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NORM INEQUALITIES RELATED TO CLARKSON INEQUALITIES*

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Abstract. Let A and B be $n \times n$ matrices. It is shown that if $p = 2$, $4 \leq p < \infty$, or $2 < p < 4$, and both $A + B$, $A - B$ are positive semidefinite, then

$$\|A+B\|_p^p + \|A-B\|_p^p \leq 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) - \left(2^{p/2}-2\right)\left|\|A\|_p - \|B\|_p\right|^p,$$

and if $p = 2$, $4 \leq p < \infty$, or $2 < p < 4$, and both A, B are positive semidefinite, then

$$\|A+B\|_p^p + \|A-B\|_p^p \geq 2\left(\|A\|_p^p + \|B\|_p^p\right) + (2^{1-p/2} - 2^{2-p})\left|\|A+B\|_p - \|A-B\|_p\right|^p.$$

These inequalities are reversed if $p = 2$, $1 \leq p \leq \frac{4}{3}$, or $\frac{4}{3} < p < 2$, and both $A + B$, $A - B$ are positive semidefinite, and if $p = 2$, $1 \leq p \leq \frac{4}{3}$, or $\frac{4}{3} < p < 2$, and both A, B are positive semidefinite, respectively. Commutative (or L_p) versions of these inequalities are also considered.

Key words. Clarkson inequalities, Hanner's inequality, Schatten p-norm, L_p function, Singular value.

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1. Introduction. Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $s_1(A), s_2(A), \ldots, s_n(A)$ denote the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$.

For $0 < p < \infty$ and $A \in \mathbb{M}_n(\mathbb{C})$, define $\|A\|_p$ by

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}.$$

So, $\|A\|_p = (\mathrm{tr}\ |A|^p)^{1/p}$. For $1 \leq p < \infty$, this is the Schatten p-norm of A.

The celebrated Clarkson inequalities for scalars assert that if $a, b \in \mathbb{C}$, then

$$2(|a|^p + |b|^p) \leq |a+b|^p + |a-b|^p \leq 2^{p-1}(|a|^p + |b|^p) \tag{1.1}$$

for $2 \leq p < \infty$, and

$$2^{p-1}(|a|^p + |b|^p) \leq |a+b|^p + |a-b|^p \leq 2(|a|^p + |b|^p) \tag{1.2}$$

for $0 < p \leq 2$.

Generalizations of the inequalities (1.1) and (1.2) to matrices can be seen as follows (see, e.g., [8]): If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$2\left(\|A\|_p^p + \|B\|_p^p\right) \leq \|A+B\|_p^p + \|A-B\|_p^p \leq 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) \tag{1.3}$$

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for $2 \leq p < \infty$, and

$$2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) \leq \|A+B\|_p^p + \|A-B\|_p^p \leq 2\left(\|A\|_p^p + \|B\|_p^p\right) \tag{1.4}$$

for $0 < p \leq 2$.

The inequalities (1.3) and (1.4) play an important role in analysis, operator theory, and mathematical physics (see, e.g., [8] and [13, p. 20]).

For generalizations of the Clarkson inequalities to several operators and unitarily invarient norms, we refer to [1], [3], [4], [7], [10], [11], and [12].

2. Preliminary results. In this section, we introduce some basic refinements of the inequalities (1.1) and (1.2). These refinements are the base of our results that are given in Sections 3 and 4. First, we start with the following lemma.

LEMMA 2.1. *Let $x \in [1, \infty)$.*

(a) *If $1 \leq r < \infty$, then $(1+x)^r \geq x^r + 2^r - 1$.*
(b) *If $0 < r \leq 1$, then $(1+x)^r \leq x^r + 2^r - 1$.*

Proof. We prove part (a), the proof of part (b) is similar. If $r = 1$, then the inequality becomes equality, so suppose that $r > 1$ and let $f(x) = (1+x)^r - x^r, x \in [1, \infty)$. Then the derivative of f is $f^{'}(x) = r\left((1+x)^{r-1} - x^{r-1}\right) \geq 0$, $x \in (1, \infty)$. So, f is increasing on $[1, \infty)$, and hence, $f(x) \geq f(1)$ for all $x \in [1, \infty)$. The last inequality implies that $(1+x)^r \geq x^r + 2^r - 1$, as required. ∎

Based on Lemma 2.1, we have the following result.

THEOREM 2.2. *Let $a, b \in [0, \infty)$.*

(b) *If $1 \leq r < \infty$, then*

$$a^r + b^r \leq (a+b)^r - (2^r - 2)\min(a^r, b^r).$$

In particular,

$$a^r + b^r \leq (a+b)^r$$

with equality if and only if $a = 0, b = 0$, or $r = 1$.

(b) *If $0 < r \leq 1$, then*

$$a^r + b^r \geq (a+b)^r - (2^r - 2)\min(a^r, b^r).$$

In particular,

$$a^r + b^r \geq (a+b)^r$$

with equality if and only if $a = 0, b = 0$, or $r = 1$.

Proof. We prove part (a), the proof of part (b) is similar. The result is trivial if either a or b is zero, so suppose that both a and b are non-zero. Let $c = \max(a, b)$ and $d = \min(a, b)$. Then $\frac{c}{d} \geq 1$ and by letting $x = \frac{c}{d}$ in Lemma 2.1(a), we have

$$\left(1 + \frac{c}{d}\right)^r \geq \left(\frac{c}{d}\right)^r + 2^r - 1,$$

and so,

$$(c+d)^r \geq c^r + d^r(2^r - 1) = c^r + d^r + d^r(2^r - 2). \tag{2.5}$$

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Since $c+d=a+b$ and $c^r+d^r=a^r+b^r$, the result follows from the inequality (2.5). Moreover, the equality conditions follow from the fact that $(2^r-2)\min(a^r,b^r)=0$ if and only if $a=0, b=0$, or $r=1$. This completes the proof of part (a). ∎

Based on Theorem 2.2, refinements of the Clarkson inequalities (1.1) and (1.2) can be seen in the next theorem. Let $c_p(s,t)=(2^{p/2}-2)\min(|s|^p,|t|^p)$ for $0<p<\infty$ and $s,t\in\mathbb{C}$.

THEOREM 2.3. *Let $a,b\in\mathbb{C}$. Then:*

(a) For $2\le p<\infty$,

$$2(|a|^p+|b|^p)+2c_p(a,b)\le|a+b|^p+|a-b|^p\le 2^{p-1}(|a|^p+|b|^p)-c_p(a+b,a-b). \tag{2.6}$$

(b) For $0<p\le 2$,

$$2^{p-1}(|a|^p+|b|^p)-c_p(a+b,a-b)\le|a+b|^p+|a-b|^p\le 2(|a|^p+|b|^p)+2c_p(a,b). \tag{2.7}$$

Proof. We prove part (a), the proof of part (b) is similar. We have

$$\begin{aligned}|a+b|^p+|a-b|^p &= (|a+b|^2)^{p/2}+(|a-b|^2)^{p/2}\\ &\le (|a+b|^2+(|a-b|^2)^{p/2}-(2^{p/2}-2)\min(|a+b|^p,|a-b|^p) \qquad \text{(by Theorem 2.2 (a))}\\ &= (2|a|^2+2|b|^2)^{p/2}-c_p(a+b,a-b)\\ &= 2^p\left(\frac{|a|^2+|b|^2}{2}\right)^{p/2}-c_p(a+b,a-b).\end{aligned} \tag{2.8}$$

The convexity of the function $f(t)=t^{p/2}$ on $[0,\infty)$ implies that

$$\left(\frac{|a|^2+|b|^2}{2}\right)^{p/2}\le\frac{(|a|^2)^{p/2}+(|b|^2)^{p/2}}{2}=\frac{|a|^p+|b|^p}{2}. \tag{2.9}$$

Now, the second inequality in (2.6) follows from the inequalities (2.8) and (2.9). On the other hand, the first inequality in (2.6) follows from the second inequality in (2.6) by replacing a and b by $\frac{a+b}{2}$ and $\frac{a-b}{2}$, respectively. ∎

Based on Theorem 2.3, the equality conditions of the inequalities (1.1) and (1.2) are given in the following result.

COROLLARY 2.4. *Let $a,b\in\mathbb{C}$, and let $0<p<\infty$. Then:*

(a) $|a+b|^p+|a-b|^p=2^{p-1}(|a|^p+|b|^p)$ *if and only if* $a=b$, $a=-b$, *or* $p=2$.
(b) $|a+b|^p+|a-b|^p=2(|a|^p+|b|^p)$ *if and only if* $a=0$, $b=0$, *or* $p=2$.

Proof. We prove parts (a) and (b) when $2\le p<\infty$. The proof of parts (a) and (b) when $0<p\le 2$ is similar. So, suppose that $2\le p<\infty$.

(a) ($\Longrightarrow$) If $|a+b|^p+|a-b|^p=2^{p-1}(|a|^p+|b|^p)$, then the second inequality in (2.6) implies that $c_p(a+b,a-b)=0$, and so $(2^{p/2}-2)\min(|a+b|^p,|a-b|^p)=0$, which implies that $a=b$, $a=-b$, or $p=2$.

($\Longleftarrow$) If $b=a$, $b=-a$, or $p=2$, then by direct computations it can be seen that $|a+b|^p+|a-b|^p=2^{p-1}(|a|^p+|b|^p)$, as required.

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(b) ($\Longrightarrow$) If $|a+b|^p + |a-b|^p = 2(|a|^p + |b|^p)$, then it follows from the first inequality in (2.6) that $c_p(a,b) = 0$, and so $(2^{p/2}-2)\min(|a|^p, |b|^p) = 0$, which implies that $a = 0$, $b = 0$, or $p = 2$.

($\Longleftarrow$) If $a = 0, b = 0$, or $p = 2$, then by direct computations it can be seen that $|a+b|^p + |a-b|^p = 2(|a|^p + |b|^p)$, as required. ∎

3. Clarkson-type inequalities for the Schatten p-norms. In this section, we give refinements of the inequalities (1.3) and (1.4) under additional conditions. These refinements are based on the following lemma (see e.g., [2] and [5]), which can be considered as a matrix version of Hanner's inequality [9]. Our new inequalities may be useful in the geometry of certain Banach spaces, regarding the uniform convexity and smoothness of these spaces.

Though we confine our discussion to matrices regarded as operators on a finite-dimensional Hilbert space, by slight modifications the inequalities in this section can be extended to operators on an infinite-dimensional Hilbert space.

LEMMA 3.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then:*

(a)
$$\|A+B\|_p^p + \|A-B\|_p^p \le (\|A\|_p + \|B\|_p)^p + \left|\|A\|_p - \|B\|_p\right|^p$$
holds if $p = 2$, $4 \le p < \infty$, or $2 < p < 4$, and both $A+B, A-B$ are positive semidefinite.

(b)
$$\|A+B\|_p^p + \|A-B\|_p^p \ge (\|A\|_p + \|B\|_p)^p + \left|\|A\|_p - \|B\|_p\right|^p$$
holds if $p = 2$, $1 \le p \le \frac{4}{3}$, or $\frac{4}{3} < p < 2$, and both $A+B, A-B$ are positive semidefinite.

Based on Theorem 2.3 and Lemma 3.1, our new refinements of the inequalities (1.3) and (1.4) can be seen in the following result.

THEOREM 3.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then:*

(a)
$$\|A+B\|_p^p + \|A-B\|_p^p \le 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) - (2^{p/2}-2)\left|\|A\|_p - \|B\|_p\right|^p \tag{3.10}$$
holds if $p = 2$, $4 \le p < \infty$, or $2 < p < 4$, and both $A+B, A-B$ are positive semidefinite.

(b)
$$\|A+B\|_p^p + \|A-B\|_p^p \ge 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) - (2^{p/2}-2)\left|\|A\|_p - \|B\|_p\right|^p \tag{3.11}$$
holds if $p = 2$, $1 \le p \le \frac{4}{3}$, or $\frac{4}{3} < p < 2$, and both $A+B, A-B$ are positive semidefinite.

Proof. We prove part (a), the proof of part (b) is similar. If $p = 2$, then the result is trivial. So, suppose that $4 \le p < \infty$ (or $2 < p < 4$, and both $A+B, A-B$ are positive semidefinite). Then

$$\begin{aligned}\|A+B\|_p^p + \|A-B\|_p^p &\le (\|A\|_p + \|B\|_p)^p + \left|\|A\|_p - \|B\|_p\right|^p \qquad \text{(by Lemma 3.1(a))}\\ &\le 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right) - c_p\left(\|A\|_p + \|B\|_p, \|A\|_p - \|B\|_p\right) \qquad \text{(by (2.6))}.\end{aligned}$$

But

$$\begin{aligned}c_p\left(\|A\|_p + \|B\|_p, \|A\|_p - \|B\|_p\right) &= (2^{p/2}-2)\min\left(\left(\|A\|_p + \|B\|_p\right)^p, \left|\|A\|_p - \|B\|_p\right|^p\right)\\ &= (2^{p/2}-2)\left|\|A\|_p - \|B\|_p\right|^p.\end{aligned}$$

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This proves the inequality (3.10). ∎

An immediate consequences of Theorem 3.2 are the following two corollaries.

Corollary 3.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then:*

(a)

$$\|A+B\|_p^p + \|A-B\|_p^p \geq 2\left(\|A\|_p^p + \|B\|_p^p\right) + (2^{1-p/2} - 2^{2-p})\left|\|A+B\|_p - \|A-B\|_p\right|^p \tag{3.12}$$

holds if $p = 2$, $4 \leq p < \infty$, or $2 < p < 4$, and both A, B are positive semidefinite.

(b)

$$\|A+B\|_p^p + \|A-B\|_p^p \leq 2\left(\|A\|_p^p + \|B\|_p^p\right) + (2^{1-p/2} - 2^{2-p})\left|\|A+B\|_p - \|A-B\|_p\right|^p \tag{3.13}$$

holds if $p = 2$, $1 \leq p \leq \frac{4}{3}$, or $\frac{4}{3} < p < 2$, and both A, B are positive semidefinite.

Proof. The inequalities (3.12) and (3.13) follows from the inequalities (3.10) and (3.11) by replacing A and B by $\frac{A+B}{2}$ and $\frac{A-B}{2}$, respectively. ∎

Corollary 3.4. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$.*

(a) *If $4 \leq p < \infty$ or $2 < p < 4$, and both $A+B, A-B$ are positive semidefinite such that $\|A+B\|_p^p + \|A-B\|_p^p = 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right)$, then $\|A\|_p = \|B\|_p$.*
(b) *If $4 \leq p < \infty$ or $2 < p < 4$, and both A, B are positive semidefinite such that $\|A+B\|_p^p + \|A-B\|_p^p = 2\left(\|A\|_p^p + \|B\|_p^p\right)$, then $\|A+B\|_p = \|A-B\|_p$.*
(c) *If $1 \leq p \leq \frac{4}{3}$ or $\frac{4}{3} < p < 2$, and both $A+B, A-B$ are positive semidefinite such that $\|A+B\|_p^p + \|A-B\|_p^p = 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right)$, then $\|A\|_p = \|B\|_p$.*
(d) *If $1 \leq p \leq \frac{4}{3}$ or $\frac{4}{3} < p < 2$, and both A, B are positive semidefinite such that $\|A+B\|_p^p + \|A-B\|_p^p = 2\left(\|A\|_p^p + \|B\|_p^p\right)$, then $\|A+B\|_p = \|A-B\|_p$.*

It should be mentioned here that Hanner's inequality for any functions $f, g \in L_p$ (see, e.g., [5]) states that

$$\|f+g\|_p^p + \|f-g\|_p^p \leq (\|f\|_p + \|g\|_p)^p + \left|\|f\|_p - \|g\|_p\right|^p \tag{3.14}$$

for $2 \leq p < \infty$. The inequality reverses for $1 \leq p \leq 2$.

Also Clarkson's inequalities for L_p functions assert that (see, e.g., [6])

$$2\left(\|f\|_p^p + \|g\|_p^p\right) \leq \|f+g\|_p^p + \|f-g\|_p^p \leq 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right) \tag{3.15}$$

for $2 \leq p < \infty$. The inequalities reverse for $1 \leq p \leq 2$.

We can give refinements of the inequalities (3.15) depending on Theorem 2.3 and the inequality (3.14).

Theorem 3.5. *Let $f, g \in L_p$. Then*

$$\|f+g\|_p^p + \|f-g\|_p^p \leq 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right) - (2^{p/2} - 2)\left|\|f\|_p - \|g\|_p\right|^p \tag{3.16}$$

and

$$\|f+g\|_p^p + \|f-g\|_p^p \geq 2\left(\|f\|_p^p + \|g\|_p^p\right) + (2^{1-p/2} - 2^{2-p})\left|\|f+g\|_p - \|f-g\|_p\right|^p \tag{3.17}$$

for $2 \leq p < \infty$. The inequalities reverse for $1 \leq p \leq 2$.

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Proof. In the case when $2 \le p < \infty$, the proof of the inequality (3.16) is similar to the proof of part (a) of Theorem 3.2, and the inequality (3.17) follows from the inequality (3.16) by replacing f and g by $\frac{f+g}{2}$ and $\frac{f-g}{2}$, respectively. ∎

4. Other related inequalities. In this section, we give further upper and lower bounds for $\|A+B\|_p^p + \|A-B\|_p^p$ by dropping some of the conditions given in Theorem 3.2 that are imposed on the values of p and on the matrices A and B. Based on Theorem 2.3, we start with the following result.

THEOREM 4.1. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$.*

(a) *If $2 \le p < \infty$, then*

$$\|A+B\|_p^p + \|A-B\|_p^p \le 2^{3p/2-2}\left(\|A\|_p^p + \|B\|_p^p\right) - nc_p(s_n(A+B), s_n(A-B)) \tag{4.18}$$

and

$$\|A+B\|_p^p + \|A-B\|_p^p \ge 2^{2-p/2}\left(\|A\|_p^p + \|B\|_p^p\right) + n2^{2-p/2}c_p(s_n(A), s_n(B)). \tag{4.19}$$

(b) *If $1 \le p \le 2$, then*

$$\|A+B\|_p^p + \|A-B\|_p^p \ge 2^{3p/2-2}\left(\|A\|_p^p + \|B\|_p^p\right) - nc_p(s_n(A+B), s_n(A-B)) \tag{4.20}$$

and

$$\|A+B\|_p^p + \|A-B\|_p^p \le 2^{2-p/2}\left(\|A\|_p^p + \|B\|_p^p\right) + n2^{2-p/2}c_p(s_n(A), s_n(B)). \tag{4.21}$$

Proof. We prove part (a), the proof of part (b) is similar.

Since $nc_p(s_n(A+B), s_n(A-B)) \le c_p\left(\|A+B\|_p, \|A-B\|_p\right)$, we have

$$\begin{aligned}
&\|A+B\|_p^p + \|A-B\|_p^p + nc_p(s_n(A+B), s_n(A-B))\\
&\le \|A+B\|_p^p + \|A-B\|_p^p + c_p\left(\|A+B\|_p, \|A-B\|_p\right)\\
&= \left(\|A+B\|_p^2\right)^{p/2} + \left(\|A-B\|_p^2\right)^{p/2} + c_p\left(\|A+B\|_p, \|A-B\|_p\right)\\
&\le \left(\|A+B\|_p^2 + \|A-B\|_p^2\right)^{p/2} \qquad \text{(by Theorem 2.2 (a))}\\
&\le 2^{p/2-1}\left(\|A+B\|_p^p + \|A-B\|_p^p\right) \qquad \text{(by the convexity of } f(t)=t^{p/2} \text{ on } [0,\infty))\\
&\le 2^{p/2-1}\left(2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right)\right) \qquad \text{(by the second inequality in (1.3))}\\
&= 2^{3p/2-2}\left(\|A\|_p^p + \|B\|_p^p\right).
\end{aligned}$$

This proves the inequality (4.18). The inequality (4.19) follows from the inequality (4.18) by replacing A and B by $\frac{A+B}{2}$ and $\frac{A-B}{2}$, respectively. ∎

To complete our work, we need the following lemma (see [2]).

LEMMA 4.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then:*

(a) $\frac{1}{2}\left(\|A+B\|_p^2 + \|A-B\|_p^2\right) \le \|A\|_p^2 + (p-1)\|B\|_p^2$ *for* $2 \le p < \infty$.

(b) $\frac{1}{2}\left(\|A+B\|_p^2 + \|A-B\|_p^2\right) \ge \|A\|_p^2 + (p-1)\|B\|_p^2$ *for* $1 \le p \le 2$.

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Other related upper and lower bounds for $\|A+B\|_p^p + \|A-B\|_p^p$ are given in the following result.

THEOREM 4.3. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$.

(a) *If* $2 \le p < \infty$, *then*

$$\|A+B\|_p^p + \|A-B\|_p^p \le 2^{p-1}\left(\|A\|_p^p + (p-1)^{p/2}\|B\|_p^p\right) - nc_p(s_n(A+B), s_n(A-B)).$$

(b) *If* $1 \le p \le 2$, *then*

$$\|A+B\|_p^p + \|A-B\|_p^p \ge 2^{p-1}\left(\|A\|_p^p + (p-1)^{p/2}\|B\|_p^p\right) - nc_p(s_n(A+B), s_n(A-B)).$$

Proof. We prove part (a), the proof of part (b) is similar. We have

$$\begin{aligned}
&\|A+B\|_p^p + \|A-B\|_p^p + nc_p(s_n(A+B), s_n(A-B)) \\
&\le \left(\|A+B\|_p^2\right)^{p/2} + \left(\|A-B\|_p^2\right)^{p/2} + c_p(\|A+B\|_p, \|A-B\|_p) \\
&\le \left(\|A+B\|_p^2 + \|A-B\|_p^2\right)^{p/2} \qquad \text{(by Theorem 2.2 (a))} \\
&\le \left(2\left(\|A\|_p^2 + (p-1)\|B\|_p^2\right)\right)^{p/2} \qquad \text{(by Lemma 4.2 (a))} \\
&= 2^{p/2}\left(\|A\|_p^2 + (p-1)\|B\|_p^2\right)^{p/2} \\
&\le 2^{p-1}\left(\|A\|_p^p + (p-1)^{p/2}\|B\|_p^p\right) \qquad \text{(by the convexity of } f(t) = t^{p/2} \text{ on } [0,\infty)),
\end{aligned}$$

as required. ∎

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