lambda_k|^2} + \lambda_k \frac{1-|\lambda_j|^2}{1-|\lambda_j\lambda_k|^2} \tag{5.4}$$

for some reordering (i, j, k) of the triple $(1, 2, 3)$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

Proof. According to [8, Section 2], for a unitarily irreducible matrix (5.2) to have an elliptical numerical range it is necessary and sufficient that

$$\lambda = \frac{\lambda_3 |x|^2 + \lambda_2 |y|^2 + \lambda_1 |z|^2 - x\overline{y}z}{|x|^2 + |y|^2 + |z|^2}$$

coincides with one of the eigenvalues λ_j. Plugging in the values of x, y, z from (5.3), we may rewrite λ as

$$\frac{\lambda_1(1-|\lambda_2|^2)(1-|\lambda_3|^2) + \lambda_2(1-|\lambda_1|^2)(1-|\lambda_3|^2) + \lambda_3(1-|\lambda_1|^2)(1-|\lambda_2|^2)}{2 - |\lambda_1|^2 - |\lambda_2|^2 - |\lambda_3|^2 + |\lambda_1\lambda_2\lambda_3|^2}.$$

Now it is straightforward to check that $\lambda = \lambda_i$ if and only if (5.4) holds. ∎

Proposition 4 and Theorem 5.3 both simplify greatly if A is singular.

THEOREM 5.4. *A singular 3-by-3 matrix A is unitarily irreducible with the norm attained on a 2-dimensional subspace if and only if it is unitarily similar to*

$$B = \omega \begin{bmatrix} 0 & \sqrt{1-|\lambda|^2} & -\lambda\sqrt{1-|\mu|^2} \\ 0 & \lambda & \sqrt{(1-|\lambda|^2)(1-|\mu|^2)} \\ 0 & 0 & \mu \end{bmatrix}, \tag{5.5}$$

where $\omega \neq 0$, $-1 < \lambda \leq 0$ and $|\mu| < 1$. Its numerical range $W(A)$ is an elliptical disk if and only if $\mu = \pm\lambda$, and has an ovular shape otherwise.

Note that for matrices (5.5) $\mathcal{L} = \mathrm{Span}\{e_2, e_3\}$, and so $W_0(B)$ is nothing but the numerical range of the right lower 2-by-2 block of B. The next three figures show the shape of $W_0(B)$ and $W(B)$ for B given by (5.5) with $\omega = 1$ for several choices of λ, μ.

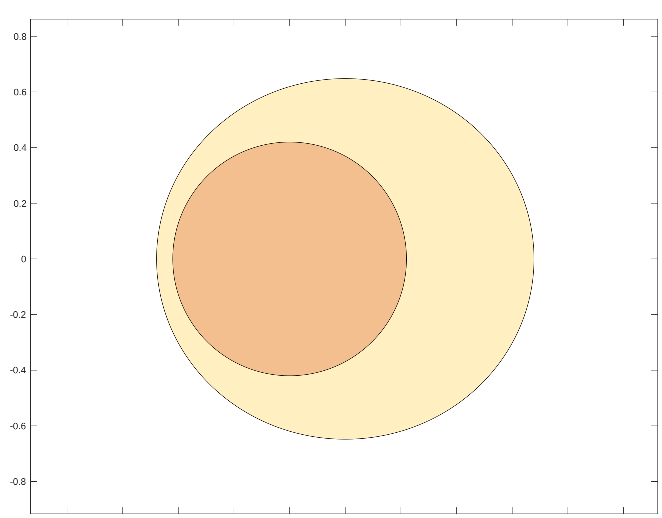

FIGURE 7. $\mu = \lambda = -2/5$. *The numerical range and maximal numerical range are both circular discs.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

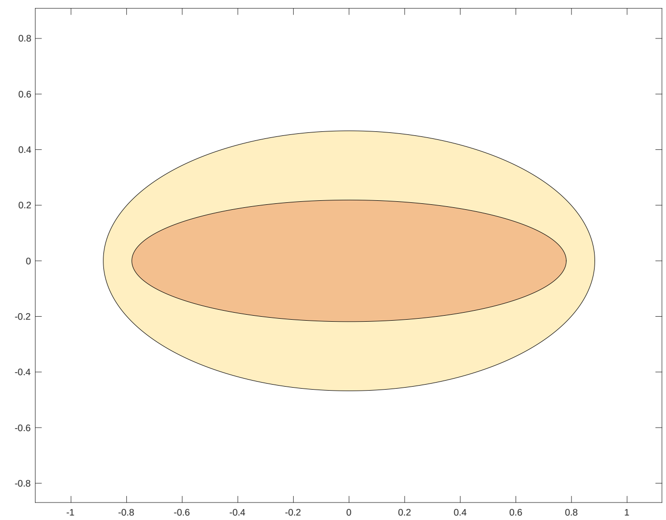

FIGURE 8. $\mu = -\lambda = 3/4$. *The numerical range and maximal numerical range are both elliptical discs.*

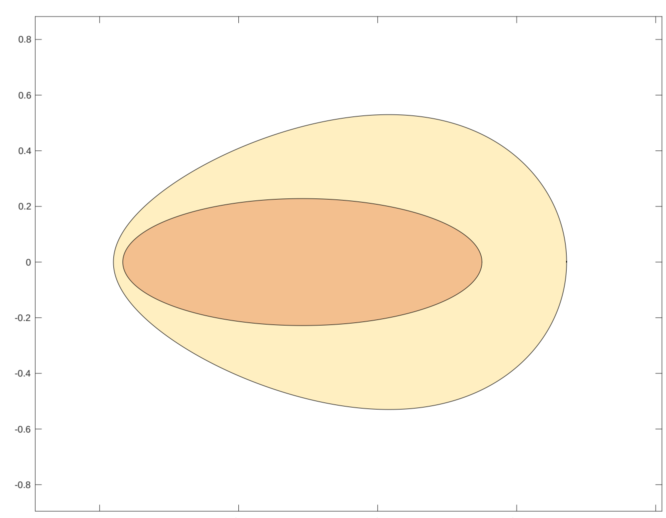

FIGURE 9. $\mu = 1/3, \lambda = -7/8$. *The numerical range is an ovular disc, and the maximal numerical range is an elliptical disc.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 288-303, June 2018.

REFERENCES

[1] E. Brown and I. Spitkovsky. On matrices with elliptical numerical ranges. *Linear Multilinear Algebra*, 52:177–193, 2004.
[2] J.-T. Chan and K. Chan. An observation about normaloid operators. *Operators and Matrices*, 11:885–890, 2017.
[3] H.-L. Gau and P.Y. Wu. Numerical range of $S(\phi)$. *Linear Multilinear Algebra*, 45(1):49–73, 1998.
[4] H.-L. Gau and P.Y. Wu. Numerical ranges and compressions of S_n-matrices. *Operators and Matrices*, 7(2):465–476, 2013.
[5] M. Goldberg and G. Zwas. On matrices having equal spectral radius and spectral norm. *Linear Algebra and its Applications*, 8:427–434, 1974.
[6] K.E. Gustafson and D.K.M. Rao. *Numerical range. The field of values of linear operators and matrices*. Springer, New York, 1997.
[7] G. Ji, N. Liu, and Z. Li. Essential numerical range and maximal numerical range of the Aluthge transform. *Linear Multilinear Algebra*, 55(4):315–322, 2007.
[8] D. Keeler, L. Rodman, and I. Spitkovsky. The numerical range of 3×3 matrices. *Linear Algebra and its Applications*, 252:115–139, 1997.
[9] R. Kippenhahn. Über den Wertevorrat einer Matrix. *Mathematische Nachrichten*, 6:193–228, 1951.
[10] R. Kippenhahn. On the numerical range of a matrix. *Linear Multilinear Algebra*, 56(1-2):185–225, 2008. Translated from the German by P.F. Zachlin and M.E. Hochstenbach.
[11] B. Mirman. Numerical ranges and Poncelet curves. *Linear Algebra and its Applications*, 281:59–85, 1998.
[12] L. Rodman and I.M. Spitkovsky. 3×3 matrices with a flat portion on the boundary of the numerical range. *Linear Algebra and its Applications*, 397:193–207, 2005.
[13] I.M. Spitkovsky. A note on the maximal numerical range. Preprint, arXiv.math.FA/1803.10516v1, 2018.
[14] J.G. Stampfli. The norm of a derivation. *Pacific Journal of Mathematics*, 33:737–747, 1970.
[15] S.-H. Tso and P.Y. Wu. Matricial ranges of quadratic operators. *Rocky Mountain Journal of Mathematics*, 29(3):1139–1152, 1999.
[16] H.-Y. Zhang, Y.-N. Dou, M.-F. Wang, and H.-K. Du. On the boundary of numerical ranges of operators. *Applied Mathematics Letters*, 24(5):620–622, 2011.