



CORRELATION MATRICES WITH THE PERRON-FROBENIUS PROPERTY*

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Abstract. This paper investigates conditions under which correlation matrices have a strictly positive dominant eigenvector. The sufficient conditions, from the Perron-Frobenius theorem, are that all the matrix entries are positive. The conditions for a correlation matrix with some negative entries to have a strictly positive dominant eigenvector are examined. The special structure of correlation matrices permits obtaining of detailed analytical results for low dimensional matrices. Some specific results for the n -by- n case are also derived. This problem was motivated by an application in portfolio theory.

Key words. Perron-Frobenius theory, Correlation matrix, Permutation matrix.

AMS subject classifications. 15A18, 15B48.

1. Introduction. The classic Perron-Frobenius theorem provides sufficient conditions for a real matrix to have a strictly positive dominant eigenvector. These conditions are that all the entries of the matrix are positive. However, the positivity condition on the matrix entries is not essential and matrices with some negative entries can have a strictly positive dominant eigenvector. Papers that discuss this topic include Tarazaga, Raydan, and Hurman [12], Noutsos [11], Elhashash and Szyld [5], and Berman et al. [2]. Other pertinent references include Handelman [6] and Johnson and Tarazaga [8].

Tarazaga, Raydan, and Hurman [12] demonstrate that matrices with negative entries can have a strictly positive dominant eigenvector. They derive a set of sufficient conditions for a symmetric matrix to have a strictly positive dominant eigenvector. Their condition is that the sum of all the matrix elements exceeds a multiple of the Frobenius norm of the matrix. Noutsos [11] shows that matrices which are eventually positive possess a strictly positive dominant eigenvector. Berman et al. [2] investigate the relation between the sign patterns of a matrix and its eventual positivity. They establish various conditions for the sign pattern of a matrix to be potentially eventually positive.

Our motivation to study the relationship between negative correlations and the existence of a strictly positive dominant eigenvector arose from an application in portfolio theory. Avellaneda and Lee [1] and Boyle [3] describe how the correlation matrix of common stock-returns can be used to construct portfolios with desirable properties. It is well-known that most of the variation in stock-returns can be explained by a single factor: the so-called market-factor. As a consequence

*Received by the editors on September 5, 2017. Accepted for publication on April 5, 2018. Handling Editor: Michael Tsatsomeros. Corresponding Author: Phelim Boyle. Research was funded by the Natural Sciences and Engineering Research Council of Canada.

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there is one eigenvalue in the correlation matrix that is significantly larger than the remaining eigenvalues. The dominant eigenvector associated with this eigenvalue can be used to construct portfolios where the portfolio weights are proportional to this eigenvector. If all the elements of the dominant eigenvector are positive, the portfolio has positive weights in each asset. If one of the elements of the dominant eigenvector is negative, the portfolio has a negative weight (or short position in finance parlance) in the corresponding asset. An analysis of these short positions is of interest because in many situations, short positions are prohibited by investment regulations.

If there are negative elements in the dominant eigenvector, they must be caused by negative entries in the correlation matrix in view of the Perron-Frobenius result. Boyle et al. [4] document the incidence and severity of negative correlations based on actual stock-returns data. They find that in some cases empirical correlation matrices with some negative entries have a strictly positive dominant eigenvector. In other cases empirical correlation matrices with some negative entries do not have a strictly positive dominant eigenvector. The characteristics of the negative entries in the correlation matrix determine the signs of its dominant eigenvector. Hence, the connection between the negative entries in the matrix and the positivity of the dominant eigenvector is relevant in the context of portfolio construction.

The current paper analyzes conditions under which certain symmetric matrices have a strictly positive dominant eigenvector. These are correlation matrices which have ones on the leading diagonal and where the absolute value of each off-diagonal entry is less than one. We formulate these conditions in terms of the non-diagonal entries in the matrix. Intuitively we would expect that the more prevalent the negative entries are and the larger their absolute magnitude, the less chance there is that the dominant eigenvector will be strictly positive. We are able to make this intuition precise by deriving analytical results for some low dimensional cases. The special structure of correlation matrices enables us to obtain explicit results.

It proved helpful in deriving, classifying and interpreting our results¹ to focus on groups of correlation matrices that are related as follows. If C is an n -by- n correlation matrix we can generate a related correlation matrix, \hat{C} , using a similarity transformation based on an n -by- n permutation matrix P . The relation is $\hat{C} = PCP'$. Both C and \hat{C} share many key properties that are of interest in our context. In particular both have the same eigenvalues and the dominant eigenvector of C is a permutation of the dominant eigenvector of \hat{C} . Hence, when analyzing matrices with a given number of negative correlations we only need to consider representative members from each distinct similarity class. For example a four-by-four correlation matrix with three negative entries in the upper triangle has twenty possible combinations. These combinations can be divided into three similarity classes so we can focus on three combinations instead of twenty.

Here are the main results of the paper. The two-by-two case has one independent correlation. The dominant eigenvector is strictly positive if and only if this correlation is positive. A three-by-three correlation matrix has three independent correlations because of symmetry. We derive a set of conditions on the matrix entries for it to have a strictly positive dominant eigenvector. The most

¹ We thank the referee for this suggestion.

interesting case is when exactly one of the correlations is negative and the other two are positive. In this case, the necessary and sufficient conditions coincide. In the case of a four-by-four correlation matrix, we have obtained a set of necessary conditions for the dominant eigenvector to be strictly positive. The conditions depend on the number of negative correlations and the similarity class. We also provide specific results for five-by-five correlation matrices and show that with appropriate modifications they hold for n -by- n matrices.

The rest of the paper is divided as follows. [Section 2](#) provides definitions and reviews some useful results. We also discuss similarity transformations of a correlation matrix based on a permutation matrix. [Section 3](#) analyzes two-by-two and three-by-three correlation matrices and derives necessary and sufficient conditions for these matrices to have a strictly positive dominant eigenvector. [Section 4](#) derives conditions for a four-by-four correlation matrix to have a strictly positive dominant eigenvector. We discuss the case of five-by-five correlation matrices in [Section 5](#) and we provide some results for the general n -by- n case in [Section 6](#). [Section 7](#) contains a brief summary.

2. Background and notation. We assume that all the correlation matrices have a unique largest eigenvalue. We also assume that each individual correlation is either positive or negative. Both assumptions are reasonable in the case of portfolio applications.

2.1. Definitions.

DEFINITION 2.1 (Correlation matrix). An n -by- n matrix C is called a *correlation matrix* if C is symmetric, $c_{ii} = 1$ ($1 \leq i \leq n$) and $c_{ij} \in (-1, 1)$ for every $(i \neq j)$.

Note that due to the symmetry C is fully specified by N off-diagonal terms in the upper triangle, where

$$N = \frac{n(n-1)}{2}.$$

Our convention in this paper is to use the number of negative entries among these N correlations to measure the number of negative entries in the matrix. If there is just one negative entry among these N terms we will say that C has one negative correlation even though there are two negative entries in C because of the symmetry. In essence we only count the number of independent correlations.

DEFINITION 2.2 (The strong Perron-Frobenius property). An n -by- n matrix, A is said to *possess the strong Perron-Frobenius property* if the *spectral radius* (i.e., dominant eigenvalue) $\rho(A)$, is positive, simple and A has a positive left and right eigenvector corresponding to $\rho(A)$. When A is symmetric, possession of a positive right eigenvector guarantees possession of a positive left eigenvector.

In this paper, we normally use the term *Perron-Frobenius property* instead of *the strong Perron-Frobenius property*.

DEFINITION 2.3 (Eventually positive matrices). An n -by- n matrix B is said to be *eventually positive* if there exists a positive integer k_0 such that $B^k > 0$ for all $k \geq k_0$.



DEFINITION 2.4 (Permutation matrices). Given a permutation $\pi \in S_n$ (S_n is the *symmetric group*), the matrix $P = [p_{ij}] \in M_n(\mathbb{R})$, defined by

$$p_{ij} = \begin{cases} 1, & j = \pi(i), \\ 0, & \text{otherwise} \end{cases}$$

is the *permutation matrix* with respect to π .

Each row of P contains a one and $(n - 1)$ zeros. Similarly each column of P contains a one and $(n - 1)$ zeros. Furthermore P is orthogonal.

2.2. Relevant results. In this subsection, we will recall the Perron-Frobenius result for positive matrices and review some results that will be useful in the sequel.

THEOREM 2.5 (Perron-Frobenius). *A sufficient condition for the dominant eigenvector of a real matrix B to be strictly positive is that all the elements of B are positive.*

The next result concerns similar classes of correlation matrices. See (Li and Pierce [9] and Horn and Johnson [7]).

THEOREM 2.6. *If C is an n -by- n correlation matrix and P is an n -by- n permutation matrix, then*

$$(2.1) \quad \hat{C} = PCP'$$

is an n -by- n correlation matrix. Furthermore \hat{C} has the same eigenvalues as C and the dominant eigenvector of \hat{C} is a permutation of the dominant eigenvector of C .

Proof. From equation (2.1), \hat{C} is symmetric. It is straightforward to show that

$$\hat{c}_{ii} = 1, \quad 1 \leq i \leq n.$$

The off-diagonal elements of \hat{C} represent a reordering of the off-diagonal elements of C . Hence, \hat{C} is a valid correlation matrix.

The matrix C can be expressed as

$$C = Q\Lambda Q',$$

where Q is an orthogonal matrix whose columns are the eigenvectors of C and Λ is a diagonal matrix with the eigenvalues of C on its main diagonal. Hence,

$$\begin{aligned} \hat{C} &= PCP' \\ &= PQ\Lambda Q'P' \\ &= PQ\Lambda(PQ)'. \end{aligned}$$

Note that PQ is orthogonal since it is the product of two orthogonal matrices. The last equation shows that C and \hat{C} have the same eigenvalues. It also shows that the eigenvectors of

\hat{C} are obtained by reordering the components of the corresponding eigenvectors of C . If C has a complete row of negative correlations, then we can show that \hat{C} will also have a complete row of negative correlations. \square

This theorem provides a simple and effective framework to organize our results. Because of this theorem we do not need to consider all possible combinations for a given number of negative correlations but only representative members of certain similarity classes.

The next result due to Mercer and Mercer [10] provides a lower bound on the dominant eigenvalue of a real symmetric matrix. The proof is based on the Cauchy interlacing theorem (see Horn and Johnson [7, §4.3]).

THEOREM 2.7. *Let $B = [b_{jk}]$ be a real symmetric n -by- n matrix with $n \geq 2$. Let λ denote its largest eigenvalue. Then*

$$\lambda \geq \frac{1}{2} \max_{1 \leq j \leq k \leq n} \left[(b_{jj} + b_{kk}) + \sqrt{(b_{jj} - b_{kk})^2 + 4b_{jk}^2} \right].$$

In the case of correlation matrices, the lower bound simplifies leading to the following corollary.

COROLLARY 2.8. *Let $C = [c_{jk}]$ be an n -by- n correlation matrix with $n \geq 2$ and let λ denote its largest eigenvalue. Then*

$$\lambda \geq 1 + \max_{1 \leq j < k \leq n} |c_{jk}|.$$

This last result will be used extensively in the rest of the paper.

The next theorem establishes a direct relation between the signs and positions of the negative entries in the matrix and the existence of a positive dominant eigenvector in one particular case.

THEOREM 2.9. *If $C = [c_{ij}]$ is an n -by- n correlation matrix with a complete row of negative correlations, then it is impossible for C to have a strictly positive dominant eigenvector.*

Proof. Let λ be the dominant eigenvalue and v its associated eigenvector. Assume that all the elements of v are positive. In view of Theorem 2.6 there is no loss of generality in assuming that all the entries in the first row (apart from $c_{11} = 1$) are negative. We show that these assumptions lead to a contradiction.

From Corollary 2.8, the largest eigenvalue must be greater than one. Hence,

$$(2.2) \quad \lambda > 1$$

and $v_i > 0$, $1 \leq i \leq n$. We know that

$$Cv = \lambda v.$$

We focus on the first row to obtain

$$v_1 + \sum_{j \neq 1}^n v_j c_{1j} = \lambda v_1,$$



which can be rewritten as

$$v_1(\lambda - 1) = \sum_{j \neq 1}^n v_j c_{1j}.$$

The left hand side of this last equation is positive because of condition (2.2) and the fact that v_1 is positive. The right hand side is negative because all the c_{1j} are negative and all the v_j 's are positive. This contradiction proves the result. \square

Theorem 2.9 is implicit in Lemma 5.5 of Berman et al. [2] but the current version is more transparent for our purposes.

3. Two-by-two and three-by-three correlation matrices. The two-by-two case is easy to handle since we can obtain explicit solutions for the eigenvalues and the eigenvectors. Suppose

$$C = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

The two eigenvalues are $(1 + a)$ and $(1 - a)$. The corresponding eigenvectors are

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

We see that C has a strictly positive dominant eigenvector if and only if a is positive. When a is negative the result is consistent with **Theorem 2.9**.

We next discuss three-by-three correlation matrices. These matrices are specified by three independent correlations. There will be either zero, one, two or three negative correlations. If there are no negative correlations then all the correlations are positive and the dominant eigenvector will be strictly positive from the Perron-Frobenius result. If there are either two or three negative correlations there has to be a row of negative correlations and **Theorem 2.9** tells us that it is impossible to have a strictly positive dominant eigenvector. Hence, we are left with the case when there is exactly one negative correlation and two positive correlations. When there is just one negative correlation it can appear in three possible positions. The three corresponding matrices form a single similarity class as per our recent example at the end of the last section. Hence, without loss of generality, we can assume that the correlation in the (1,2) position is negative.

Denote the correlations by a, b and c and define

$$C = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

THEOREM 3.1. *Let C be a three-by-three correlation matrix with exactly one negative correlation (say a). Then C has a strictly positive dominant eigenvector if and only if*

$$(3.1) \quad |a| < \min\{b, c\}.$$

(The intuition from condition (3.1) is that the impact of the negative correlation is less important than that of the two positive correlations.)

Proof. Let λ be the largest eigenvalue of C and let

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

be its corresponding eigenvector. In addition, let $\kappa = \lambda - 1$.

The relationship between C, κ and v is

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (\kappa + 1) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

This is equivalent to

$$(3.2) \quad \begin{cases} \kappa v_1 - av_2 - bv_3 &= 0, \\ -av_1 + \kappa v_2 - cv_3 &= 0, \\ -bv_1 - cv_2 + \kappa v_3 &= 0. \end{cases}$$

We know from Corollary 2.8 that

$$(3.3) \quad \kappa \geq \max\{|a|, |b|, |c|\}.$$

Proceed by contradiction and assume that

$$(3.4) \quad |a| \geq \min\{b, c\},$$

and that all the elements of v are positive.

When a is negative equations (3.2) can be written as

$$(3.5) \quad \kappa v_1 + |a|v_2 = bv_3,$$

$$(3.6) \quad |a|v_1 + \kappa v_2 = cv_3,$$

$$(3.7) \quad \kappa v_3 = bv_1 + cv_2.$$

Adding (3.5) and (3.6), we obtain

$$(\kappa + |a|)(v_1 + v_2) = (b + c)v_3,$$

which implies

$$(3.8) \quad v_1 + v_2 = \left(\frac{b + c}{\kappa + |a|} \right) v_3.$$



From (3.7), we have

$$(3.9) \quad v_3 = \left(\frac{b}{\kappa}\right) v_1 + \left(\frac{c}{\kappa}\right) v_2.$$

Using the last two equations and conditions (3.3) and (3.4) we obtain

$$(3.10) \quad v_1 + v_2 \geq v_3,$$

$$(3.11) \quad v_1 + v_2 \leq v_3.$$

These two equations provide a contradiction unless there is equality in (3.10) and (3.11). We will show that our assumptions rule out this case. If there is equality in both equations then this implies

$$\kappa = b = c = |a| = m.$$

However for $0 < m < 1$ the matrix

$$C = \begin{pmatrix} 1 & -m & m \\ -m & 1 & m \\ m & m & 1 \end{pmatrix},$$

does not have a unique largest eigenvalue since its eigenvalues are

$$(1 + m), \quad (1 + m), \quad (1 - 2m).$$

Throughout the paper, we assumed a unique largest eigenvalue so this case is excluded. Hence, the assumptions that equation (3.4) is satisfied and that v is strictly positive lead to a contradiction. This completes the necessity part. We next prove sufficiency.

The first step is to show why $v_3 \neq 0$. We proceed by contradiction. If $v_3 = 0$, equations (3.5) and (3.6) become

$$\begin{aligned} \kappa v_1 + |a| v_2 &= 0, \\ |a| v_1 + \kappa v_2 &= 0. \end{aligned}$$

These equations show that $v_1 = 0$ implies $v_2 = 0$ and that $v_2 = 0$ implies $v_1 = 0$. Since all three components (v_1, v_2, v_3) cannot be zero, this means that both v_1 and v_2 are non zero. Hence,

$$\frac{v_1}{v_2} = -\frac{|a|}{\kappa} = -\frac{\kappa}{|a|}.$$

This last equation implies that

$$\kappa = |a|.$$

From this result and equation (3.1), it follows

$$\kappa = |a| < \min\{b, c\},$$

which contradicts Corollary 2.8.

Since v and $-v$ are both principal eigenvectors we can also assume that $v_3 > 0$. Divide equations (3.5) and (3.6) across by v_3 to obtain

$$\begin{aligned}\kappa w_1 + |a|w_2 &= b, \\ |a|w_1 + \kappa w_2 &= c,\end{aligned}$$

where

$$w_1 = \frac{v_1}{v_3}, \quad w_2 = \frac{v_2}{v_3}.$$

By virtue of Corollary 2.8 and our assumptions, it follows

$$\kappa^2 > |a|^2.$$

Hence, w_1 and w_2 are given by

$$w_1 = \frac{\kappa b - |a|c}{\kappa^2 - |a|^2}, \quad w_2 = \frac{\kappa c - |a|b}{\kappa^2 - |a|^2}.$$

The numerators in these last two expressions are both strictly positive by Corollary 2.8 and our assumptions. Hence, both w_1 and w_2 are strictly positive. This implies v_1, v_2 and v_3 are also strictly positive. \square

In summary, a three-by-three correlation matrix with a single negative correlation has a strictly positive dominant eigenvector if and only if the absolute value of the negative correlation is less than the smaller of the two positive correlations. If there are two or three negative correlations then it is impossible for a three-by-three correlation matrix to have a strictly positive dominant eigenvector.

4. Four by four correlation matrices. This section examines the conditions for a four-by-four correlation matrix to have a strictly positive dominant eigenvector. Let C be a four-by-four correlation matrix with six independent correlations.

$$(4.1) \quad C = \begin{pmatrix} 1 & a & b & c \\ a & 1 & d & e \\ b & d & 1 & f \\ c & e & f & 1 \end{pmatrix}.$$

As a first step we classify the matrices by the number of negative correlations. Since each correlation can be either positive or negative and there are six correlations the total number of combinations is $2^6 = 64$. Table 1 shows the number of combinations that correspond to a given number of negative correlations.

In the second step, we further classify these matrices into similarity classes. Table 2 shows the numbers in each similarity class corresponding to a given number of negative correlations.

For example, when there are two negative correlations, there are fifteen different combinations. These fifteen combinations can be grouped into two distinct similarity classes. Class I contains

TABLE 1

Four-by-four correlation matrices: Distribution of possible combinations by number of negative correlations.

Number of negative correlations n_c	Number of combinations that produce n_c negative correlations
0	1
1	6
2	15
3	20
4	15
5	6
6	1
Total	64

TABLE 2

Four-by-four correlation matrices: Distribution of similarity classes by number of negative correlations.

Number of negative correlations	Number in Class I	Number in Class II	Number in Class III	Total Number in all Classes
0	1	na	na	1
1	6	na	na	6
2	12	3	na	15
3	12	4	4	20
4	12	3	na	15
5	6	na	na	6
6	1	na	na	1
Total				64

12 members and Class II contains 3 members. We only need to find conditions for representative members from these two classes.

Let λ be the largest eigenvalue of C . Let

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

be its corresponding eigenvector. As before $\kappa = \lambda - 1$.

The four linear equations in this case are

$$(4.2) \quad \begin{cases} \kappa v_1 - av_2 - bv_3 - cv_4 = 0, \\ -av_1 + \kappa v_2 - dv_3 - ev_4 = 0, \\ -bv_1 - dv_2 + \kappa v_3 - fv_4 = 0, \\ -cv_1 - ev_2 - fv_3 + \kappa v_4 = 0. \end{cases}$$

From [Corollary 2.8](#), we have the following relation between κ and the absolute values of the six correlations.

$$(4.3) \quad \kappa \geq \max \{|a|, |b|, |c|, |d|, |e|, |f|\}.$$

If none of the correlations is negative, then by the Perron-Frobenius theorem, the matrix has a strictly positive dominant eigenvector. It is convenient to present the results according to the number of negative correlations.

4.1. One negative entry. When there is just one negative entry in a four-by-four correlation matrix we are able to derive both necessary and sufficient conditions. In this case, there is just one similarity class with six members. Without loss of generality we can assume the negative correlation occupies the $(1, 3)$ position so that b is negative. The next theorem gives the necessary condition.

THEOREM 4.1 (4-by-4 matrix with one negative entry: Necessary Condition). *Let C be a four-by-four correlation matrix with exactly one negative correlation (say b). A necessary condition for C to have a strictly positive, dominant eigenvector is that the sum of the six independent correlations is positive i.e*

$$(4.4) \quad a + b + c + d + e + f > 0.$$

Proof. Proceed by contradiction. Assume that

$$(4.5) \quad a + b + c + d + e + f \leq 0,$$

and that v is strictly positive.

We have

$$(4.6) \quad \kappa \geq |b| \geq a + c + d + e + f.$$

From [\(4.2\)](#), we have

$$(4.7) \quad \begin{cases} \kappa v_1 + |b|v_3 &= av_2 + cv_4, \\ \kappa v_2 &= av_1 + dv_3 + ev_4, \\ |b|v_1 + \kappa v_3 &= dv_2 + fv_4, \\ \kappa v_4 &= cv_1 + ev_2 + fv_3. \end{cases}$$

Adding these four equations we obtain

$$[\kappa + |b| - (a + c)]v_1 + [\kappa - (a + d + e)]v_2 + [\kappa + |b| - (d + f)]v_3 + [\kappa - (c + e + f)]v_4 = 0.$$

Each term in square brackets is positive by equation [\(4.6\)](#) and if all the v 's are positive this last equation gives the required contradiction. \square



We now turn to the sufficient condition.

THEOREM 4.2 (4-by-4 matrix with one negative entry: Sufficient Condition). *Let C be a four-by-four correlation matrix with exactly one negative correlation (say b). A sufficient condition for C to have a strictly positive, dominant eigenvector is that*

$$(4.8) \quad |b| < \min \{a, c, d, e, f\}.$$

Proof. The four linear equations in this case are given by the system (4.7). Since the eigenvector cannot be zero it must have at least one component that is not zero. Say it is v_2 . We can assume $v_2 > 0$. Dividing each equation in the system (4.7) by v_2 we obtain

$$(4.9) \quad \begin{cases} \kappa w_1 + |b|w_2 &= a + cw_3, \\ \kappa &= aw_1 + dw_2 + ew_3, \\ |b|w_1 + \kappa w_2 &= d + fw_3, \\ \kappa w_3 &= cw_1 + e + fw_2, \end{cases}$$

where

$$w_1 = \frac{v_1}{v_2}, \quad w_2 = \frac{v_3}{v_2}, \quad w_3 = \frac{v_4}{v_2}.$$

Using the first and third equations in (4.9) to eliminate w_1 we obtain

$$(4.10) \quad w_2 = \left(\frac{\kappa f - |b|c}{\kappa^2 - |b|^2} \right) w_3 + \left(\frac{\kappa d - a|b|}{\kappa^2 - |b|^2} \right).$$

Note that the denominator, $(\kappa^2 - |b|^2)$ is strictly positive and that the terms in the large parentheses are also positive by (4.8) and (4.3).

In the same way, we can also use the first and third equations in (4.9) to eliminate w_2 to obtain

$$(4.11) \quad w_1 = \left(\frac{\kappa c - |b|f}{\kappa^2 - |b|^2} \right) w_3 + \left(\frac{\kappa a - |b|d}{\kappa^2 - |b|^2} \right).$$

Once again the terms in the large parentheses are positive.

We use the last equation in (4.9) together with (4.10) and (4.11) to derive the following expression for w_3

$$(4.12) \quad w_3 = \left(\frac{e(\kappa^2 - |b|^2) + c(\kappa a - |b|d) + f(\kappa d - a|b|)}{\kappa^3 - \kappa(b^2 + c^2 + f^2) - 2bcf} \right).$$

The numerator of this expression is positive and we will now show that the denominator is also positive. Note that the matrix

$$\begin{bmatrix} 1 & b & c \\ b & 1 & f \\ c & f & 1 \end{bmatrix},$$

is a submatrix of C with characteristic equation.

$$\phi(y) = y^3 - y(b^2 + c^2 + f^2) - 2bcf.$$

Because of the Cauchy interlacing theorem and our assumptions $\phi(\kappa) > 0$. This means that the denominator of (4.12) is positive. Hence, w_3 is positive. By virtue of (4.10) and (4.11), both w_1 and w_2 are also positive. Since $v_2 > 0$ this means that C has a strictly positive dominant eigenvector and the proof is complete. \square

We now discuss the case when two of the six correlations are negative.

4.2. Two negative correlations. From Table 2, there are two similarity classes. The first similarity class contains 12 combinations and the pair (a, c) is a representative combination. The second similarity class contains 3 combinations and the pair (a, f) is a representative combination.

THEOREM 4.3 (4-by-4 matrix with two negative correlations, Class I). *Let C be a four-by-four correlation matrix with two negative correlations that belongs to the first similarity class. We assume the negative pair is (a, c) . Then a necessary condition for C to have a strictly positive, dominant eigenvector is that the sum of the six independent correlations is positive*

$$(4.13) \quad a + b + c + d + e + f > 0.$$

Proof. Proceed by contradiction. Assume that all the elements of v are positive and assume that

$$(4.14) \quad |a| + |c| \geq b + d + e + f.$$

From (4.2), with a and c negative, we have

$$(4.15) \quad \begin{cases} \kappa v_1 + |a|v_2 + |c|v_4 = bv_3, \\ |a|v_1 + \kappa v_2 = dv_3 + ev_4, \\ \kappa v_3 = bv_1 + dv_2 + fv_4, \\ |c|v_1 + \kappa v_4 = ev_2 + fv_3. \end{cases}$$

Adding the first two equations and the last one, we get

$$(\kappa + |a| + |c|)v_1 + (\kappa + |a| - e)v_2 + (\kappa + |c| - e)v_4 = (b + d + f)v_3.$$

Hence, we have

$$(4.16) \quad v_3 = \left(\frac{\kappa + |a| + |c|}{b + d + f} \right) v_1 + \left(\frac{\kappa + |a| - e}{b + d + f} \right) v_2 + \left(\frac{\kappa + |c| - e}{b + d + f} \right) v_4.$$

Based on our assumptions the coefficient of v_1 is greater than one. The coefficient of v_2 is greater than or equal to one as is the coefficient of v_4 . Hence,

$$(4.17) \quad v_3 > v_1 + v_2 + v_4.$$

From the third equation of the system (4.15), we obtain

$$v_3 = \left(\frac{b}{\kappa} \right) v_1 + \left(\frac{d}{\kappa} \right) v_2 + \left(\frac{f}{\kappa} \right) v_4.$$



Since each term in the big parentheses is positive and less than or equal to one and all the v 's are assumed positive, we have

$$(4.18) \quad v_3 \leq v_1 + v_2 + v_4.$$

Equations (4.17) and (4.18) provide a contradiction. This completes the proof. \square

Now we deal with the second similarity class that contains three combinations.

THEOREM 4.4 (4-by-4 matrix with two negative correlations, Class II). *Let C be a four-by-four correlation matrix with two negative correlations that belongs to the second similarity class. We assume the negative pair is (a, f) . Then a necessary condition for C to have a strictly positive, dominant eigenvector is that*

$$(4.19) \quad \max\{(c+e), (b+d), (b+c), (d+e)\} > |a| + |f|.$$

Proof. Proceed by contradiction. Assume that all the elements of v are positive and assume that

$$(4.20) \quad |a| + |f| \geq \max\{(c+e), (b+d), (b+c), (d+e)\}.$$

From (4.2), with a and f negative, we have

$$(4.21) \quad \begin{cases} \kappa v_1 + |a|v_2 &= bv_3 + cv_4, \\ |a|v_1 + \kappa v_2 &= dv_3 + ev_4, \\ \kappa v_3 + |f|v_4 &= bv_1 + dv_2, \\ |f|v_3 + \kappa v_4 &= cv_1 + ev_2. \end{cases}$$

Adding the first two equations and dividing across by $(\kappa + |a|)$ we obtain

$$(v_1 + v_2) = \frac{(b+d)}{(\kappa + |a|)}v_3 + \frac{(c+e)}{(\kappa + |a|)}v_4.$$

This implies that

$$(4.22) \quad v_1 + v_2 \leq v_3 + v_4.$$

Adding the last two equations and dividing across by $(\kappa + |f|)$ we obtain

$$(v_3 + v_4) = \frac{(b+c)}{(\kappa + |f|)}v_1 + \frac{(c+e)}{(\kappa + |f|)}v_2.$$

This implies that

$$(4.23) \quad v_3 + v_4 \leq v_1 + v_2.$$

The contradiction proof goes through unless there is an equality sign in both equations (4.22) and (4.23). We can rule this out by noting that if both have an equality sign, this implies

$$|a| = b = c = d = e = |f| = m.$$

The corresponding matrix

$$C = \begin{pmatrix} 1 & -m & m & m \\ -m & 1 & m & m \\ m & m & 1 & -m \\ m & m & -m & 1 \end{pmatrix},$$

has eigenvalues

$$(1 + m), (1 + m), (1 + m), (1 - 3m),$$

and this is ruled out by our assumption that there is a unique largest eigenvalue. \square

It is instructive to compare the necessary conditions in [Theorem 4.3](#) with the necessary conditions in [Theorem 4.4](#). Condition [\(4.13\)](#) which assumes the sum of all the correlations is positive implies condition [\(4.19\)](#). There is a sense in which condition [\(4.19\)](#) is more informative. Note that if a necessary condition is satisfied the matrix *may or may not* have a strictly positive dominant eigenvector, whereas if a necessary condition is violated we know that the matrix *does not have* a strictly positive dominant eigenvector. The following numerical example illustrates this point. We simulated one million random (four-by-four) correlation matrices where a and f are negative and b, c, d, e are positive. [Table 3](#) provides the numbers which satisfy and the numbers which do not satisfy each of the two conditions.

TABLE 3

Simulation results of 1,000,000 four-by-four correlation matrices where a and f are negative and b, c, d, e are positive. The table shows how many matrices satisfy and how many matrices do not satisfy each of the two conditions.

Condition	Number of matrices that satisfy the condition	Number of matrices that do not satisfy the condition
$b + c + d + e > a + f $	919,430	80,570
$\max\{(c + e), (b + d), (b + c), (d + e)\} > a + f $	733,362	266,638

[Table 3](#) shows that if the first condition is violated there are 80,481 matrices that do not have a strictly positive dominant eigenvector. However if the second condition is violated there are 266,038 matrices that do not have a strictly positive dominant eigenvector. In this sense, the second condition is more informative than the first.

4.3. Three negative entries. We saw in [Table 2](#) that a four-by-four correlation matrix with three negative entries has twenty possible combinations that divide into three similarity classes.

- Class I. The four triplets in Class I are abc , ade , bdf and cef . They all have a complete row of negative correlations.
- Class II. The four triplets in Class II are def , bcf , ace and abd . They are the complements of Class I.
- Class III. The twelve remaining combinations. A representative triplet in Class III is abe .

Hence, we analyze these three separately.



Each of the four members of Class I has a complete row of negative correlations. By virtue of [Theorem 2.9](#), none of them will have a strictly positive dominant eigenvector.

We now turn to Class II. The necessary condition in this case is that the sum of the six correlations is positive.

THEOREM 4.5 (Three negative entries: Class II). *Let C be a four-by-four correlation matrix belonging to Class II with three negative correlations (say a, c, e). A necessary condition for C to have a strictly positive, dominant eigenvector is that the sum of the six correlations is positive*

$$(4.24) \quad a + b + c + d + e + f > 0.$$

Proof. Proceed by contradiction. Assume that

$$(4.25) \quad |a| + |c| + |e| \geq b + d + f,$$

and assume that v is strictly positive.

From [\(4.2\)](#), we have

$$(4.26) \quad \begin{cases} \kappa v_1 + |a|v_2 + |c|v_4 = bv_3, \\ |a|v_1 + \kappa v_2 + |e|v_4 = dv_3, \\ \kappa v_3 = bv_1 + dv_2 + fv_4, \\ |c|v_1 + |e|v_2 + \kappa v_4 = fv_3. \end{cases}$$

Adding the first, second and last equation yields

$$(\kappa + |a| + |c|)v_1 + (\kappa + |a| + |e|)v_2 + (\kappa + |c| + |e|)v_4 = (b + d + f)v_3.$$

This means that

$$v_3 = \left(\frac{\kappa + |a| + |c|}{b + d + f} \right) v_1 + \left(\frac{\kappa + |a| + |e|}{b + d + f} \right) v_2 + \left(\frac{\kappa + |c| + |e|}{b + d + f} \right) v_4.$$

Using the same logic as before this last equation implies

$$(4.27) \quad v_3 \geq v_1 + v_2 + v_4.$$

From the third equation of the system [\(4.26\)](#), we have

$$v_3 = \left(\frac{b}{\kappa} \right) v_1 + \left(\frac{d}{\kappa} \right) v_2 + \left(\frac{f}{\kappa} \right) v_4,$$

and using [\(4.3\)](#), we obtain

$$(4.28) \quad v_3 \leq v_1 + v_2 + v_4.$$

Equation [\(4.27\)](#) and equation [\(4.28\)](#) provide a contradiction unless both hold as equalities. We can rule out this possibility with the same approach we used in the proof of the last theorem. This completes the proof. \square

We now turn to Class III. We consider the case where a, b, e are negative.

THEOREM 4.6 (Three negative entries: Class III). *Let C be a four-by-four correlation matrix belonging to Class III with three negative correlations (say a, b, e). A necessary condition for C to have a strictly positive, dominant eigenvector is that at least one of the following conditions are satisfied*

$$(4.29) \quad (d + f) > (\kappa_1 + |b|), \quad (c + f) > (\kappa_1 + |e|),$$

where

$$\kappa_1 = \max\{|a|, |b|, c, d, |e|, f\}.$$

Proof. We assume this condition is false so that *both* of the following conditions are satisfied

$$(4.30) \quad (\kappa_1 + |b|) \geq (d + f), \quad (\kappa_1 + |e|) \geq (c + f),$$

and that v is strictly positive.

We have from (4.3)

$$(4.31) \quad \kappa \geq \max(|a|, |b|, c, d, |e|, f) = \kappa_1.$$

From the system (4.2), we have

$$(4.32) \quad \begin{cases} \kappa v_1 + |a|v_2 + |b|v_3 &= cv_4, \\ |a|v_1 + \kappa v_2 + |e|v_4 &= dv_3, \\ |b|v_1 + \kappa v_3 &= dv_2 + fv_4, \\ |e|v_2 + \kappa v_4 &= cv_1 + fv_3. \end{cases}$$

Adding the four equations, we get

$$(4.33) \quad [\kappa + |a| + |b| - c]v_1 + [\kappa + |a| + |e| - d]v_2 + [\kappa + |b| - (d + f)]v_3 + [\kappa + |e| - (c + f)]v_4 = 0.$$

The coefficients of v_1 and v_2 are obviously positive and the coefficients of v_3 and v_4 are non negative from (4.30) and (4.31). Hence, the left hand side of the expression is positive. This establishes a contradiction and completes the proof. \square

4.4. Four negative entries. When a four-by-four correlation matrix has four negative entries, we will show that it is impossible for it to have a strictly positive dominant eigenvector. Twelve of the fifteen combinations belong to Class I and they all have a complete negative row and hence cannot have a strictly positive dominant eigenvector. The remaining three combinations belong to Class II. To illustrate this point we derive the result for the case where a, b, e and f are the negative correlations.

THEOREM 4.7 (4-by-4 matrix with four negative entries). *Let C be a four-by-four correlation matrix belonging to Class II with four negative correlations (say a, b, e, f). It is impossible for C to have a strictly positive, dominant eigenvector.*

Proof. The proof follows the usual approach.

When a, b, e and f are negative we get from the system (4.2)

$$(4.34) \quad \begin{cases} \kappa v_1 + |a|v_2 + |b|v_3 &= cv_4, \\ |a|v_1 + \kappa v_2 + |e|v_4 &= dv_3, \\ |b|v_1 + \kappa v_3 + |f|v_4 &= dv_2, \\ |e|v_2 + |f|v_3 + \kappa v_4 &= cv_1. \end{cases}$$

By adding the first equation to the last one, we obtain

$$(4.35) \quad (\kappa - c)(v_1 + v_4) + (|a| + |e|)v_2 + (|b| + |f|)v_3 = 0.$$

We know from equation (4.3) that $\kappa \geq c$. Hence, this last equation provides a contradiction. Therefore, when a, b, e and f are the only negative correlations, we have shown that the corresponding correlation matrix C cannot have a strictly positive dominant eigenvector. This completes the proof. \square

When a four-by-four correlation matrix has five or six negative correlations, each of the possible combinations will give a rise to a complete row of negative correlations. In this case, it is impossible to have a strictly positive dominant eigenvector because of Theorem 2.9. Combining this with the results of Theorem 4.7 we can state that if a four-by-four correlation matrix has four or more negative correlations it cannot have the Perron-Frobenius property.

We conclude this section with a short summary. We studied the impact of negative correlations on the positivity of the dominant eigenvector for four-by-four correlation matrices. We derived necessary and sufficient conditions for the existence of a strictly positive dominant eigenvector when there is one negative correlation. When there are two or three negative correlations the necessary conditions vary with the similarity class. If the matrix has four or more negative correlations it cannot have a strictly positive dominant eigenvector. In this case, the magnitude of the negative entries is immaterial.

5. Five by five correlation matrices. This section examines the conditions for a five-by-five correlation matrix to have a strictly positive dominant eigenvector. The nature of these conditions varies with the number of negative correlations and also with the similarity class. Let C be a five-by-five correlation matrix with ten independent correlations.

$$C = \begin{pmatrix} 1 & a & b & c & d \\ a & 1 & e & f & g \\ b & e & 1 & h & i \\ c & f & h & 1 & j \\ d & g & i & j & 1 \end{pmatrix}.$$

Table 4 provides summary information on five-by-five correlation matrices. We provide a two stage classification of these matrices. First we classify them by the numbers of negative correlations.

There are ten correlations and each correlation can be positive or negative. Hence, there are

$$1024 = 2^{10},$$

different possible combinations. To interpret Table 4 consider the fourth row. It corresponds to the case of three negative correlations, and therefore, seven positive correlations. There are a total of 120 different possible combinations with three positive correlations and seven negative correlations since

$$\binom{10}{3} = \frac{10!}{(7!)(3!)} = 120.$$

These 120 combinations can be organized into four different similarity classes. The numbers in these similarity classes are 60, 30, 20 and 10 making up the 120 combinations.

The entire set of 1024 combinations can be divided into 34 different similarity classes and in general the conditions for each similarity class to have a strictly positive dominant eigenvector are different. Hence, we only consider a few important cases. These correspond to the case where there is only one negative correlation at one extreme and cases where there are sufficient negative correlations to form a complete row at the other.

TABLE 4
Analysis of 5-by-5 correlation matrix by number of negative correlations and by number of similarity classes.

Number of negative correlations n_c	Number of combinations that produce n_c negative correlations	Number of similarity classes
0	1	1
1	10	1
2	45	2
3	120	4
4	210	6
5	252	6
6	210	6
7	120	4
8	45	2
9	10	1
10	1	1
Totals	1024	34

We now turn to the case when there is just one negative correlation. There is just one similarity class containing ten combinations. Without loss of generality we can assume a is negative and all the other correlations are positive. The necessary condition in this case is given in the next theorem.

THEOREM 5.1 (5-by-5 matrix with one negative entry: Necessary Condition). *Let C be a five-by-five correlation matrix with exactly one negative correlation (say a). A necessary condition for C to have a strictly positive, dominant eigenvector is that the sum of the ten independent correlations*

is positive.

$$(5.1) \quad a + b + c + d + e + f + g + h + i + j > 0.$$

Proof. Proceed by contradiction and assume

$$(5.2) \quad |a| \geq b + c + d + e + f + h + i + j.$$

From the usual equations, we have

$$(5.3) \quad \begin{cases} \kappa v_1 + |a|v_2 = & bv_3 + cv_4 + dv_5, \\ |a|v_1 + \kappa v_2 = & ev_3 + fv_4 + gv_5, \\ \kappa v_3 = & bv_1 + ev_2 + hv_4 + iv_5, \\ \kappa v_4 = & cv_1 + fv_2 + hv_3 + jv_5, \\ \kappa v_5 = & dv_1 + gv_2 + iv_3 + jv_4. \end{cases}$$

Adding these five equations we obtain

$$(5.4) \quad \psi_1 v_1 + \psi_2 v_2 + \psi_3 v_3 + \psi_4 v_4 + \psi_5 v_5 = 0.$$

From our assumptions, all the ψ 's are positive, and if we assume that all the v 's are also positive, we obtain a contradiction. \square

The sufficient condition when a is the only negative correlation is given in the next theorem.

THEOREM 5.2 (5-by-5 matrix with one negative entry: Sufficient Condition). *Let C be a five-by-five correlation matrix with exactly one negative correlation (say a). A sufficient condition for C to have a strictly positive, dominant eigenvector is that the absolute value of the negative correlation is less than the smallest of the positive correlations.*

$$(5.5) \quad |a| < \min \{b, c, d, e, f, g, h, i, j\}.$$

The proof is lengthy and involves some tedious algebra. It may be helpful to provide a road map. We assume one of components of the dominant eigenvector (say v_3) is positive. We divide our five equations in (5.3) by v_3 to get equations in four unknowns. Label these unknowns as w_1, w_2, w_3, w_4 . We show that w_1 and w_2 have the same sign and that w_3 and w_4 also have the same sign. From this, we can prove that all the w 's are positive, and hence, conclude that all the elements of v are positive.

Proof. We assume $v_3 > 0$. Divide the system (5.3) by v_3 to obtain.

$$(5.6) \quad \begin{cases} \kappa w_1 + |a|w_2 = b + cw_3 + dw_4, \\ |a|w_1 + \kappa w_2 = e + fw_3 + gw_4, \\ \kappa = bw_1 + ew_2 + hw_3 + iw_4, \\ \kappa w_3 = cw_1 + fw_2 + h + jw_4, \\ \kappa w_4 = dw_1 + gw_2 + i + jw_3. \end{cases}$$

where

$$w_1 = \frac{v_1}{v_3}, \quad w_2 = \frac{v_2}{v_3}, \quad w_3 = \frac{v_4}{v_3}, \quad w_4 = \frac{v_5}{v_3}.$$

The first two equations in (5.6) are:

$$(5.7) \quad \begin{cases} \kappa w_1 + |a|w_2 &= b + cw_3 + dw_4, \\ |a|w_1 + \kappa w_2 &= e + fw_3 + gw_4. \end{cases}$$

From (5.7), we can obtain expressions for w_2 and w_1 in terms of w_3 and w_4 .

$$(5.8) \quad w_2 = \frac{(\kappa e - |a|b) + (\kappa f - |a|c)w_3 + (\kappa g - |a|d)w_4}{\kappa^2 - |a|^2},$$

and

$$(5.9) \quad w_1 = \frac{(\kappa b - |a|e) + (\kappa c - |a|f)w_3 + (\kappa d - |a|g)w_4}{\kappa^2 - |a|^2}.$$

The final two equations in (5.6) are

$$(5.10) \quad \begin{aligned} \kappa w_3 &= cw_1 + fw_2 + h + jw_4, \\ \kappa w_4 &= dw_1 + gw_2 + i + jw_3. \end{aligned}$$

From these last two equations, we obtain expressions for w_3 and w_4 in terms of w_1 and w_2 .

$$(5.11) \quad \begin{aligned} w_3 &= \frac{(\kappa h + ij) + (\kappa c + jd)w_1 + (\kappa f + gj)w_2}{(\kappa^2 - j^2)}, \\ w_4 &= \frac{(\kappa i + hj) + (\kappa d + jc)w_1 + (\kappa g + fj)w_2}{(\kappa^2 - j^2)}. \end{aligned}$$

Using equations (5.8), (5.9), (5.10) and (5.11), we obtain two equations for w_3 and w_4 .

$$(5.12) \quad \begin{aligned} h_{11}w_3 + h_{12}w_4 &= b_1, \\ h_{21}w_3 + h_{22}w_4 &= b_2, \end{aligned}$$

and two equations for w_1 and w_2 .

$$(5.13) \quad \begin{aligned} g_{11}w_1 + g_{12}w_2 &= c_1, \\ g_{21}w_1 + g_{22}w_2 &= c_2, \end{aligned}$$

where

$$\begin{aligned} h_{11} &= ((\kappa^2 - |a|^2)(\kappa^2 - j^2) - (\kappa c + jd)(\kappa c - |a|f) - (\kappa f + gj)(\kappa f - |a|c)), \\ h_{12} &= -((\kappa c + jd)(\kappa d - |a|g) + (\kappa f + gj)(\kappa g - |a|d)), \\ h_{21} &= -((\kappa d + jc)(\kappa c - |a|f) + (\kappa g + fj)(\kappa f - |a|c)), \\ h_{22} &= ((\kappa^2 - |a|^2)(\kappa^2 - j^2) - (\kappa d + jc)(\kappa d - |a|g) - (\kappa g + fj)(\kappa g - |a|d)), \\ b_1 &= (\kappa h + ij)(\kappa^2 - |a|^2) + (\kappa c + jd)(\kappa b - |a|e) + (\kappa f + gj)(\kappa e - |a|b), \\ b_2 &= (\kappa i + hj)(\kappa^2 - |a|^2) + (\kappa d + jc)(\kappa b - |a|e) + (\kappa g + fj)(\kappa e - |a|b), \end{aligned}$$



and

$$\begin{aligned} g_{11} &= -((\kappa c + jd)(\kappa f - |a|c) + (\kappa d + jc)(\kappa g - |a|d)), \\ g_{12} &= ((\kappa^2 - |a|^2)(\kappa^2 - j^2) - (\kappa f + gj)(\kappa f - |a|c) - (\kappa g + fj)(\kappa g - |a|d)), \\ g_{21} &= ((\kappa^2 - |a|^2)(\kappa^2 - j^2) - (\kappa c + jd)(\kappa c - |a|f) - (\kappa d + jc)(\kappa d - |a|g)), \\ g_{22} &= -((\kappa f + gj)(\kappa c - |a|f) + (\kappa g + fj)(\kappa d - |a|g)), \\ c_1 &= (\kappa e - |a|b)(\kappa^2 - j^2) + (\kappa h + ij)(\kappa f - |a|c) + (\kappa i + hj)(\kappa g - |a|d), \\ c_2 &= (\kappa b - |a|e)(\kappa^2 - j^2) + (\kappa h + ij)(\kappa c - |a|f) + (\kappa i + hj)(\kappa d - |a|g). \end{aligned}$$

We will concentrate on finding the solution to equation (5.12) since the solution procedure for (5.13) is similar. From inspection, we have

$$h_{12} < 0, \quad h_{21} < 0, \quad b_1 > 0, \quad b_2 > 0.$$

From Cramer's rule, the solutions for w_3 and w_4 are

$$(5.14) \quad w_3 = \frac{\det \begin{pmatrix} b_1 & h_{12} \\ b_2 & h_{22} \end{pmatrix}}{\det \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}}, \quad w_4 = \frac{\det \begin{pmatrix} h_{11} & b_1 \\ h_{21} & b_2 \end{pmatrix}}{\det \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}}.$$

We now show that w_3 and w_4 have same sign. From (5.14), the expressions for w_3 and w_4 have the same denominator

$$D = h_{11}h_{22} - |h_{12}||h_{21}|,$$

hence, their signs are determined by the numerators. We will show that both numerators in (5.14) are positive.

The numerator of the expression for w_3 is

$$b_1h_{22} + |h_{12}|b_2.$$

A sufficient condition for this last expression to be positive is $h_{22} > 0$. The numerator of the expression for w_4 is

$$b_2h_{11} + |h_{21}|b_1.$$

A sufficient condition for this last expression to be positive is $h_{11} > 0$. We now show that h_{11} is strictly positive. Noting that $a = -|a|$, It is convenient to write h_{11} as

$$(5.15) \quad h_{11} = (\kappa^2 - a^2)(\kappa^2 - j^2) - (\kappa c + jd)(\kappa c + af) - (\kappa f + gj)(\kappa f + ac).$$

Consider the matrix

$$D = \begin{pmatrix} 1 & a & c & d \\ a & 1 & f & g \\ c & f & 1 & j \\ d & g & j & 1 \end{pmatrix}.$$

In this case, the characteristic equation is

$$\begin{aligned}\phi(y) = & y^4 - (a^2 + c^2 + d^2 + f^2 + g^2 + j^2)y^2 \\ & - 2(acf + adg + cdj + fgj)y - 2(acgj + adfj + cdfg).\end{aligned}$$

The matrix D is a sub matrix of the five-by-five matrix C . Hence, $\phi(\kappa) > 0$. Now we rewrite h_{11} in terms of ϕ . Use y instead of κ so that

$$(5.16) \quad h_{11}(y) = (y^2 - a^2)(y^2 - j^2) - (yc + jd)(yc + af) - (yf + gj)(yf + ac).$$

Manipulating the algebra we have

$$h_{11}(y) = \phi(y) + (d^2 + g^2)y^2 + a^2j^2 + y(cjd + fgj + 2adg) + (jda f + gjac + 2cdfg).$$

This can be written as

$$h_{11}(y) = \phi(y) + a^2j^2 + 2cdfg + [(d^2 + g^2)y^2 - 2|a|dgy] + [y(cjd + fgj) - (jd|a|f + gj|a|c)].$$

The first three terms are positive. We now show that the last two terms in the square brackets are also positive.

First we have

$$(d^2 + g^2)y^2 - 2|a|dgy = dy(yd - |a|g) + gy(yg - |a|d) > 0,$$

because

$$y > g, \quad d > |a|, \quad y > d, \quad g > |a|.$$

Turning to the second term

$$\begin{aligned}y(cjd + fgj) - (jd|a|f + gj|a|c) &= jd(yc - |a|f) + gj(yf - |a|c) \\ &> 0,\end{aligned}$$

because

$$y > f, \quad c > |a|, \quad y > c, \quad f > |a|.$$

In the same way, we show that h_{22} is also positive. Hence, w_3 and w_4 have the same sign. We can use similar arguments to show that w_1 and w_2 also have the same sign.

Hence, there are only four possibilities.

\mathcal{P}_1 : Both w_3 and w_4 are positive and both w_1 and w_2 are positive.

\mathcal{P}_2 : Both w_3 and w_4 are positive and both w_1 and w_2 are negative.

\mathcal{P}_3 : Both w_3 and w_4 are negative and both w_1 and w_2 are positive.

\mathcal{P}_4 : Both w_3 and w_4 are negative and both w_1 and w_2 are negative.



We now show that possibilities $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ cannot occur.

Possibility: \mathcal{P}_2

By adding the two first equations of (5.6), we have

$$(\kappa + |a|)(w_1 + w_2) = (b + e) + (c + f)w_3 + (d + g)w_4.$$

However this shows that \mathcal{P}_2 cannot occur.

Possibility: \mathcal{P}_3

By adding the two last equation of (5.6), we have

$$(\kappa - j)(w_3 + w_4) = (h + i) + (c + d)w_1 + (f + g)w_2.$$

However this shows that \mathcal{P}_3 cannot occur.

Possibility: \mathcal{P}_4

From the middle equation in (5.6), we have

$$\kappa = bw_1 + ew_2 + hw_3 + iw_4.$$

However this shows that \mathcal{P}_4 cannot occur.

Hence, only Possibility \mathcal{P}_1 can occur. Therefore, w_1, w_2, w_3 and w_4 are positive which implies that v_1, v_2, v_4 and v_5 are also positive. \square

From Table 4, we see that five-by-five correlation matrices with two or more negative correlations constitute 32 similarity classes. Since each similarity class can give rise to its own set of conditions we do not analyze these classes individually. However we can obtain clear cut results if there is a complete row of negative correlations. If this occurs the matrix cannot have a strictly positive dominant eigenvector. We need at least four negative correlations to have a complete row of negative correlations. If there are exactly four negative correlations we can obtain a complete row of negative correlations in five possible ways and they all belong to the same similarity class. However if there are four negative correlations the remaining 205 combinations representing five other similarity classes do not have a complete row of negative correlations.

We summarize the results for the incidence of complete rows of negative correlation in Table 5.

In this section, we analyzed five-by-five correlation matrices. These matrices give rise to 34 different similarity classes. We derived necessary and sufficient conditions for such matrices to have a strictly positive dominant eigenvector when there is one negative correlation. We also provided details of the incidence of complete rows of negative correlations.

6. General case: n -by- n correlation matrices. From our preceding analysis, we know that the conditions for the general n -by- n case will vary by both the number of negative elements and the similarity class. It is impractical to analyze all the possible similarity classes. Instead we obtained some specific results similar to those obtained in the last section for five-by-five matrices.

In particular, we derive necessary and sufficient conditions for an n -by- n correlation matrix to have a strictly positive dominant eigenvector when there is just one negative correlation. We also

TABLE 5

Analysis of 5-by-5 correlation matrix by number of negative correlations and by number of complete rows of negative correlations

Number of negative correlations n_c	Number of combinations that produce n_c negative correlations	Number of combinations that produce complete rows of negative correlations
0	1	0
1	10	0
2	45	0
3	120	0
4	210	5
5	252	30
6	210	75
7	120	90
8	45	45
9	10	10
10	1	1
Totals	1024	256

obtained some results for the number of random matrices which have a strictly positive dominant eigenvector. The correlation matrix in the general case is

$$(6.1) \quad C = \begin{bmatrix} 1 & c_{1,2} & c_{1,3} & c_{1,4} & \cdots & c_{1,n} \\ c_{1,2} & 1 & c_{2,3} & c_{2,4} & \cdots & c_{2,n} \\ c_{1,3} & c_{2,3} & 1 & c_{3,4} & \cdots & c_{3,n} \\ c_{1,4} & c_{2,4} & c_{3,4} & 1 & \cdots & c_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,n} & c_{2,n} & c_{3,n} & \cdots & c_{n-1,n} & 1 \end{bmatrix}.$$

We now use the notation $c_{i,j}$ instead of c_{ij} to denote the correlation coefficients. By including the separating comma, the notation becomes more transparent when we have terms like $c_{n-1,n}$. Where there is exactly one negative correlation there are n members of the similarity class. In this case we can assume the negative correlation is in position $c_{1,2}$.

Necessary conditions: one negative correlation.

THEOREM 6.1 (*n -by- n matrix with one negative entry: Necessary Condition*). *Let C be an n -by- n correlation matrix with exactly one negative correlation (say $c_{1,2}$). A necessary condition for C to have a strictly positive, dominant eigenvector is that the sum of the n correlations is positive:*

$$(6.2) \quad \sum_{i=1}^{(n-1)} \sum_{j=(i+1)}^n c_{i,j} > 0.$$

Proof. We assume that C has a strictly positive dominant eigenvector and derive a contradiction. Assume (6.2) is false so that

$$(6.3) \quad \sum_{i=1}^{(n-1)} \sum_{j=(i+1)}^n c_{i,j} \leq 0.$$

We assume that $c_{1,2} < 0$ so that (6.3) implies

$$(6.4) \quad |c_{1,2}| \geq \sum_{k=3}^n c_{1,k} + \sum_{i=2}^{(n-1)} \sum_{j=(i+1)}^n c_{i,j}.$$

The n linear equations in this case are

$$(6.5) \quad \begin{cases} \kappa v_1 + |c_{1,2}|v_2 &= c_{1,3}v_3 + c_{1,4}v_4 + \cdots + c_{1,n}v_n, \\ |c_{1,2}|v_1 + \kappa v_2 &= c_{2,3}v_3 + c_{2,4}v_4 + \cdots + c_{2,n}v_n, \\ \kappa v_3 &= c_{1,3}v_1 + c_{2,3}v_2 + \cdots + c_{3,n}v_n, \\ \kappa v_4 &= c_{1,4}v_1 + c_{2,4}v_2 + \cdots + c_{4,n}v_n, \\ \vdots &= \vdots \\ \kappa v_n &= c_{1,n}v_1 + c_{2,n}v_2 + \cdots + c_{n-1,n}v_n. \end{cases}$$

Adding together all n equations we obtain

$$\sum_{i=1}^n \psi_i v_i = 0.$$

From our assumptions, each ψ_i is positive and so we have a contradiction. This completes the proof. \square

Sufficient conditions: One negative correlation.

THEOREM 6.2 (n -by- n matrix with one negative entry: Sufficient Condition). *Let C be an n -by- n correlation matrix with exactly one negative correlation (say $c_{1,2}$). A sufficient condition for C to have a strictly positive, dominant eigenvector is:*

$$(6.6) \quad |c_{1,2}| < \min \{c_{i,j} : i \neq 1, j \neq 2\},$$

and

$$(6.7) \quad 2|c_{1,2}| < \sum_{j=3}^n c_{1,j} c_{2,j}.$$

Proof. It is straightforward to show that if conditions (6.6) and (6.7) are satisfied the matrix C^2 has all its entries positive. This means that C^2 has a strictly positive dominant eigenvector by the Perron-Frobenius theorem. Since C and C^2 have the same dominant eigenvector the proof follows. \square

The proof of this theorem is very much simpler than the corresponding result for the five-by-five case because of the additional condition (6.7). However for $n \geq 5$, virtually all the correlation matrices that satisfy (6.6) also satisfy (6.7). Hence, very little is lost by assuming (6.7).

To illustrate this point for $n = 5, 6, 7$ and 8, we conducted a numerical experiment. We generated a large² number of random correlation matrices that satisfied condition (6.6). We now explain the generation procedure using the five-by-five case. First we generate ten independent random numbers that are uniformly distributed in $(0, 1)$. Label these as

$$r(1), r(2), \dots, r(10).$$

Suppose $r(j)$ is the smallest of these. Switch $r(j)$ and $r(1)$ and change the sign of the first member of the revised sequence. Using our early notation the revised set of correlations can be represented as

$$-|a|, b, c, d, e, f, g, h, i, j,$$

where

$$|a| < \min \{b, c, d, e, f, g, h, i, j\}.$$

Note that under this construction, condition (6.6) is automatically satisfied. We repeated this procedure ten million times and for each simulation we checked if condition (6.7) was satisfied. It turns out that a very high percentage of the ten million simulations that satisfy condition (6.6) also satisfy condition (6.7) for the five-by-five case and the numbers become even higher with increasing n . Table 6 illustrates this point.

TABLE 6
Simulations to illustrate the extent to which correlation matrices satisfying condition (6.6) also satisfy (6.7).

Value of n	Number of simulations that satisfy (6.6)	Number of simulations that satisfy (6.7)
5	10,000,000	9,971,693
6	10,000,000	9,999,524
7	10,000,000	9,999,996
8	10,000,000	10,000,000

Fraction of n -by- n matrices with Perron-Frobenius property. There is a simple formula for the proportion of (random) correlation matrices which have a strictly positive dominant eigenvector. This proportion declines with increasing n . Suppose the N correlations are independently and identically distributed and each is drawn from the uniform distribution on $(-1, 1)$. We can assume that the first component of the dominant eigenvector is positive. The probability that each of the remaining components is positive is one half. Since there are $(n - 1)$ of them, the probability that all of them are positive is $\frac{1}{2^{n-1}}$. We can confirm by direct calculation that this formula works for $n = 2$ and $n = 3$. We have confirmed numerically that it is valid for high dimensional matrices up to order ten.

² Ten million.

Any n -by- n correlation matrix can be represented by a point in N -dimensional space. The entire set of these matrices corresponds to the N -dimensional hypercube $(-1, 1)^N$ with volume 2^N . The subset of these matrices which have a strictly positive dominant eigenvalue occupies a region of this hypercube with volume

$$2^{\frac{n^2-n+2}{2}}.$$

Note that

$$\frac{2^{\frac{n^2-n+2}{2}}}{2^N} = \frac{1}{2^{n-1}}.$$

The proportion of n -by- n random matrices that *do not* have a strictly positive dominant eigenvector is therefore

$$\frac{2^{n-1} - 1}{2^{n-1}}.$$

Table 7 shows the proportions of random matrices that have and the proportions of random matrices that do not have a strictly positive dominant eigenvector for different values of n . For $n = 3$ the proportion with the Perron-Frobenius property is 25% but by the time we reach eight-by-eight matrices the proportion is less than one percent.

TABLE 7

Proportions of random matrices with and without a strictly positive dominant eigenvector for various values of n .

n	Proportion of random matrices that have a strictly positive dominant eigenvector	Proportion of random matrices that do not have a strictly positive dominant eigenvector
3	0.2500	0.7500
4	0.1250	0.8750
5	0.0625	0.9375
6	0.0313	0.9688
7	0.0156	0.9844
8	0.0078	0.9922

7. Summary. Correlation matrices play an important role in multivariate statistics. In the context of certain portfolio problems, the dominant eigenvector of the correlation matrix of stock-returns plays a key role. It is of both practical and theoretical interest to understand the conditions under which the elements of this eigenvector are strictly positive. This paper has examined how the characteristics of the negative entries in a correlation matrix are related to the positivity of the dominant eigenvector. The matrix groupings induced by similarity transformations based on the permutation matrix provide a natural framework for our analysis. We derived detailed results for two-by-two, three-by-three and four-by-four correlation matrices. We also obtained some specific results for higher order correlation matrices when there is just one negative correlation. We established a negative row condition which provides a sufficient condition for a correlation matrix *not to have* a strictly positive dominant eigenvector.



Acknowledgements. The authors are grateful to the referee for constructive comments. They are also grateful for feedback received from seminar participants at the University of Waterloo, Georgia State University, and the African Institute for Mathematical Sciences in M'bour, Senegal. They thank Yun En Tan, Yingluo Xun, Zhiyi (Joey) Shen and Danqiao Guo for research assistance. They also thank Junichi Imai, Philip Knight and Shui Feng for comments on earlier versions. Boyle acknowledges support from the Natural Sciences and Engineering Research Council of Canada. Bocar thanks Cheikh Ndao for research guidance.

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