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# BLOCK REPRESENTATION AND SPECTRAL PROPERTIES OF CONSTANT SUM MATRICES*

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**Abstract.** An equivalent representation of constant sum matrices in terms of block-structured matrices is given in this paper. This provides an easy way of constructing all constant sum matrices, including those with further symmetry properties. The block representation gives a convenient description of the dihedral equivalence of such matrices. It is also shown how it can be used to study their spectral properties, giving explicit formulae for eigenvalues and eigenvectors in special situations, as well as for quasi-inverses when these exist.

**Key words.** Constant sum matrix, Eigenvalue, Quasi-inverse matrix.

**AMS subject classifications.** 15B99, 15B51, 15A18, 15A09.

## 1. Introduction.

In this paper, we study square matrices with the following symmetry properties.

Definition 1.1. An $n \times n$ matrix $M \in \mathbb{R}^{n\times n}$ is called *a constant sum matrix of weight* $w$ if each of its rows and columns sums to $nw$. If, in addition, each of its two principal diagonals also adds up to $nw$, it is called a *diagonal constant sum matrix.* We call $M$ a *non-diagonal constant sum matrix* if it is a constant sum matrix which is not a diagonal constant sum matrix (see [9]).

Matrices of this type, often with additional properties, are of interest in various areas of mathematics. For example, a constant sum matrix is called a *doubly stochastic matrix* if the entries are non-negative and the sums are equal to 1, an *alternating sign matrix* if it is a matrix with entries in $\{-1, 0, 1\}$ where each row and column sums to 1 and the nonzero entries in each row and column alternate in sign (see [2]), and a *semimagic square* if its entries are integers (see [1], [13]). The following two centre-point symmetry types of semimagic square matrices are of interest.

Definition 1.2. Let $M$ be an $n \times n$ constant sum matrix of weight $w$.

(a) The matrix $M$ is called *associated* if each entry and its mirror entry with respect to the centre of the matrix add to $2w$, i.e., if

$$M_{ij} + M_{n+1-i,n+1-j} = 2w \qquad (i, j \in \{1, \dots, n\}).$$

(b) The matrix $M$ is called *balanced* (or *centro-symmetric*) if each entry is equal to its mirror entry with respect to the centre of the matrix, i.e.,

$$M_{ij} - M_{n+1-i,n+1-j} = 0 \qquad (i, j \in \{1, \dots, n\}).$$

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These properties can be equivalently expressed in a more convenient way using the following conventions. Let $1_n \in \mathbb{R}^n$ be the vector which has all entries equal to 1. Let $\mathcal{E}_n = (1)_{i,j=1}^n \in \mathbb{R}^{n\times n}$ be the $n\times n$ matrix which has all entries equal to 1. Moreover, let $\mathcal{J}_n = (\delta_{i,n+1-j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$, where $\delta$ is the Kronecker delta symbol, be the $n\times n$ matrix which has entries 1 on the antidiagonal and 0 otherwise. Later, we shall also use the notations $0_n$ for the null vector in $\mathbb{R}^n$, $\mathcal{O}_n = (0)_{i,j=1}^n$ for the $n\times n$ null matrix, and $\mathcal{I}_n = (\delta_{ij})_{i,j=1}^n$ for the $n\times n$ unit matrix.

Note that we consider the elements of $\mathbb{R}^n$ as column vectors, i.e., matrices with $n$ rows and a single column, throughout; row vectors are represented in the form $v^T$, where $v \in \mathbb{R}^n$.

Lemma 1.3. *Let $M \in \mathbb{R}^{n\times n}$.*

*(a) The following statements are equivalent.*
- *(i) $M$ is a constant sum matrix of weight $w$;*
- *(ii) $M\mathcal{E}_n = nw\mathcal{E}_n = M^T\mathcal{E}_n$;*
- *(iii) $1_n$ is an eigenvector, with eigenvalue $nw$, for both $M$ and its transpose $M^T$.*

*(b) If $M$ is a constant sum matrix, then it is associated if and only if $M + \mathcal{J}_n M \mathcal{J}_n = 2w\mathcal{E}_n$.*
*(c) If $M$ is a constant sum matrix, then it is balanced if and only if $M = \mathcal{J}_n M \mathcal{J}_n$.*

Let $S_n$ be the set of all $n\times n$ constant sum matrices, and $A_n, B_n$ the subsets of associated and balanced constant sum matrices, respectively. Then $S_n$ is a subalgebra of the standard $n\times n$ matrix algebra, and $A_n$ and $B_n$ are vector subspaces of $S_n$. Moreover, $B_n$ is a subalgebra of $S_n$, and

$$A_n A_n \subset B_n, \qquad A_n B_n \subset A_n, \qquad B_n A_n \subset A_n \tag{1.1}$$

(see [10], [4] Lemma 1.1; it was already observed in [12] that all odd powers of an associated $3\times 3$ magic square were associated). Furthermore, $A_n$ and $B_n$ generate the whole algebra $S_n$ in the following way. Denoting by $S_n^o$ the subset (in fact, subalgebra) of weight 0 constant sum matrices, and setting $A_n^o = A_n \cap S_n^o$, $B_n^o = B_n \cap S_n^o$, we have

$$A_n \cap B_n = \mathbb{R}\mathcal{E}_n, \qquad A_n^o \cap B_n^o = \{\mathcal{O}_n\}, \quad \text{and} \quad S_n = A_n^o + B_n^o + \mathbb{R}\mathcal{E}_n,$$

where every $n\times n$ constant sum matrix of weight $w$ can be written as a sum of unique elements of $A_n^o$ and $B_n^o$, and $w\mathcal{E}_n$ (see [4] Lemma 2.3).

In Section 2 of the present paper, we introduce an equivalent *block representation* of constant sum matrices of even dimension by means of conjugation with a fixed symmetric involution matrix $\mathcal{X}_n$. In the transformed representation, split into a $2\times 2$ array of block matrix components, the associated and balanced types correspond to the off-diagonal and on-diagonal blocks, respectively, so the above decomposition of constant sum matrices can easily be read off. Moreover, the block representation provides a simple way of constructing all constant sum matrices, and since the rationality of entries is preserved in the transformation, also helps in the construction of semimagic squares. We mention in passing that the block representation also makes other matrix symmetries more transparently accessible [6].

In Section 3, we provide the block representation for constant sum matrices of odd dimension. Apart from some technical differences, it has the same essential properties and advantages as in the even-dimensional case.

The remainder of the paper is dedicated to using the block representation as a tool to study properties of constant sum matrices. In Section 4, we show how the dihedral symmetries are reflected in the block

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representation. This is of interest when the matrix is considered as an array of numbers irrespective of its overall orientation, e.g. when classifying semimagic squares.

Section 5 is dedicated to the question when a constant sum matrix is a diagonal constant sum matrix. We show how this property (which all associated constant sum matrices have due to their structure) is characterised in terms of the block representation (Theorem 5.1), and then use this to show that some power of a diagonal constant sum matrix is either a non-diagonal constant sum matrix or has all entries equal.

In Section 6, we consider associated constant sum matrices and their (balanced) squares, focusing on the case of matrices with two rank-1 blocks, for which we give explicit eigenvalue formulae. Section 7 presents the explicit construction of a two-sided eigenvector matrix (whose columns are right eigenvectors while its rows are left eigenvectors) for the square of any rank 1 + 1 associated constant sum matrix, concluding with concrete examples. In Section 8, we find maximal ranks for (weighted and unweighted) associated and balanced constant sum matrices of even or odd dimension and give an explicit formula, based on the block representation, for their quasi-inverses if they exist.

**2. Block representation of even dimensional constant sum matrices.** We first consider $2n \times 2n$ matrices, $n \in \mathbb{N}$. Let

$$\mathcal{X}_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Clearly $\mathcal{X}_{2n}^2 = \mathcal{I}_{2n}$ and $\mathcal{X}_{2n}^T = \mathcal{X}_{2n}$, so $\mathcal{X}_{2n}$ is an orthogonal symmetric involution (see [7] pp. 165–166). Conjugation with the matrix $\mathcal{X}_{2n}$ gives rise to a block representation of matrices in $S_{2n}$ in which the symmetry type can easily be read off. This also provides a convenient and systematic way of constructing constant sum matrices with (or without) centre-point symmetries.

Theorem 2.1. *Let $M \in \mathbb{R}^{2n \times 2n}$.*

*(a) The matrix $M \in A_{2n}^o$ if and only if*

$$M = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & V^T \\ W & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n},$$

*where $V, W \in \mathbb{R}^{n \times n}$ have row sum 0, i.e.,*

$$V 1_n = 0_n, \qquad W 1_n = 0_n.$$

*(b) The matrix $M \in B_{2n}^o$ if and only if*

$$M = \mathcal{X}_{2n} \begin{pmatrix} Y & \mathcal{O}_n \\ \mathcal{O}_n & Z \end{pmatrix} \mathcal{X}_{2n},$$

*where $Y \in S_n^o$ and $Z \in \mathbb{R}^{n \times n}$.*

*(c) The weight matrix $\mathcal{E}_{2n}$ satisfies*

$$\mathcal{E}_{2n} = \mathcal{X}_{2n} \begin{pmatrix} 2\mathcal{E}_n & \mathcal{O}_n \\ \mathcal{O}_n & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n}.$$

In view of the decomposition of general constant sum matrices mentioned above, we see that, as a consequence of Theorem 2.1, any even-dimensional, weight $w$ constant sum matrix has, after conjugation

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with $\mathcal{X}_{2n}$, the block representation

$$\begin{pmatrix} Y + 2w\mathcal{E}_n & V^T \\ W & Z \end{pmatrix},$$

with a weight 0 constant sum matrix $Y$, matrices $V, W$ with row sum 0 and a matrix $Z$ which can be any $n \times n$ matrix. Evidently, this block representation clearly shows the decomposition into an associated and a balanced matrix, corresponding to setting the two diagonal or the two off-diagonal blocks equal to 0, respectively.

From the block representation, it is very straightforward to generate all matrices in $A^o_{2n}$ (and hence, by adding a multiple of $\mathcal{E}_{2n}$, in $A_{2n}$); indeed, the conditions on the matrices $V, W$ can very easily be satisfied, as $n-1$ columns can be arbitrary when the last column is chosen so that the rows add to 0. The construction of a general matrix in $B_{2n}$ is a bit more complicated, as $Z$ can be chosen arbitrarily, but $Y$ must be a constant sum matrix. At least this reduces the dimension of the problem from $2n$ to $n$.

*Proof of Theorem* 2.1. We begin by writing the matrix $M$ in the form

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where $A, B, C, D \in \mathbb{R}^{n\times n}$; then

$$\mathcal{J}_{2n} M \mathcal{J}_{2n} = \begin{pmatrix} \mathcal{J}_n D \mathcal{J}_n & \mathcal{J}_n B \mathcal{J}_n \\ \mathcal{J}_n C \mathcal{J}_n & \mathcal{J}_n A \mathcal{J}_n \end{pmatrix}.$$

(a) Assume that $M \in A^o_{2n}$. Then, by Lemma 1.3 (b), we have $\mathcal{O}_{2n} = M + \mathcal{J}_{2n} M \mathcal{J}_{2n}$, which is equivalent to $B = -\mathcal{J}_n C \mathcal{J}_n$, $D = -\mathcal{J}_n A \mathcal{J}_n$. Hence,

$$\mathcal{X}_{2n} M \mathcal{X}_{2n} = \begin{pmatrix} \mathcal{O}_n & A\mathcal{J}_n - C \\ \mathcal{J}_n A + \mathcal{J}_n C \mathcal{J}_n & \mathcal{O}_n \end{pmatrix}. \tag{2.2}$$

By Lemma 1.3 (a), $M$ is a weight 0 constant sum matrix if and only if $1_{2n}$ is an eigenvector of both $M$ and $M^T$ for eigenvalue 0. In view of Lemma 4.1, this is equivalent to

$$\mathcal{X}_{2n} 1_{2n} = \sqrt{2} \begin{pmatrix} 1_n \\ 0_n \end{pmatrix} \tag{2.3}$$

being an eigenvector, for eigenvalue 0, of both $\mathcal{X}_{2n} M \mathcal{X}_{2n}$ and $(\mathcal{X}_{2n} M \mathcal{X}_{2n})^T$. From (2.2), this corresponds to the conditions

$$\mathcal{J}_n(A + C\mathcal{J}_n)1_n = 0_n, \qquad (A\mathcal{J}_n - C)^T 1_n = 0_n.$$

Thus, $V = (A\mathcal{J}_n - C)^T$ and $W = \mathcal{J}_n A + \mathcal{J}_n C \mathcal{J}_n$ will have row sum 0.

Conversely, given $V, W \in \mathbb{R}^{n\times n}$ with row sum 0, we take $A = \frac{1}{2}(V^T \mathcal{J}_n + \mathcal{J}_n W)$, $C = \frac{1}{2}(\mathcal{J}_n W \mathcal{J}_n - V^T)$ and further $B = -\mathcal{J}_n C \mathcal{J}_n$, $D = -\mathcal{J}_n A \mathcal{J}_n$, to construct a weight 0 constant sum matrix.

(b) Assume $M \in B^o_n$. Then, by Lemma 1.3 (c), we have $\mathcal{O}_{2n} = M - \mathcal{J}_{2n} M \mathcal{J}_{2n}$, which is equivalent to $B = \mathcal{J}_n C \mathcal{J}_n$, $D = \mathcal{J}_n A \mathcal{J}_n$. Hence,

$$\mathcal{X}_{2n} M \mathcal{X}_{2n} = \begin{pmatrix} A + C\mathcal{J}_n & \mathcal{O}_n \\ \mathcal{O}_n & \mathcal{J}_n A \mathcal{J}_n - \mathcal{J} C \end{pmatrix}. \tag{2.4}$$

As before, the condition that $M$ is a constant sum matrix of weight 0 means that (2.3) is an eigenvector of both $\mathcal{X}_{2n} M \mathcal{X}_{2n}$ and $(\mathcal{X}_{2n} M \mathcal{X}_{2n})^T$ for eigenvalue 0; by (2.4) this is equivalent to

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$$(A + C\mathcal{J}_n)1_n = 0_n, \qquad (A + C\mathcal{J})^T 1_n = 0_n.$$

By Lemma 1.3 (a), $Y = A + C\mathcal{J}_n \in S_n^o$.

Conversely, given $Y \in S_n^o$ and $Z \in \mathbb{R}^{n\times n}$, we take $A = \frac{1}{2}(Y + \mathcal{J}_n Z \mathcal{J}_n)$, $C = \frac{1}{2}(Y\mathcal{J}_n - \mathcal{J}_n Z)$, and further $B = \mathcal{J}_n C \mathcal{J}_n$, $D = \mathcal{J}_n A \mathcal{J}_n$, to construct a weight 0 balanced constant sum matrix.

(c) This is a straightforward calculation. ∎

**3. Block representation of odd dimensional constant sum matrices.** We now consider $(2n+1)\times(2n+1)$ matrices, $n \in \mathbb{N}$. Let

$$\mathcal{X}_{2n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}}\mathcal{I}_n & 0_n & \frac{1}{\sqrt{2}}\mathcal{J}_n \\ 0_n^T & 1 & 0_n^T \\ \frac{1}{\sqrt{2}}\mathcal{J}_n & 0_n & -\frac{1}{\sqrt{2}}\mathcal{I}_n \end{pmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)}.$$

(The matrix $\mathcal{X}_{2n+1}$ turns into $\mathcal{X}_{2n}$ when its central row and column are deleted.) The matrix $\mathcal{X}_{2n+1}$ is a symmetric involution, i.e., $\mathcal{X}_{2n+1}^T = \mathcal{X}_{2n+1}$ and $\mathcal{X}_{2n+1}^2 = \mathcal{I}_{2n+1}$. Conjugation with $\mathcal{X}_{2n+1}$ gives rise to the following block representation of odd-dimensional constant sum matrices. We emphasise that despite the presence of $\sqrt{2}$ in the matrix $\mathcal{X}_{2n+1}$, the constant sum matrix will have rational entries if and only if its block representation has rational entries. This is important for the application to the construction of semimagic square matrices.

Theorem 3.1. *Let $M \in \mathbb{R}^{(2n+1)\times(2n+1)}$.*

*(a) The matrix $M \in A_{2n+1}^o$ if and only if*

$$M = \mathcal{X}_{2n+1} \begin{pmatrix} \mathcal{O}_n & 0_n & V^T \\ 0_n^T & 0 & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n+1}$$

*with matrices $V, W \in \mathbb{R}^{n\times n}$. Moreover, $M$ will have rational entries if and only if $V$ and $W$ have rational entries.*

*(b) The matrix $M \in B_{2n+1}^o$ if and only if*

$$M = \mathcal{X}_{2n+1} \begin{pmatrix} Y & -\sqrt{2}Y1_n & \mathcal{O}_n \\ -\sqrt{2}(Y^T 1_n)^T & 2\,1_n^T Y 1_n & 0_n^T \\ \mathcal{O}_n & 0_n & Z \end{pmatrix} \mathcal{X}_{2n+1}$$

*with matrices $Y, Z \in \mathbb{R}^{n\times n}$. Moreover, $M$ will have rational entries if and only if $Y$ and $Z$ have rational entries.*

*(c) The matrix $\mathcal{E}_{2n+1}$ satisfies*

$$\mathcal{E}_{2n+1} = \mathcal{X}_{2n+1} \begin{pmatrix} 2\mathcal{E}_n & \sqrt{2}\,1_n & \mathcal{O}_n \\ \sqrt{2}\,1_n^T & 1 & 0_n^T \\ \mathcal{O}_n & 0_n & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n+1}.$$

Note that there are no conditions on the matrices $V, W, Y$ and $Z$ in Theorem 3.1, so the block representation gives a very simple way of constructing all odd-dimensional constant sum matrices (with or without

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centre-point symmetries). Indeed, it is evident from the theorem that the general element of $S_{2n+1}$ will be of the form

$$M = \mathcal{X}_{2n+1} \begin{pmatrix} Y + 2w\mathcal{E}_n & \sqrt{2}(w\mathcal{I}_n - Y)1_n & V^T \\ \sqrt{2}((w\mathcal{I}_n - Y)1_n)^T & w + 2\,1_n^T Y 1_n & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & Z \end{pmatrix} \mathcal{X}_{2n+1}$$

with arbitrary $V, W, Y, Z \in \mathbb{R}^{n\times n}$. Note that, in contrast to the even-dimensional case, adding the weight $w$ to $M$ is not equivalent to adding a weight to the matrix $Y$.

*Proof of Theorem* 3.1. Writing $M$ in the form

$$M = \begin{pmatrix} A & v & C \\ w^T & x & y^T \\ B & z & D \end{pmatrix}$$

with $x \in \mathbb{R}$, $v, w, y, z \in \mathbb{R}^n$ and $A, B, C, D \in \mathbb{R}^{n\times n}$, we find

$$\mathcal{J}_{2n+1} M \mathcal{J}_{2n+1} = \begin{pmatrix} \mathcal{J}_n D \mathcal{J}_n & \mathcal{J}_n z & \mathcal{J}_n B \mathcal{J}_n \\ y^T \mathcal{J}_n & x & w^T \mathcal{J}_n \\ \mathcal{J}_n C \mathcal{J}_n & \mathcal{J}_n v & \mathcal{J}_n A \mathcal{J}_n \end{pmatrix}.$$

(a) The weight 0 association condition $\mathcal{O}_{2n+1} = M + \mathcal{J}_{2n+1} M \mathcal{J}_{2n+1}$ of Lemma 1.3 (b) hence gives $x = 0$, $y = -\mathcal{J}_n w$, $z = -\mathcal{J}_n v$, $B = -\mathcal{J}_n C \mathcal{J}_n$ and $D = -\mathcal{J}_n A \mathcal{J}_n$. Thus, if M is a weight 0 associated constant sum matrix, then

$$\mathcal{X}_{2n+1} M \mathcal{X}_{2n+1} = \begin{pmatrix} \mathcal{O}_n & 0_n & A\mathcal{J}_n - C \\ 0_n^T & 0 & \sqrt{2} w^T \mathcal{J}_n \\ \mathcal{J}_n A + \mathcal{J}_n C \mathcal{J}_n & \sqrt{2}\mathcal{J}_n v & \mathcal{O}_n \end{pmatrix}.$$

By Lemma 1.3 (a), $M \in S^o_{2n+1}$ if and only if $1_{2n+1}$ is an eigenvector with eigenvalue 0 of $M$ and of $M^T$. Since

$$\mathcal{X}_{2n+1} 1_{2n+1} = \begin{pmatrix} \sqrt{2}\,1_n \\ 1 \\ 0_n \end{pmatrix}, \tag{3.5}$$

we see that

$$0_{2n+1} = M 1_{2n+1} = M \mathcal{X}_{2n+1} \mathcal{X}_{2n+1} 1_{2n+1}$$

if and only if $0_n = W\sqrt{2}\,1_n + \sqrt{2}\mathcal{J}_n v$, where we set $W = \mathcal{J}_n A + \mathcal{J}_n C \mathcal{J}_n$; this gives $\mathcal{J}_n v = -W1_n$. Similarly, setting $V = (A\mathcal{J}_n - C)^T$,

$$0_{2n+1} = M^T 1_{2n+1} = M^T \mathcal{X}_{2n+1} \mathcal{X}_{2n+1} 1_{2n+1}$$

if and only if $0_n = V\sqrt{2}\,1_n + \sqrt{2}\mathcal{J}_n w$, and hence, $w^T \mathcal{J}_n = -(V1_n)^T$.

Conversely, given matrices $V, W \in \mathbb{R}^{n\times n}$, we take $A = \frac{1}{2}(\mathcal{J}_n W + V^T \mathcal{J}_n)$, $C = \frac{1}{2}(\mathcal{J}_n W \mathcal{J}_n - V^T)$ and further $B = -\mathcal{J}_n C \mathcal{J}_n$, $D = -\mathcal{J}_n A \mathcal{J}_n$, $x = 0$, $v = -\mathcal{J}_n W 1_n$, $w = -\mathcal{J}_n V 1_n$, $y = V1_n$ and $z = W1_n$ to

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construct a weight 0 associated constant sum matrix. From these formulae, it is evident that $M$ has rational entries if and only if $V, W$ have rational entries.

(b) The balanced condition $\mathcal{O}_{2n+1} = M - \mathcal{J}_{2n+1} M \mathcal{J}_{2n+1}$ of Lemma 1.3 (c) gives $z = \mathcal{J}_n v$, $y = \mathcal{J}_n w$, $B = \mathcal{J}_n C \mathcal{J}_n$ and $D = \mathcal{J}_n A \mathcal{J}_n$; there is no condition on $x$. Then

$$\mathcal{X}_{2n+1} M \mathcal{X}_{2n+1} = \begin{pmatrix} A + C\mathcal{J}_n & \sqrt{2}v & \mathcal{O}_n \\ \sqrt{2}w^T & x & 0_n^T \\ \mathcal{O}_n & 0_n & \mathcal{J}_n A \mathcal{J}_n - \mathcal{J}_n C \end{pmatrix}$$

and hence, using Lemma 1.3 (a) and (3.5), $0_{2n+1} = M 1_{2n+1}$ if and only if

$$0_{n+1} = \begin{pmatrix} \sqrt{2}((A + C\mathcal{J}_n)1_n + v) \\ 2w^T 1_n + x \end{pmatrix},$$

giving $x = -2w^T 1_n$ and $v = -Y 1_n$, where we set $Y = A + C\mathcal{J}_n$. Similarly, $0_{2n+1} = M^T 1_{2n+1}$ if and only if

$$0_{n+1} = \begin{pmatrix} \sqrt{2}(Y^T 1_n + w) \\ 2v^T 1_n + x \end{pmatrix},$$

so $x = -2v^T 1_n$ and $w = -Y^T 1_n$. This determines $v$ and $w$ and gives two conditions on $x$, which turn out to be each equivalent to $x = 2\,1_n^T Y 1_n$.

For the converse, we take $A = \frac{1}{2}(Y + \mathcal{J}_n Z \mathcal{J}_n)$, $C = \frac{1}{2}(Y\mathcal{J}_n - \mathcal{J}_n Z)$, and further $B = \mathcal{J}_n C \mathcal{J}_n$, $D = \mathcal{J}_n A \mathcal{J}_n$, $v = -Y 1_n$, $w = -Y^T 1_n$, $x = 2\,1_n^T Y 1_n$, $y = -\mathcal{J}_n Y^T 1_n$ and $z = -\mathcal{J}_n Y 1_n$ to construct a weight 0 balanced constant sum matrix. From these formulae it is evident that $M \in \mathbb{Q}^{(2n+1)\times(2n+1)}$ if and only if $Y, Z \in \mathbb{Q}^{n\times n}$.

(c) This is a straightforward calculation. ∎

**4. Dihedral symmetry.** As the matrix $\mathcal{X}_n$ is symmetric (i.e., equal to its transpose) and involutory (i.e., its own inverse matrix), the following observation is straightforward.

LEMMA 4.1. *(a) The block representation is an algebra isomorphism; indeed,*

$$\mathcal{X}_n(\alpha M + N)\mathcal{X}_n = \alpha \mathcal{X}_n M \mathcal{X}_n + \mathcal{X}_n N \mathcal{X}_n \qquad (\alpha \in \mathbb{R},\ M, N \in \mathbb{R}^{n\times n})$$

*and*

$$\mathcal{X}_n(MN)\mathcal{X}_n = \mathcal{X}_n M \mathcal{X}_n \mathcal{X}_n N \mathcal{X}_n \qquad (M, N \in \mathbb{R}^{n\times n}).$$

*(b) The block representation of the transposed matrix is the transpose of the block representation of the original matrix; indeed,*

$$\mathcal{X}_n M^T \mathcal{X}_n = (\mathcal{X}_n M \mathcal{X}_n)^T \qquad (M \in \mathbb{R}^{n\times n}).$$

The unique decomposition of an $n \times n$ constant sum matrix into weightless associated and balanced constant sum matrices and a multiple of $\mathcal{E}_n$ can easily be read off its block representation. Associated and balanced constant sum matrices are clearly identified by the presence and position of null blocks in their block representation. Moreover, the block representation can be used to characterise other matrix symmetries in a convenient manner. For example, in the study of semimagic or magic squares, it is usually the relative

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arrangement of numbers in the square which is the object of interest, not their fixed positioning in a matrix; therefore matrices which differ only by rotation or reflection will be identified with one another. In other words, the equivalence classes in $S_n$ with respect to the dihedral group for the $n\times n$ square will be considered. In the block representation, the action of the dihedral group translates into transposition and sign inversion of the right or bottom half of the block matrix. Thus, we have the following result.

THEOREM 4.2. *Let $M \in S_n$ and*

$$M = \mathcal{X}_n \begin{pmatrix} \tilde{Y} & \tilde{V}^T \\ \tilde{W} & Z \end{pmatrix} \mathcal{X}_n \tag{4.6}$$

*its block representation. Then the dihedral equivalence class of $M$ is*

$$\left\{ \mathcal{X}_n \begin{pmatrix} \tilde{Y} & s\tilde{V}^T \\ t\tilde{W} & stZ \end{pmatrix} \mathcal{X}_n, \mathcal{X}_n \begin{pmatrix} \tilde{Y}^T & s\tilde{W}^T \\ t\tilde{V} & stZ^T \end{pmatrix} \mathcal{X}_n \mid s,t \in \{1,-1\} \right\}.$$

REMARK 4.3. If $n$ is even, then $\tilde{Y}, \tilde{V}, \tilde{W}$ are the matrices $Y, V, W$ of Theorem 2.1; if $n$ is odd, then they include parts of the centre row and column in the block representation of Theorem 3.1. (In particular, $\tilde{V}$ and $\tilde{W}$ will then no longer be square matrices.) In either case, $Z$ is the matrix denoted by the same letter in Theorems 2.1, 3.1.

*Proof of Theorem 4.2.* The dihedral group is generated by the two operations of reflection along the diagonal and reflection along the horizontal centreline. In matrix terms, these correspond to taking the transpose of the matrix and to left multiplication with the matrix $\mathcal{J}_n$, respectively. Transposition carries over directly to the block representation, as seen in Lemma 4.1 (b). By the product formula in Lemma 4.1 (a), left multiplication of $M$ with $\mathcal{J}_n$ translates into left multiplication of its block representation with the matrix

$$\mathcal{X}_n \mathcal{J}_n \mathcal{X}_n = \begin{pmatrix} \mathcal{I}_k & \mathcal{O}_k \\ \mathcal{O}_k & -\mathcal{I}_k \end{pmatrix}$$

in the even-dimensional case $n = 2k$, and

$$\mathcal{X}_n \mathcal{J}_n \mathcal{X}_n = \begin{pmatrix} \mathcal{I}_{k+1} & \mathcal{O}_{k+1,k} \\ \mathcal{O}_{k,k+1} & -\mathcal{I}_k \end{pmatrix}$$

in the odd-dimensional case $n = 2k+1$; here $\mathcal{O}_{k,l}$ denotes the null matrix with $k$ rows and $l$ columns. Hence,

$$\mathcal{J}_n M = \mathcal{X}_n \begin{pmatrix} \tilde{Y} & \tilde{V}^T \\ -\tilde{W} & -Z \end{pmatrix} \mathcal{X}_n. \tag{4.7}$$

Similarly, reflection of $M$ along the vertical centreline corresponds to multiplication with $\mathcal{J}_n$ on the right, and hence to inverting the sign of the rightmost $k$ columns in the block representation. (This operation is of course derived from the group generators as $(\mathcal{J}_n M^T)^T = M\mathcal{J}_n$.) ∎

**5. Diagonal constant sum matrices.** In a diagonal constant sum matrix, its two main diagonals also need to add up to the row and column sum. For associated constant sum matrices, this is always the case. However, for balanced constant sum matrices, which, since $\mathcal{E}_n$ is a diagonal constant sum matrix, we can assume without loss of generality to have weight 0, this gives an additional condition.

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The sum of the diagonal entries of the matrix $M$ is equal to its trace, $\operatorname{tr} M$; the sum of the entries on the second diagonal is the trace of the matrix after reflection along the horizontal (or vertical) centreline, i.e., equal to $\operatorname{tr}(\mathcal{J}_n M)$. As the trace of a product of matrices is invariant under cyclic permutations of its factors, and hence, the trace of a constant sum matrix is equal to the trace of its block representation, we see that for a weight 0 balanced constant sum matrix $M$, using the representation (4.6),

$$\operatorname{tr} M = \operatorname{tr}(\mathcal{X}_n \begin{pmatrix} \tilde{Y} & \mathcal{O} \\ \mathcal{O} & Z \end{pmatrix} \mathcal{X}_n) = \operatorname{tr} \tilde{Y} + \operatorname{tr} Z,$$

and by (4.7),

$$\operatorname{tr}(\mathcal{J}_n M) = \operatorname{tr}(\mathcal{X}_n \begin{pmatrix} \tilde{Y} & \mathcal{O} \\ \mathcal{O} & -Z \end{pmatrix} \mathcal{X}_n) = \operatorname{tr} \tilde{Y} - \operatorname{tr} Z.$$

As both must vanish for the matrix to be a weight 0 diagonal constant sum matrix, this means that the traces of $\tilde{Y}$ and of $Z$ must separately be 0. This gives rise to the following statement.

Theorem 5.1. *(a) A $2n \times 2n$ matrix $M$ is a balanced diagonal constant sum matrix of weight $w$ if and only if*

$$M = \mathcal{X}_{2n} \begin{pmatrix} Y & \mathcal{O}_n \\ \mathcal{O}_n & Z \end{pmatrix} \mathcal{X}_{2n},$$

*where $Y$ is an $n \times n$ constant sum matrix of weight $2w$ and $\operatorname{tr} Y = 2nw$, and $Z$ is any $n \times n$ matrix with $\operatorname{tr} Z = 0$.*

*(b) A $(2n+1) \times (2n+1)$ matrix $M$ is a balanced diagonal constant sum matrix of weight $w$ if and only if*

$$M = \mathcal{X}_{2n+1} \begin{pmatrix} \tilde{Y} & \mathcal{O}_{n+1,n} \\ \mathcal{O}_{n,n+1} & Z \end{pmatrix} \mathcal{X}_{2n+1},$$

*where*

$$\tilde{Y} = \begin{pmatrix} Y + 2w\mathcal{E}_n & \sqrt{2}(w\mathcal{I}_n - Y)1_n \\ \sqrt{2}((w\mathcal{I}_n - Y)1_n)^T & w + 2\,1_n^T Y 1_n \end{pmatrix},$$

*$Y, Z$ are any $n \times n$ matrices such that $\operatorname{tr} Y = -2\,1_n^T Y 1_n$ (corresponding to $\operatorname{tr} \tilde{Y} = (2n+1)w$) and $\operatorname{tr} Z = 0$.*

Remark 5.2. The condition on the matrix $Y$ in Theorem 5.1 (a) almost makes it a diagonal constant sum matrix, but not quite. For example, the matrix

$$Y = \begin{pmatrix} 1 & 2 & -3 \\ -6 & 1 & 5 \\ 5 & -3 & -2 \end{pmatrix}$$

is a weight 0 constant sum matrix with vanishing trace, but its second diagonal does not add up to 0. Nevertheless, this matrix is a suitable upper block $Y$ for the block representation of a $6 \times 6$ balanced diagonal constant sum matrix, choosing any $3 \times 3$ matrix $Z$ of vanishing trace for the lower block.

Remark 5.3. Bearing in mind that $1_n^T Y 1_n$ is just the sum of all entries of $Y$, the condition on the matrix $Y$ in Theorem 5.1 (b) means that the off-diagonal entries of $Y$ sum to $-\frac{3}{2}$ times its trace. For example,

$$Y = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

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has this property, and picking a matrix $Z$ with vanishing trace, we obtain a balanced diagonal constant sum matrix by the calculation

$$\mathcal{X}_5 \begin{pmatrix} 1 & -2 & \sqrt{2} & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix} \mathcal{X}_5 = \begin{pmatrix} 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 1 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \end{pmatrix}.$$

Remark 5.4. Theorem 5.1 gives a simple method of constructing balanced diagonal constant sum matrices. General diagonal constant sum matrices can be obtained by adding any associated constant sum matrix, or equivalently by filling in the off-diagonal blocks in the block representation with matrices satisfying the conditions in Theorem 2.1 (a) or 3.1 (a).

Definition 5.5. We call an $n \times n$ diagonal constant sum matrix *trivial* if it is a multiple of the matrix $\mathcal{E}_n$.

Any associated constant sum matrix is a diagonal constant sum matrix; by (1.1), its square will be a balanced constant sum matrix. One may wonder whether it is again a diagonal constant sum matrix. For the case of even-dimensional associated diagonal constant sum matrices with rank 1 blocks (and, without loss of generality, weight 0), the following alternative holds.

Theorem 5.6. *Consider the (non-trivial, weight 0) $2n \times 2n$ associated diagonal constant sum matrix*

$$M = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & V^T \\ W & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n}$$

*with rank 1 blocks $V, W$. Then exactly one of the following statements is true.*

*(a) $M^2$ has rank 2 and is a non-diagonal constant sum matrix, and all its powers are non-diagonal constant sum matrices;*
*(b) $M^2$ has rank 0 or 1 and is a nilpotent (see [7] p. 4) diagonal constant sum matrix, in fact $M^4 = \mathcal{O}_{2n}$.*

*Proof.* There are vectors $u, v, x, y \in \mathbb{R}^n \setminus \{0_n\}$ such that $V = uv^T$, $W = xy^T$. The block representation of $M^2$ is

$$M^2 = \mathcal{X}_{2n} \begin{pmatrix} V^T W & \mathcal{O}_n \\ \mathcal{O}_n & WV^T \end{pmatrix} \mathcal{X}_{2n} \tag{5.8}$$

with blocks

$$V^T W = vu^T xy^T, \qquad WV^T = xy^T vu^T.$$

If $u^T x = 0$ and $y^T v \neq 0$, then $V^T W = \mathcal{O}_n$ and $WV^T$ is a rank 1 matrix; it only has eigenvalue 0. Hence, both $V^T W$ and $WV^T$ have vanishing trace, so by Theorem 5.1, $M^2$ is a diagonal constant sum matrix. Furthermore, $WV^T WV^T = \mathcal{O}_n$, so the matrix is nilpotent.

The case where $y^T v = 0$ and $u^T x \neq 0$ is analogous. If both $u^T x = 0$ and $y^T v = 0$, then $M^2 = \mathcal{O}_{2n}$.

If both $u^T x \neq 0$ and $y^T v \neq 0$, then both $V^T W$ and $WV^T$ have a non-zero eigenvalue $(u^T x)(y^T v)$ with eigenvector $v \neq 0_n, x \neq 0_n$, respectively. Hence, each block in (5.8) has rank 1 and a non-zero trace, so $M^2$

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has rank 2 and, by Theorem 5.1, is a non-diagonal constant sum matrix. For any $N \in \mathbb{N}$, its $N$th power has block representation

$$M^{2N} = \mathcal{X}_{2n} \begin{pmatrix} (V^T W)^N & \mathcal{O}_n \\ \mathcal{O}_n & (WV^T)^N \end{pmatrix} \mathcal{X}_{2n},$$

and as both blocks have the non-zero eigenvalue $((u^T x)(v^T y))^N$, $M^{2N}$ is a non-diagonal constant sum matrix. ◻

It is an open question whether Theorem 5.6 generalises to higher-rank matrices. It is generally true that if $WV^T = \mathcal{O}_n$, then $V^T W$ has no non-zero eigenvalues, so $M^2$ will then be a nilpotent diagonal constant sum matrix; and similarly if $V^T W = \mathcal{O}_n$.

We have, however, the following general result.

Theorem 5.7. *Let $M$ be a balanced diagonal constant sum matrix of size $m \times m$, where $m = 2n$ or $m = 2n - 1$. Then there is $N \in \{1, \dots, n\}$ such that $M^N$ is either trivial or a non-diagonal constant sum matrix.*

Remark 5.8. Higher powers of $M$ may again be diagonal constant sum matrices. For example, for the balanced diagonal constant sum matrix

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = \mathcal{X}_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \mathcal{X}_4$$

the even powers are non-diagonal, the odd powers diagonal constant sum matrices. On the other hand, triviality, i.e., the property of being a multiple of $\mathcal{E}$, obviously persists to all higher powers.

*Proof of Theorem* 5.7. We can assume without loss of generality that $M$ has weight 0. By Theorem 2.1 (b) or Theorem 3.1 (b),

$$M = \mathcal{X}_m \begin{pmatrix} \tilde{Y} & \mathcal{O} \\ \mathcal{O} & Z \end{pmatrix} \mathcal{X}_m,$$

where $\tilde{Y}$ is an $n \times n$ matrix and $Z$ is an $n \times n$ or an $(n-1) \times (n-1)$ matrix, depending on whether $m$ is even or odd. If we assume that the powers $M, M^2, \dots, M^n$ are all diagonal constant sum matrices, then $\tilde{Y}, \tilde{Y}^2, \dots, \tilde{Y}^n$ and $Z, Z^2, \dots, Z^n$ all have trace 0, by Theorem 5.1. This implies that $\tilde{Y}$ and $Z$ are nilpotent (see [5] Lemma 6.16, bearing in mind the Cayley-Hamilton theorem); in particular, $\tilde{Y}^n = \mathcal{O}$, $Z^n = \mathcal{O}$, and hence, $M^n = \mathcal{O}_m$. ◻

For the case of a general diagonal constant sum matrix, we can apply the idea of the proof of Theorem 5.7 directly to the *whole* matrix, using the fact that for a weight 0 diagonal constant sum matrix it is necessary, though not sufficient, that its trace vanish. This gives the following general statement, with a less tight upper bound on the number $N$.

Theorem 5.9. *Let $M$ be a diagonal constant sum matrix. Then there is some positive integer $N$, not greater than the dimension of $M$, such that $M^N$ is either trivial or a non-diagonal constant sum matrix.*

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**6. Properties of constant sum matrices with rank 1 blocks.** Constant sum matrices with rank 1 blocks, as described in Theorem 5.6, are of interest since the case where the entries of $v$ and $y$ are $\pm 1$, such that they sum to 0, and the absolute values of the entries of $u$ and $x$ form a sum-and-distance system (see [8]), gives rise to traditional magic squares with consecutive integer entries. In Examples 7.3 and 7.4 of Section 7 below, the matrices $\frac{1}{2}(M + 65\mathcal{E}_8)$ are of this form.

When considering the spectral properties of constant sum matrices, it is sufficient to study the weightless case. Indeed, if $M_0$ is a weightless $n \times n$ constant sum matrix, then by Lemma 1.3 (a), $1_n$ is an eigenvector of $M_0$ with eigenvalue 0. As $\mathcal{E}_n = 1_n 1_n^T$, it follows by a theorem of Brauer (see [11] Theorem 2; [3] Theorem 27) that the weighted matrix $M_0 + w\mathcal{E}_n$ will have the same non-zero eigenvalues (including multiplicities) as $M_0$, and the additional eigenvalue $nw$, while the multiplicity of the eigenvalue 0 is reduced by 1. Moreover, the eigenvectors of $M_0 + w\mathcal{E}_n$ will be independent of $w$. In particular, $\operatorname{rk}(M_0 + w\mathcal{E}_n) = \operatorname{rk} M_0 + 1$.

Starting from the block representation of a weightless associated constant sum matrix $M_0$, and then considering the block matrix for its square $M_0^2$, we have

$$M_0 = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & V^T \\ W & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n}, \qquad M_0^2 = \mathcal{X}_{2n} \begin{pmatrix} V^T W & \mathcal{O}_n \\ \mathcal{O}_n & W V^T \end{pmatrix} \mathcal{X}_{2n}.$$

Hence, $M_0^2$ is the direct sum of the matrices $V^T W$ and $W V^T$, so it will be diagonalisable if both $V^T W$ and $W V^T$ are diagonalisable, in particular if they are symmetric. This motivates the following definition.

Definition 6.1. We call the associated constant sum matrix $M$ *parasymmetric* if its square $M^2$ is a symmetric matrix.

In terms of the block representation, parasymmetry can be characterised as follows if the two constituent blocks of $M$ have rank 1. Example 7.3 in Section 7 below gives an example of a parasymmetric constant sum matrix of this form.

Lemma 6.2. *Let $V, W \in \mathbb{R}^{n \times n}$ have row sum 0 and rank 1, and consider the associated constant sum matrix*

$$M = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & V^T \\ W & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n}.$$

*If $M^2 \neq \mathcal{O}_{2n}$, then $M$ is parasymmetric if and only if $W$ is a multiple of $V$.*

*Proof.* We can write $V = uv^T$, $W = xy^T$ with non-null vectors $u, v, x, y \in \mathbb{R}^n \setminus \{0_n\}$; then $V^T = vu^T$ and $W^T = yx^T$. Now if $V^T W$ is symmetric, then

$$(u^T x) v y^T = V^T W = W^T V = (x^T u) y v^T,$$

so either $u^T x = 0$ or $vy^T = yv^T$; but in the latter case we see, multiplying by $y$ on the right, that $v(y^T y) = y(v^T y)$, so $v$ and $y$ are linearly dependent. Similarly, if $W V^T$ is symmetric, then either $v^T y = 0$ or $u$ and $x$ are linearly dependent.

Now, if $u^T x = 0$, then also $v^T y = 0$ (since otherwise, by the above, the non-null vectors $u, x$ would be simultaneously orthogonal and linearly dependent), and vice versa; and in this situation $V^T W = \mathcal{O}_n = W V^T$, which would imply $M^2 = \mathcal{O}_{2n}$.

Hence, we find that $x, y$ are multiples of $u, v$, respectively, so $W$ is a multiple of $V$. The converse statement is obvious. ∎

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More generally, a matrix is diagonalisable by conjugation with an orthogonal matrix if it commutes with its transpose, i.e., if it is *normal.* If $M$ is normal, then it is easy to see that $M^2$ is normal, too; the converse is not so clear. In analogy to parasymmetry, we call the matrix $M$ *paranormal* if $M^2$ is normal. However, this turns out to be no more general than parasymmetry as far as associated constant sum matrices with rank 1 blocks are concerned, as the following result shows.

Lemma 6.3. *Let $M \in \mathbb{R}^{2n\times 2n}$ be a weightless associated constant sum matrix with rank 1 blocks $V, W$ such that $M^2 \neq \mathcal{O}_{2n}$. If $M$ is paranormal, then it is parasymmetric.*

*Proof.* Let $V, W$ and $u, v, x, y$ be as in the proof of Lemma 6.2. Then the paranormality means that

$$V^TWW^TV = W^TVV^TW, \qquad WV^TVW^T = VW^TWV^T;$$

in terms of the generating vectors, this gives the two identities

$$\begin{aligned} v(u^Tx)(y^Ty)(x^Tu)v^T &= y(x^Tu)(v^Tv)(u^Tx)y^T, \\ x(y^Tv)(u^Tu)(v^Ty)x^T &= u(v^Ty)(x^Tx)(y^Tv)u^T, \end{aligned}$$

and by the same reasoning as above, this implies that $u^Tx = 0$ or $v, y$ are linearly dependent, and that $v^Ty = 0$ or $x, u$ are linearly dependent. As before, the cases of orthogonality can only occur together and then give a trivial $M^2$; it follows that $V, W$ are linearly dependent and hence that the matrix is parasymmetric. ∎

The following observation on the eigenvalues of a squared weightless associated constant sum matrix with rank 1 blocks follows from the proof of Theorem 5.6.

Theorem 6.4. *If $M$ is the weightless associated constant sum matrix*

$$M = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n} \tag{6.9}$$

*with rank 1 block components $V = uv^T$, $W = xy^T$, then $M^2$ has eigenvalues 0 and $(u^Tx)(y^Tv) = \operatorname{tr} V^TW$. In particular, if $V, W$ have integer entries, then the eigenvalues of $M^2$ are integers.*

Remark 6.5. If the eigenvalue $(u^Tx)(y^Tv)$ is non-zero, then its (algebraic and geometric) multiplicity will be 2, as it will be an eigenvalue of both the upper left and the lower right blocks in the block representation of $M^2$, with eigenvectors $\begin{pmatrix} v \\ 0_n \end{pmatrix}$ and $\begin{pmatrix} 0_n \\ x \end{pmatrix}$, respectively. However, when exactly one of the products $V^TW, WV^T$ vanishes, then its geometric multiplicity will only be 1. This can happen in the non-parasymmetric case, as the non-orthogonality of generating vectors for non-trivial diagonal constant sums, found in the proof of Lemma 6.2, need not hold in this case. Note that in this situation, this eigenvalue will indeed be 0, so the matrix $M^2$ will have eigenvalue 0 only with algebraic multiplicity $2n$ and geometric multiplicity $2n-1$. For example, consider

$$V = uv^T = \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix} (1, -1, -1, 1), \qquad W = xy^T = 8 \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix} (1, -1, 1, -1);$$

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here $x$ is a multiple of $u$, but $y^T v = 0$, so $WV^T = 0$. The resulting weightless associated constant sum matrix

$$M = \frac{1}{2}\begin{pmatrix} 63 & -61 & 53 & -55 & -57 & 59 & -51 & 49 \\ -47 & 45 & -37 & 39 & 41 & -43 & 35 & -33 \\ -31 & 29 & -21 & 23 & 25 & -27 & 19 & -17 \\ 15 & -13 & 5 & -7 & -9 & 11 & -3 & 1 \\ -1 & 3 & -11 & 9 & 7 & -5 & 13 & -15 \\ 17 & -19 & 27 & -25 & -23 & 21 & -29 & 31 \\ 33 & -35 & 43 & -41 & -39 & 37 & -45 & 47 \\ -49 & 51 & -59 & 57 & 55 & -53 & 61 & -63 \end{pmatrix}$$

has rank 2, but its square $M^2$ only has rank 1.

Remark 6.6. If $(u^T x)(y^T v) \neq 0$, then corresponding linearly independent (right) eigenvectors of $M^2$ will be

$$\sqrt{2}\mathcal{X}_{2n}\begin{pmatrix} v \\ 0_n \end{pmatrix} = \begin{pmatrix} v \\ \mathcal{J}_n v \end{pmatrix}, \qquad -\sqrt{2}\mathcal{X}_{2n}\begin{pmatrix} 0_n \\ x \end{pmatrix} = \begin{pmatrix} -\mathcal{J}_n x \\ x \end{pmatrix}; \tag{6.10}$$

the first of these is even, the other odd under reflection (i.e., multiplication with $\mathcal{J}_{2n}$). These eigenvectors are orthogonal for structural reasons, reflecting the fact that they belong to different direct summands in the block representation of $M^2$. Clearly, if $v, x$ have integer entries, then so do these eigenvectors.

We note that the left eigenvectors (i.e., eigenvectors of $(M^2)^T$) are given by

$$\begin{pmatrix} y \\ \mathcal{J}_n y \end{pmatrix}, \begin{pmatrix} -\mathcal{J}_n u \\ u \end{pmatrix}; \tag{6.11}$$

again, these eigenvectors are structurally orthogonal.

Remark 6.7. In the parasymmetric case $y = v$, $x = ku$, where $u, v \in \mathbb{R}^n \setminus \{0\}$ and $k \neq 0$, the matrix $M^2$ always has a non-zero eigenvalue $k(u^T u)(v^T v)$.

Remark 6.8. In the situation of Theorem 6.4 with $u^T x \neq 0$, $y^T v \neq 0$, the matrix $M$ has the two simple non-zero eigenvalues $\sqrt{(u^T x)(y^T v)}, -\sqrt{(u^T x)(y^T v)}$. Indeed, any eigenvector of the block representation

$$\begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix}$$

for non-zero eigenvalue $\lambda$ can easily be seen to be of the form $\begin{pmatrix} \alpha v \\ \beta x \end{pmatrix}$ with $\alpha, \beta \neq 0$, and hence, $\lambda^2 = (u^T x)(y^T v)$. As the trace of the matrix vanishes and 0 is the only other eigenvalue, both signs of the square root occur. The coefficients for the eigenvectors can be chosen as $\alpha = \lambda$, $\beta = y^T v$, so if $u, v, x, y$ are integer vectors and $\lambda$ is an integer, then there are integer eigenvectors as well. In the parasymmetric case with integer vectors $u, v$ and integer parasymmetry factor $k$, the eigenvalues will be integers if and only if the square-free part of $k$ is equal to the square-free part of the product $(v^T v)(u^T u)$.

**7. A two-sided eigenvector matrix.** In this section, we show a construction yielding a two-sided, regular eigenvector matrix for $M^2$, where $M$ is the rank $1+1$ associated constant sum matrix as defined in (6.9). Here 'two-sided' means that the columns of the matrix are right eigenvectors of $M^2$ while its rows are

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left eigenvectors of $M^2$. We begin by considering the two right eigenvectors (6.10) of $M^2$ corresponding to the non-zero eigenvalue $\lambda = (u^T x)(y^T v)$. These eigenvectors are placed side by side to form a $2n \times 2$ matrix

$$P_1 = \begin{pmatrix} B_1 \\ A_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} v & -\mathcal{J}_n x \\ \mathcal{J}_n v & x \end{pmatrix},$$

where $A_1 = \begin{pmatrix} v_n & -x_1 \\ v_n & x_1 \end{pmatrix}$ and $B_1$ and $C_1$ are $(n-1) \times 2$ matrices such that $C_1 = \mathcal{J}_{n-1} B_1 \sigma_3$ (with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$). We make the assumption that $v_n, x_1 \neq 0$, so that the matrix $A_1$ is regular. The construction below can be generalised to the case where $A_1$ is any regular matrix composed of two rows of $P_1$ (and indeed one can always find two linearly independent rows of $P_1$ because its columns are linearly independent), but we take the centre rows in the following for simplicity.

Now define

$$\tilde{P}_1 = -P_1 A_1^{-1} = \begin{pmatrix} -B_1 A_1^{-1} \\ -\mathcal{I}_2 \\ -C_1 A_1^{-1} \end{pmatrix}. \tag{7.12}$$

Similarly, starting from the matrix of left eigenvectors (6.11), we set

$$P_2 = \begin{pmatrix} B_2 \\ A_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} y & -\mathcal{J}_n u \\ \mathcal{J}_n y & u \end{pmatrix},$$

and assuming $y_n, u_1 \neq 0$, so that $A_2 = \begin{pmatrix} y_n & -u_1 \\ y_n & u_1 \end{pmatrix}$ is regular, we define

$$\tilde{P}_2 = -P_2 A_2^{-1} = \begin{pmatrix} -B_2 A_2^{-1} \\ -\mathcal{I}_2 \\ -C_2 A_2^{-1} \end{pmatrix}. \tag{7.13}$$

The columns of $\tilde{P}_1$ and $\tilde{P}_2$ will still be linearly independent eigenvectors, for eigenvalue $\lambda$, of $M^2$ and of $(M^2)^T$, respectively, but in general they will no longer be orthogonal unless $v_n^2 = x_1^2$ and $y_n^2 = u_1^2$. However, due to the special structure of our chosen matrices $A_j$, they will have the symmetry that the second column is the reversal of the first. Indeed, $\mathcal{J}_2 A_j \sigma_3 = A_j$, so $\sigma_3 A_j^{-1} \mathcal{J}_2 = A_j^{-1}$ $(j \in \{1, 2\})$, and it follows that

$$\mathcal{J}_{2n} \tilde{P}_j \mathcal{J}_2 = \begin{pmatrix} -\mathcal{J}_{n-1} C_j A_j^{-1} \\ -\mathcal{J}_2 \\ -\mathcal{J}_{n-1} B_j A_j^{-1} \end{pmatrix} \mathcal{J}_2 = \begin{pmatrix} -B_j \sigma_3 A_j^{-1} \mathcal{J}_2 \\ -\mathcal{I}_2 \\ -C_j \sigma_3 A_j^{-1} \mathcal{J}_2 \end{pmatrix} = \tilde{P}_j \qquad (j \in \{1, 2\}).$$

Hence, swapping the columns of $\tilde{P}_j$ is equivalent to turning them upside down.

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The matrices $\tilde{P}_1$ and $\tilde{P}_2$ have the following remarkable connection with the matrix $M$. Using the notation $(\cdot \mid \cdot \mid \cdot)$ to express that the three matrices are juxtaposed to form one $2n \times 2n$ matrix, we calculate

$$\begin{aligned}(7.14)\qquad &(\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \ \mathcal{O}_{2n,n-1})M \\ &= -\frac{1}{2}\, P_1 \left(\mathcal{O}_{2,n-1} \mid A_1^{-1}\mathcal{X}_2 \mid \mathcal{O}_{2,n-1}\right) \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \\ &= -\frac{1}{2}\, P_1 \left( A_1^{-1} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} y^T \mid A_1^{-1} \begin{pmatrix} v_n \\ v_n \end{pmatrix} u^T \right) \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \begin{pmatrix} v & 0_n \\ 0_n & -x \end{pmatrix} \begin{pmatrix} 0_n^T & u^T \\ -y^T & 0_n^T \end{pmatrix} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} = -M,\end{aligned}$$

and similarly $(\mathcal{O}_{2n,n-1} \mid \tilde{P}_2 \mid \mathcal{O}_{2n,n-1})M^T = -M^T$.

THEOREM 7.1. *Let $u, v, x, y \in \mathbb{R}^n$ such that $\lambda := (u^T x)(y^T v) \neq 0$ and $u_1, v_n, x_1, y_n$ are non-zero. Let $M$ be the matrix (6.9), and let $\tilde{P}_1$ and $\tilde{P}_2$ be defined as in (7.12), (7.13). Then*

$$P = \mathcal{I}_{2n} + (\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) + \begin{pmatrix} \mathcal{O}_{n-1,2n} \\ \tilde{P}_2^T \\ \mathcal{O}_{n-1,2n} \end{pmatrix}$$

*is a two-sided eigenvector matrix for $M^2$, so that*

$$M^2 P = P\,\mathrm{diag}(0_{n-1}, \lambda 1_2, 0_{n-1}) \text{ and } PM^2 = \mathrm{diag}(0_{n-1}, \lambda 1_2, 0_{n-1})P.$$

*Moreover, $P$ has the inverse*

$$P^{-1} = \mathrm{diag}(1_{n-1}, 0_2, 1_{n-1}) - \frac{M^2}{\lambda}.$$

*Proof.* Using the second of the above identities and the fact that the columns of $\tilde{P}_1$ are eigenvectors of $M^2$, we find

$$\begin{aligned} M^2 P &= M^2 + M^2(\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) + MM \begin{pmatrix} \mathcal{O}_{n-1,2n} \\ \tilde{P}_2^T \\ \mathcal{O}_{n-1,2n} \end{pmatrix} \\ &= M^2 + \lambda(\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) + M(-M);\end{aligned}$$

on the other hand, since the central $2 \times 2$ part of $\tilde{P}_2$ is equal to $-\mathcal{I}_2$, we have

$$\begin{aligned}(7.15)\qquad & P\,\mathrm{diag}(0_{n-1}, \lambda 1_2, 0_{n-1}) \\ &= \mathrm{diag}(0_{n-1}, \lambda 1_2, 0_{n-1}) + \lambda(\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) - \mathrm{diag}(0_{n-1}, \lambda 1_2, 0_{n-1}).\end{aligned}$$

The relation for $PM^2$ follows by a pair of completely analogous calculations.

To verify the formula for the inverse $P^{-1}$, we note that

$$\begin{aligned} &(\mathrm{diag}(1_{n-1}, 0_2, 1_{n-1}) - \frac{M^2}{\lambda})P = \mathrm{diag}(1_{n-1}, 0_2, 1_{n-1})P - P\,\mathrm{diag}(0_{n-1}, 1_2, 0_{n-1}) \\ &= \mathrm{diag}(1_{n-1}, 0_2, 1_{n-1}) + \Big((\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) + \mathrm{diag}(0_{n-1}, 1_2, 0_{n-1})\Big) + \mathcal{O}_{2n} \\ &\quad - \mathrm{diag}(0_{n-1}, 1_2, 0_{n-1}) - (\mathcal{O}_{2n,n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{2n,n-1}) + \mathrm{diag}(0_{n-1}, 1_2, 0_{n-1}) = \mathcal{I}_{2n}.\end{aligned}$$

The opposite product follows similarly. ∎

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REMARK 7.2. In the parasymmetric case, we have $y = v, x = ku$, which, following the above construction, gives rise to the eigenvector matrices

$$P_1 = \begin{pmatrix} y & -\mathcal{J}_n ku \\ \mathcal{J}_n y & ku \end{pmatrix} = P_2 \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

However, in this situation the vector $\begin{pmatrix} -\mathcal{J}_n u \\ u \end{pmatrix}$ will be an eigenvector just as well as $\begin{pmatrix} -\mathcal{J}_n ku \\ ku \end{pmatrix}$, so we can begin with this vector and take $P_1 = P_2$ instead of the above. Hence, in this instance, $P = \mathcal{I}_{2n} + (\mathcal{O}_{n-1} \mid \tilde{P}_1 \mid \mathcal{O}_{n-1}) + (\mathcal{O}_{n-1} \mid \tilde{P}_2 \mid \mathcal{O}_{n-1})^T$ will be a symmetric matrix.

We now illustrate these results with a parasymmetric and a non-parasymmetric example.

EXAMPLE 7.3. Let $u^T = (11, -13, -19, 21)$, $x = 2u$ and $v^T = y^T = (-1, 1, 1, -1)$ be our vectors in $\mathbb{R}^4$, and

$$M = \mathcal{X}_8 \begin{pmatrix} \mathcal{O}_4 & vu^T \\ xy^T & \mathcal{O}_4 \end{pmatrix} \mathcal{X}_8 = \frac{1}{2} \begin{pmatrix} -63 & 61 & 55 & -53 & -31 & 29 & 23 & -21 \\ 59 & -57 & -51 & 49 & 27 & -25 & -19 & 17 \\ 47 & -45 & -39 & 37 & 15 & -13 & -7 & 5 \\ -43 & 41 & 35 & -33 & -11 & 9 & 3 & -1 \\ 1 & -3 & -9 & 11 & 33 & -35 & -41 & 43 \\ -5 & 7 & 13 & -15 & -37 & 39 & 45 & -47 \\ -17 & 19 & 25 & -27 & -49 & 51 & 57 & -59 \\ 21 & -23 & -29 & 31 & 53 & -55 & -61 & 63 \end{pmatrix}.$$

Then $M^2$ has the symmetric block representation

$$M^2 = 8\mathcal{X}_8 \begin{pmatrix} 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 121 & -143 & -209 & 231 \\ 0 & 0 & 0 & 0 & -143 & 169 & 247 & -273 \\ 0 & 0 & 0 & 0 & -209 & 247 & 361 & -399 \\ 0 & 0 & 0 & 0 & 231 & -273 & -399 & 441 \end{pmatrix} \mathcal{X}_8.$$

The matrix $M^2$ has the eigenvectors (6.10) (dividing $b_2$ by $k$, as suggested in the above remark)

$$b_1 = \begin{pmatrix} v \\ \mathcal{J}_n v \end{pmatrix}, \qquad b_2 = \begin{pmatrix} -\mathcal{J}_n u \\ u \end{pmatrix}$$

for the eigenvalue $\lambda = k(u^T u)(v^T v) = 8736$.

Applying the construction of Theorem 7.1, we obtain the rational symmetric (left and right) eigenvector matrix

$$P = \frac{1}{11} \begin{pmatrix} 11 & 0 & 0 & -16 & 5 & 0 & 0 & 0 \\ 0 & 11 & 0 & 15 & -4 & 0 & 0 & 0 \\ 0 & 0 & 11 & 12 & -1 & 0 & 0 & 0 \\ -16 & 15 & 12 & -11 & 0 & -1 & -4 & 5 \\ 5 & -4 & -1 & 0 & -11 & 12 & 15 & -16 \\ 0 & 0 & 0 & -1 & 12 & 11 & 0 & 0 \\ 0 & 0 & 0 & -4 & 15 & 0 & 11 & 0 \\ 0 & 0 & 0 & 5 & -16 & 0 & 0 & 11 \end{pmatrix}.$$

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We note that Theorem 7.1 also gives the inverse for $P$, indeed

$$\begin{aligned} P^{-1} &= \operatorname{diag}(1_3, 0_2, 1_3) - \frac{M^2}{\lambda} \\ &= \frac{8}{\lambda}\begin{pmatrix} 735 & 336 & 273 & -252 & -21 & 0 & -63 & 84 \\ 336 & 775 & -260 & 241 & 32 & -13 & 44 & -63 \\ 273 & -260 & 871 & 208 & 65 & -52 & -13 & 0 \\ -252 & 241 & 208 & -197 & -76 & 65 & 32 & -21 \\ -21 & 32 & 65 & -76 & -197 & 208 & 241 & -252 \\ 0 & -13 & -52 & 65 & 208 & 871 & -260 & 273 \\ -63 & 44 & -13 & 32 & 241 & -260 & 775 & 336 \\ 84 & -63 & 0 & -21 & -252 & 273 & 336 & 735 \end{pmatrix}. \end{aligned}$$

EXAMPLE 7.4. Let $u^T = (10, -14, -18, 22)$, $x^T = (23, -25, -39, 41)$ and $v^T = y^T = (-1, 1, 1, -1) \in \mathbb{R}^4$, and

$$M = \mathcal{X}_8 \begin{pmatrix} \mathcal{O}_4 & vu^T \\ xy^T & \mathcal{O}_4 \end{pmatrix} \mathcal{X}_8 = \begin{pmatrix} -63 & 59 & 55 & -51 & -31 & 27 & 23 & -19 \\ 61 & -57 & -53 & 49 & 29 & -25 & -21 & 17 \\ 47 & -43 & -39 & 35 & 15 & -11 & -7 & 3 \\ -45 & 41 & 37 & -33 & -13 & 9 & 5 & -1 \\ 1 & -5 & -9 & 13 & 33 & -37 & -41 & 45 \\ -3 & 7 & 11 & -15 & -35 & 39 & 43 & -47 \\ -17 & 21 & 25 & -29 & -49 & 53 & 57 & -61 \\ 19 & -23 & -27 & 31 & 51 & -55 & -59 & 63 \end{pmatrix}.$$

Then $M^2$ has the block representation

$$M^2 = 8\mathcal{X}_8 \begin{pmatrix} 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 115 & -161 & -207 & 253 \\ 0 & 0 & 0 & 0 & -125 & 175 & 225 & -275 \\ 0 & 0 & 0 & 0 & -195 & 273 & 351 & -429 \\ 0 & 0 & 0 & 0 & 205 & -287 & -369 & 451 \end{pmatrix} \mathcal{X}_8,$$

which is clearly non-symmetric. The non-zero eigenvalue is $\lambda = (u^T x)(y^T v) = 8736$, and the (right and left) eigenvector matrix constructed as in Theorem 7.1 has the form

$$P = \frac{1}{115}\begin{pmatrix} 115 & 0 & 0 & -160 & 45 & 0 & 0 & 0 \\ 0 & 115 & 0 & 155 & -40 & 0 & 0 & 0 \\ 0 & 0 & 115 & 120 & -5 & 0 & 0 & 0 \\ -184 & 161 & 138 & -115 & 0 & -23 & -46 & 69 \\ 69 & -46 & -23 & 0 & -115 & 138 & 161 & -184 \\ 0 & 0 & 0 & -5 & 120 & 115 & 0 & 0 \\ 0 & 0 & 0 & -40 & 155 & 0 & 115 & 0 \\ 0 & 0 & 0 & 45 & -160 & 0 & 0 & 115 \end{pmatrix}.$$

**8. Ranks and quasi-inverses.** In the beginning of Section 6, we observed that adding a non-zero weight increases the rank of a weightless constant sum matrix by 1. However, for an even-dimensional associated constant sum matrix, this can never lead to full rank. Indeed, since the block component matrices $V, W$ in the block representation

$$\begin{pmatrix} 2w\mathcal{E}_n & V^T \\ W & \mathcal{O}_n \end{pmatrix}$$

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have row sum 0 and therefore rank $\leq n-1$, the total rank of the matrix is at most $2n-2$ if the weight is 0, and $2n-1$ otherwise. In particular, an even-dimensional associated constant sum matrix is never regular and does not have an inverse.

An $n \times n$ constant sum matrix with weight zero will always have the vector $1_n$ in its null space and therefore cannot be regular; however we can define a (left or right) *quasi-inverse* to be a matrix which multiplies the given matrix (to the left or right) to give the weightless part of the (constant sum) unit matrix,

$$\mathcal{U}_n := \mathcal{I}_n - \frac{1}{n}\,\mathcal{E}_n.$$

An even-dimensional associated constant sum matrix, weighted or not, will have neither a left nor a right quasi-inverse, as can be seen from the block representation

$$\mathcal{U}_{2n} = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{U}_n & \mathcal{O}_n \\ \mathcal{O}_n & \mathcal{I}_n \end{pmatrix} \mathcal{X}_{2n}; \tag{8.16}$$

the upper right-hand block of a right quasi-inverse of the block representation would have to be a right inverse of $W$, while the lower left-hand block of a left quasi-inverse would have to be a left inverse of $V$, and clearly neither is possible.

In the odd-dimensional case, the block representation is (see Theorem 3.1)

$$\begin{pmatrix} 2w\mathcal{E}_n & w\sqrt{2}\,1_n & V^T \\ w\sqrt{2}\,1_n^T & w & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & \mathcal{O}_n \end{pmatrix},$$

where the matrices $V, W$ may have full rank $n$. Hence, the maximal rank is $2n$ if the weight is 0 and $2n+1$ otherwise; in the latter case, the matrix has full rank and therefore an inverse.

For a rank $2n$, $(2n+1)\times(2n+1)$ associated constant sum matrix with weight 0, a right quasi-inverse can always be constructed, bearing in mind that $V, W$ have full rank $n$ and the matrix $\mathcal{I}_n + 2\mathcal{E}_n$ is regular, as $-\frac{1}{2}$ is not an eigenvalue of $\mathcal{E}_n$. Indeed, we find

$$\begin{pmatrix} 0 & 0 & V^T \\ 0 & 0 & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & V'^T \\ 0 & 0 & -\sqrt{2}(V'1_n)^T \\ W' & -\sqrt{2}W'1_n & 0 \end{pmatrix} = \mathcal{X}_{2n+1}\mathcal{U}_{2n+1}\mathcal{X}_{2n+1},$$

taking $W' := (V^T)^{-1}(1 - \frac{2\mathcal{E}_n}{2n+1})$, $V'^T := (1+2\mathcal{E}_n)^{-1}W^{-1}$. Note that the quasi-inverse is again the block structure matrix of a weight 0 associated constant sum matrix. Here

$$\mathcal{X}_{2n+1}\mathcal{U}_{2n+1}\mathcal{X}_{2n+1} = \begin{pmatrix} 1 - \frac{2\mathcal{E}_n}{2n+1} & -\sqrt{2}\frac{1_n}{2n+1} & \mathcal{O}_n \\ -\sqrt{2}\frac{1_n^T}{2n+1} & 1 - \frac{1}{2n+1} & 0_n^T \\ \mathcal{O}_n & 0_n & \mathcal{I}_n \end{pmatrix}.$$

In summary, we have the following statement.

Theorem 8.1. *(a) Even-dimensional associated constant sum matrices, with or without weight, never have full rank, nor a quasi-inverse.*

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*(b) Odd-dimensional associated constant sum matrices may have full rank if weighted; if the weight is 0, then the maximal rank is 1 less than the dimension, and in this case left and right quasi-inverses exist.*

Turning to the case of (weightless) balanced constant sum matrices, we note that in the block representation for the odd-dimensional case,

$$\begin{pmatrix} Y & -\sqrt{2}Y1_n & 0 \\ -\sqrt{2}1_n^TY & 2\,1_n^TY1_n & 0 \\ 0 & 0 & Z \end{pmatrix},$$

the top left $(n+1)\times(n+1)$ matrix $\tilde{Y}$ has maximal rank $n$ (when $Y$ has rank $n$), because the $n+1$st row is a linear combination of the first $n$ rows. Therefore the maximal rank of the balanced constant sum matrix, achieved if both $Y, Z$ have full rank $n$, is $2n$, and there is no inverse. However, there is always a quasi-inverse in this case; with $Y' := (1+2\mathcal{E}_n)^{-1}Y^{-1}(I_n - \frac{2}{2n+1}\mathcal{E}_n)$,

$$\begin{pmatrix} Y & -\sqrt{2}Y1_n \\ -\sqrt{2}\,1_n^TY & 2\,1_n^TY1_n \end{pmatrix}\begin{pmatrix} Y' & -\sqrt{2}Y'1_n \\ -\sqrt{2}\,1_n^TY' & 2\,1_n^TY'1_n \end{pmatrix} = \begin{pmatrix} \mathcal{I}_n - \frac{2}{2n+1}\mathcal{E}_n & \frac{-\sqrt{2}}{2n+1}1_n \\ \frac{-\sqrt{2}}{2n+1}1_n^T & 1-\frac{1}{2n+1} \end{pmatrix},$$

and completing a $(2n+1)\times(2n+1)$ matrix with $Z^{-1}$ in the lower right corner, we obtain the block representation of a right quasi-inverse which is again a weightless balanced constant sum matrix.

The block representation of a $2n$-dimensional weightless balanced constant sum matrix is

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix},$$

where $Y$ is a weightless constant sum matrix. Hence, the maximal possible rank is $2n-1$, and there will not be an inverse. Regarding a quasi-inverse, we see from (8.16) that $Z$ must be invertible and $Y$ must have a quasi-inverse. If $n$ is odd and $Y$ has maximal rank $n-1$ and is either associated or balanced, then there exists a quasi-inverse by the above considerations. If $n$ is even and $Y$ is associated, then no quasi-inverse exists. If $n$ is even and $Y$ is balanced, then we can apply these considerations recursively to the block structure of $Y$.

Unfortunately, the case of mixed type, i.e., of a general weightless constant sum matrix, seems rather more difficult to analyse; this case will generally occur when applying the above reduction to an even-dimensional balanced constant sum matrix.

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