



## CHARACTERIZING GRAPHS OF MAXIMUM PRINCIPAL RATIO\*

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**Abstract.** The principal ratio of a connected graph, denoted  $\gamma(G)$ , is the ratio of the maximum and minimum entries of its Perron eigenvector. Cioabă and Gregory (2007) conjectured that the graph on  $n$  vertices maximizing  $\gamma(G)$  is a kite graph, that is, a complete graph with a pendant path. In this paper, their conjecture is proved.

**Key words.** Spectral radius, Irregular graph, Eigenvectors.

**AMS subject classifications.** 05C50, 15A18.

**1. Introduction.** Several measures of graph irregularity have been proposed to evaluate how far a graph is from being regular. In this paper, we determine the extremal graphs with respect to one such irregularity measure, answering a conjecture of Cioabă and Gregory [5].

All graphs in this paper will be simple and undirected, and all eigenvalues are of the adjacency matrix of the graph. For a connected graph  $G$ , the eigenvector corresponding to its largest eigenvalue, the *principal eigenvector*, can be taken to have all positive entries. If  $\mathbf{x}$  is this eigenvector, let  $x_{\min}$  and  $x_{\max}$  be the smallest and largest eigenvector entries, respectively. Then define the *principal ratio*,  $\gamma(G)$  to be

$$\gamma(G) = \frac{x_{\max}}{x_{\min}}.$$

Note that  $\gamma(G) \geq 1$  with equality exactly when  $G$  is regular, and it therefore can be considered as a measure of graph irregularity.

Let  $P_r \cdot K_s$  be the graph attained by identifying an end vertex of a path on  $r$  vertices to any vertex of a complete graph on  $s$  vertices. This has been called a *kite graph* or a *lollipop graph*. Cioabă and Gregory [5] conjectured that the connected graph on  $n$  vertices maximizing  $\gamma$  is a kite graph. Our main theorem proves this conjecture for  $n$  large enough.

**THEOREM 1.1.** *For sufficiently large  $n$ , the connected graph  $G$  on  $n$  vertices with largest principal ratio is a kite graph.*

We note that Brightwell and Winkler [4] showed that a kite graph maximizes the expected hitting time of a random walk. Other irregularity measures for graphs have been well-studied. Bell [3] studied the irregularity measure  $\epsilon(G) := \lambda_1(G) - \bar{d}(G)$ , the difference between the spectral radius and the average degree of  $G$ . He determined the extremal graph over all (not necessarily connected) graphs on  $n$  vertices and  $e$  edges. It is not known what the extremal connected graph is, and Aouchiche et al [2] conjectured that this

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extremal graph is a “pineapple”, that is, a complete graph with pendant vertices added to a single vertex. Bell also studied the *variance* of a graph,

$$\text{var}(G) = \frac{1}{n} \sum_{v \in V(G)} |d_v - \bar{d}|^2.$$

Albertson [1] defined a measure of irregularity by

$$\sum_{uv \in E(G)} |d(u) - d(v)|,$$

and the extremal graphs were characterized by Hansen and Mélot [6].

Nikiforov [9] proved several inequalities comparing  $\text{var}(G)$ ,  $\epsilon(G)$  and  $s(G) := \sum_v |d(u) - \bar{d}|$ . Bell showed that  $\epsilon(G)$  and  $\text{var}(G)$  are incomparable in general [3]. Finally, bounds on  $\gamma(G)$  have been given in [5, 7, 8, 10, 11].

**2. Preliminaries.** Throughout this paper  $G$  will be a connected simple graph on  $n$  vertices. The eigenvectors and eigenvalues of  $G$  are those of the adjacency matrix  $A$  of  $G$ . The vector  $v$  will be the eigenvector corresponding to the largest eigenvalue  $\lambda_1$ , and we take  $v$  to be scaled so that its largest entry is 1. Let  $x_1$  and  $x_k$  be the vertices with smallest and largest eigenvector entries, respectively, and if several such vertices exist then we pick any of them arbitrarily. Let  $x_1, x_2, \dots, x_k$  be a shortest path between  $x_1$  and  $x_k$ . Let  $\gamma(G)$  be the principal ratio of  $G$ . We will abuse notation so that for any vertex  $x$ , the symbol  $x$  will refer also to  $v(x)$ , the value of the eigenvector entry of  $x$ . For example, with this notation the eigenvector equation becomes

$$\lambda v = \sum_{w \sim v} w.$$

We will make use of the Rayleigh quotient characterization of the largest eigenvalue of a graph,

$$(2.1) \quad \lambda_1(G) = \max_{0 \neq v} \frac{v^T A(G)v}{v^T v}.$$

Recall that the vertices  $v_1, v_2, \dots, v_m$  are a *pendant path* if the induced graph on these vertices is a path and furthermore if, in  $G$ ,  $v_1$  has degree 1 and the vertices  $v_2, \dots, v_{m-1}$  have degree 2 (note there is no requirement on the degree of  $v_m$ ).

LEMMA 2.1. *If  $\lambda_1 \geq 2$  and  $\sigma = (\lambda_1 + \sqrt{\lambda_1^2 - 4})/2$ , then for  $1 \leq j \leq k$ ,*

$$\gamma(G) \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}.$$

*Moreover we have equality if the vertices  $x_1, x_2, \dots, x_j$  are a pendant path.*

*Proof.* We have the following system of inequalities:

$$\begin{aligned} \lambda_1 x_1 &\geq x_2 \\ \lambda_1 x_2 &\geq x_1 + x_3 \\ \lambda_1 x_3 &\geq x_2 + x_4 \\ &\vdots \\ \lambda_1 x_{j-1} &\geq x_j + x_{j-2}. \end{aligned}$$

The first inequality implies that

$$x_1 \geq \frac{1}{\lambda_1} x_2.$$

Plugging this into the second equation and rearranging gives

$$x_2 \geq \frac{\lambda_1}{\lambda_1^2 - 1} x_3.$$

Now assume that

$$x_i \geq \frac{u_{i-1}}{u_i} x_{i+1},$$

with  $u_j$  positive for all  $j < i$ . Then

$$\lambda_1 x_{i+1} \geq x_i + x_{i+2},$$

implies that

$$x_{i+1} \geq \frac{u_i}{\lambda_1 u_i - u_{i-1}} x_{i+2}.$$

Note that  $\lambda_1 u_i - u_{i-1}$  must be positive since  $\lambda_1 x_{i+1} \geq x_i + x_{i+2} \geq \frac{u_{i-1}}{u_i} x_{i+1} + x_{i+2} > \frac{u_{i-1}}{u_i} x_{i+1}$  as  $x_j$  is positive for all  $j$ . Therefore, we may choose the coefficients  $u_i$  to satisfy the recurrence

$$u_{i+1} = \lambda_1 u_i - u_{i-1}.$$

Solving this and using the initial conditions  $u_0 = 1$ ,  $u_1 = \lambda$ , we get

$$u_i = \frac{\sigma^{i+1} - \sigma^{-i-1}}{\sigma - \sigma^{-1}}.$$

In particular,  $u_i$  is always positive, a fact implicitly used above. Finally this gives,

$$x_1 \geq \frac{u_0}{u_1} x_2 \geq \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} x_3 \geq \dots \geq \frac{x_j}{u_{j-1}}.$$

Hence,

$$\gamma(G) = \frac{x_k}{x_1} = \frac{1}{x_1} \leq \frac{\sigma^j - \sigma^{-j}}{\sigma - \sigma^{-1}} x_j^{-1}.$$

If these vertices are a pendant path, then we have equality throughout. □

We will also use the following lemma which comes from the paper of Cioabă and Gregory [5].

LEMMA 2.2. For  $r \geq 2$  and  $s \geq 3$ ,

$$s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}.$$

In the remainder of the paper, we prove Theorem 1.1. We now give a sketch of the proof that is contained in Section 3.

1. We show that the vertices  $x_1, x_2, \dots, x_{k-2}$  are a pendant path and that  $x_k$  is connected to all of the vertices in  $G$  that are not on this path (Lemma 3.2).
2. Next we prove that the length of the path is approximately  $n - n/\log(n)$  (Lemma 3.3).
3. We show that  $x_{k-2}$  has degree exactly 2 (Lemma 3.6), which extends our pendant path to  $x_1, x_2, \dots, x_{k-1}$ . To do this, we find conditions under which adding or deleting edges increases the principal ratio (Lemma 3.4).

4. Next we show that  $x_{k-1}$  also has degree exactly 2 (Lemma 3.8). At this point, we can deduce that our extremal graph is either a kite graph or a graph obtained from a kite graph by removing some edges from the clique. We show that adding in any missing edges will increase the principal ratio, and hence, the extremal graph is exactly a kite graph.

**3. Proof of Theorem 1.1.** Let  $G$  be the graph with maximal principal ratio among all connected graphs on  $n$  vertices, and let  $k$  be the number of vertices in a shortest path between the vertices with smallest and largest eigenvalue entries. As above, let  $x_1, \dots, x_k$  be the vertices of the shortest path, where  $\gamma(G) = x_k/x_1$ . Let  $C$  be the set of vertices not on this shortest path, so  $|C| = n - k$ . Note that there is no graph with  $n - k = 1$ , as the endpoints of a path have the same principal eigenvector entry. Also  $\lambda_1(G) \geq 2$ , otherwise  $P_{n-2} \cdot K_3$  would have larger principal ratio. Finally note that  $k$  is strictly larger than 1, otherwise  $x_k = x_1$  and  $G$  would be regular.

LEMMA 3.1.  $\lambda_1(G) > n - k$ .

*Proof.* Let  $H$  be the graph  $P_k \cdot K_{n-k+1}$ . It is straightforward to see that in  $H$ , the smallest entry of the principal eigenvector is the vertex of degree 1 and the largest is the vertex of degree  $n - k + 1$ . Also note that in  $H$ , the vertices on the path  $P_k$  form a pendant path. By maximality we know that  $\gamma(G) \geq \gamma(H)$ . Combining this with Lemma 2.1, we get

$$\frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}} \geq \gamma(G) \geq \gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}},$$

where  $\sigma_H = \left( \lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4} \right) / 2$ . Now the function

$$f(x) = \frac{x^k - x^{-k}}{x - x^{-1}}$$

is increasing when  $x \geq 1$ . Hence, we have  $\sigma \geq \sigma_H$ , and so  $\lambda_1(G) \geq \lambda_1(H) > n - k$ . □

LEMMA 3.2.  $x_1, x_2, \dots, x_{k-2}$  are a pendant path in  $G$ , and  $x_k$  is connected to every vertex in  $G$  that is not on this path.

*Proof.* By our choice of scaling,  $x_k = 1$ . From Lemma 3.1

$$n - k < \lambda_1(G) = \sum_{y \sim x_k} y \leq |N(x_k)|.$$

Now  $|N(x_k)|$  is an integer, so we have  $|N(x_k)| \geq n - k + 1$ . Moreover because  $x_1, x_2, \dots, x_k$  is an induced path, we must have that  $|N(x_k)| = n - k + 1$  exactly, and hence, the  $N(x_k) = C \cup \{x_{k-1}\}$ . It follows that  $x_1, x_2, \dots, x_{k-3}$  have no neighbors off the path, as otherwise there would be a shorter path between  $x_1$  and  $x_k$ . □

LEMMA 3.3. For the extremal graph  $G$ , we have  $n - k = (1 + o(1)) \frac{n}{\log n}$ .

*Proof.* Let  $H$  be the graph  $P_j \cdot K_{n-j+1}$  where  $j = \left\lfloor n - \frac{n}{\log n} \right\rfloor$ , and let  $G$  be the connected graph on  $n$  vertices with maximum principal ratio. Let  $x_1, \dots, x_k$  be a shortest path from  $x_1$  to  $x_k$  where  $\gamma(G) = \frac{x_k}{x_1}$ . By Lemma 3.2, we have

$$\lambda_1(G) \leq \Delta(G) \leq n - k + 1.$$

By the eigenvector equation, this gives that

$$(3.2) \quad \gamma(G) \leq (n - k + 1)^k.$$

Now, Lemma 2.1 gives that

$$\gamma(H) = \frac{\sigma_H^j - \sigma_H^{-j}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) + \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Now by Lemma 2.2,  $s - 1 + \frac{1}{s(s-1)} < \lambda_1(P_r \cdot K_s) < s - 1 + \frac{1}{(s-1)^2}$ , and so we have  $\lambda_1(H) = \frac{n}{\log n} + o(1)$ , so we may choose  $n$  large enough that  $\frac{n}{\log n} + 1 > \sigma_H - \sigma_H^{-1} > \frac{n-1}{\log n}$ . By maximality of  $\gamma(G)$ , we have

$$(n - k + 1)^k \geq \gamma(G) \geq \gamma(H) \geq \left(\frac{n}{\log n}\right)^{n - \frac{n-1}{\log n} - 2}.$$

Thus,  $n - k = (1 + o(1))\frac{n}{\log n}$ . □

For the remainder of this paper we will explore the structure of  $G$  by showing that if certain edges are missing, adding them would increase the principal ratio, and so by maximality these edges must already be present in  $G$ . We have established that the vertices  $x_1, x_2, \dots, x_{k-2}$  are a pendant path, and so we have

$$(3.3) \quad \gamma(G) = \frac{\sigma^{k-2} - \sigma^{-k+2}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-2}}.$$

We will not add any edges that affect this path, and so the above equality will remain true. The change in  $\gamma$  is then completely determined by the change in  $\lambda_1$  and the change in  $x_{k-2}$ . The next lemma gives conditions on these two parameters under which  $\gamma$  will increase or decrease.

LEMMA 3.4. *Let  $x_1, x_2, \dots, x_{m-1}$  form a pendant path in  $G$ , where  $n - m = (1 + o(1))n/\log(n)$ . Let  $G_+$  be a graph obtained from  $G$  by adding some edges from  $x_{m-1}$  to  $V(G) \setminus \{x_1, \dots, x_{m-1}\}$ , where the addition of these edges does not affect which vertex has largest principal eigenvector entry. Let  $\lambda_1^+$  be the largest eigenvalue of  $G_+$  with leading eigenvector entry for vertex  $x$  denoted  $x^+$ , also normalized to have maximum entry one. Define  $\delta_1$  and  $\delta_2$  such that  $\lambda_1^+ = (1 + \delta_1)\lambda_1$  and  $x_{m-1}^+ = (1 + \delta_2)x_{m-1}$ . Then,*

- $\gamma(G_+) > \gamma(G)$  whenever  $\delta_1 > 4\delta_2/n$ ;
- $\gamma(G_+) < \gamma(G)$  whenever  $\delta_1 \exp(2\delta_1 \lambda_1 \log n) < \delta_2/3n$ .

*Proof.* We have

$$\sigma = \lambda_1 - \lambda_1^{-1} - \lambda_1^{-3} - 2\lambda_1^{-5} - \dots - \frac{2}{2n-3} \binom{2n-2}{n} \lambda_1^{-(2n-1)} - \dots$$

So,

$$\lambda_1^+ - \lambda_1 < \sigma_+ - \sigma < \lambda_1^+ - \lambda_1 - 2((\lambda_1^+)^{-1} - \lambda_1^{-1})$$

when  $\lambda_1$  is sufficiently large, which is guaranteed by Lemma 3.3. Plugging in  $\lambda_1^+ = (1 + \delta_1)\lambda_1$ , we get

$$\delta_1 \lambda_1 < \sigma_+ - \sigma < \delta_1 \lambda_1 + 2\lambda_1^{-1}(1 - (1 + \delta_1)^{-1}) < \delta_1 \lambda_1 + \delta_1.$$

In particular,

$$(1 + \delta_1/2)\sigma < \sigma_+ < (1 + 2\delta_1)\sigma.$$

To prove part (i), we wish to find a lower bound in the change in the first factor of (3.3). Let

$$f(x) = \frac{x^{m-1} - x^{-m+1}}{x - x^{-1}}.$$

Then  $2mx^{m-3} > f'(x) > (m-2)x^{m-3} - mx^{m-5}$ , and using that  $n - m \sim n/\log(n)$  and  $\sigma \sim \lambda_1$  which goes to infinity with  $n$ , we get  $f'(x) \gtrsim (m-2)x^{m-3}$ . By linearization and because  $f(\sigma) \sim \sigma^{m-2}$ , it follows that

$$\frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}} \geq \left(1 + \frac{\delta_1(m-3)}{2}\right) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}}.$$

Hence, if

$$\frac{\delta_1(m-3)}{2} > \delta_2,$$

then  $\gamma(G_+) > \gamma(G)$ . In particular it is sufficient that  $\delta_1 > 4\delta_2/n$ .

To prove part (ii), recall from above that  $f'(x) < 2mx^{m-3}$ . Then, when  $x = (1 + o(1))(n/\log(n))$

$$\begin{aligned} f'(x + \varepsilon) &< 2m(x + \varepsilon)^{m-3} \\ &= 2mx^{m-3} \left(1 + \frac{\varepsilon}{x}\right)^{m-3} \\ &\leq 2mx^{m-3} \exp\left(\frac{m\varepsilon}{x}\right) \\ &\leq 2nx^{m-3} \exp(2\log(n)\varepsilon). \end{aligned}$$

So, for  $0 < \varepsilon < \delta_1\lambda_1$ , we have

$$f'(x + \varepsilon) < 2nx^{m-3} \exp(2\log(n)\delta_1\lambda_1).$$

Hence,

$$(1 + 3n \exp(2\delta_1\lambda_1 \log n)\delta_1) \frac{\sigma^{m-1} - \sigma^{-m+1}}{\sigma - \sigma^{-1}} > \frac{\sigma_+^{m-1} - \sigma_+^{-m+1}}{\sigma_+ - \sigma_+^{-1}}. \quad \square$$

LEMMA 3.5. For every subset of  $U$  of  $N(x_k)$ , we have

$$|U| - 1 < \sum_{y \in U} y \leq |U|.$$

Consequently, there is at most one vertex in the neighborhood of  $x_k$  with eigenvector entry smaller than  $1/2$ .

*Proof.* The upper bound follows from  $y \leq 1$ , and the lower bound from the inequalities

$$\sum_{y \in N(x_k) \setminus U} y \leq |N(x_k)| - |U|,$$

and

$$\sum_{y \in N(x_k)} y = \lambda_1(G) > |N(x_k)| - 1. \quad \square$$

LEMMA 3.6. The vertex  $x_{k-2}$  has degree exactly 2 in  $G$ .

*Proof.* Assume to the contrary. Let  $U = N(x_{k-2}) \cap N(x_k)$ . Then  $|U| \geq 2$ , so by Lemma 3.5 we have

$$\sum_{y \in U} y > |U| - 1 \geq 1.$$

Now, by the same argument as the in the proof of Lemma 2.1, we have that

$$\gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \left( \sum_{y \in U} y \right)^{-1}.$$

Let  $H = P_{k-1} \cdot K_{n-k+2}$ . Then by maximality of  $\gamma(G)$  we have

$$\frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} > \gamma(G) \geq \gamma(H) = \frac{\sigma_H^{k-1} - \sigma_H^{-k+1}}{\sigma_H - \sigma_H^{-1}}.$$

So  $\sigma > \sigma_H$ , which means  $\lambda_1(G) > \lambda_1(H) > n - k + 1$ . This means that  $\Delta(G) > n - k + 1$ , but we have established that  $\Delta(G) = n - k + 1$ .  $\square$

We now know that  $x_1, x_2, \dots, x_{k-1}$  is a pendant path in  $G$ , and so (3.3) becomes

$$(3.4) \quad \gamma(G) = \frac{\sigma^{k-1} - \sigma^{-k+1}}{\sigma - \sigma^{-1}} \frac{1}{x_{k-1}}.$$

LEMMA 3.7. *The vertex  $x_{k-1}$  has degree less than  $11|C|/\sqrt{\log n}$ .*

*Proof.* Assume to the contrary that the degree of  $x_{k-1}$  is at least  $11|C|/\sqrt{\log n}$ . Let  $G_+$  the graph obtained form  $G$  with an additional edge from  $x_{k-1}$  to a vertex  $z \in C$  with  $z \geq 1/2$ . Let  $\lambda_1^+ = \lambda_1(G_+)$  and let  $x^+$  be the principal eigenvector entry of vertex  $x$  in  $G_+$ , where this eigenvector is normalized to have  $x_k^+ = 1$ .

*Change in  $\lambda_1$ :* By (2.1), we have  $\lambda_1^+ - \lambda_1 \geq 2 \frac{x_{k-1}z}{\|v\|_2^2}$ . A crude upper bound on  $\|v\|_2^2$  is

$$\|v\|_2^2 \leq 1 + \sum_{y \sim x_k} y + \frac{2}{\lambda_1} + \frac{4}{\lambda_1^2} + \dots < 2\lambda_1.$$

We also have that  $z \geq 1/2$  so

$$\lambda_1^+ \geq \left( 1 + \frac{x_{k-1}}{2\lambda_1^2} \right) \lambda_1.$$

*Change in  $x_{k-1}$ :* Let  $U = N(x_{k-1} \cap C)$ . By the eigenvector equation, we have

$$x_{k-1} = \frac{1}{\lambda_1} \left( x_{k-2} + x_k + \sum_{y \in U} y \right) \quad \text{and} \quad x_{k-1}^+ = \frac{1}{\lambda_1^+} \left( x_{k-2}^+ + x_k^+ + z^+ + \sum_{y \in U} y^+ \right).$$

Subtracting these, and using that  $\lambda_1 < \lambda_1^+$  and  $x_k = x_k^+ = 1$ , we get

$$x_{k-1}^+ - x_{k-1} \leq \frac{1}{\lambda_1} \left( x_{k-2}^+ - x_{k-2} + z^+ + \sum_{y \in U} y^+ - y \right).$$

By Lemma 3.5, we have  $\sum_{y \in U} y^+ - y \leq 1$ . We also have  $x_{k-2}^+ - x_{k-2} < 1$  and  $z^+ \leq 1$ . Hence,  $x_{k-1}^+ - x_{k-1} \leq 3/\lambda_1$ , or

$$x_{k-1}^+ \geq \left( 1 + \frac{3}{\lambda_1 x_{k-1}} \right) x_{k-1}.$$

We can only apply Lemma 3.4 if  $x_k^+$  is the largest eigenvector entry in  $G_+$ . So we must consider two cases.

*Case 1:* If in  $G^+$  the largest eigenvector entry is still attained by vertex  $x_k$ , then we can apply Lemma 3.4, and see that  $\gamma(G^+) > \gamma(G)$  if

$$\frac{x_{k-1}}{2\lambda_1^2} \geq \frac{12}{\lambda_1 x_{k-1} n},$$

or equivalently,

$$x_{k-1}^2 \geq \frac{24\lambda_1}{n}.$$

We have that  $\lambda_1 = (1 + o(1))(n - n/\log(n))$ , so it suffices for

$$(3.5) \quad x_{k-1} \geq \frac{5}{\sqrt{\log n}}.$$

We know that

$$x_{k-1} > \frac{|U| - 1}{2\lambda_1}.$$

By assumption,

$$|U| + 2 = N(x_{k-1}) \geq 11|C|/\sqrt{\log n}.$$

Equation (3.5) follows from this, so  $\gamma(G^+) > \gamma(G)$ .

*Case 2:* Say the largest eigenvector entry of  $G^+$  is no longer attained by vertex  $x_k$ . It is easy to see that the largest eigenvector entry is not attained by a vertex with degree less than or equal to 2, and comparing the neighborhood of any vertex in  $C$  with the neighborhood of  $x_k$  we can see that  $x_k \geq y$  for all  $y \in C$ . So the largest eigenvector entry must be attained by  $x_{k-1}$ . Then (3.4) no longer holds, instead we have

$$(3.6) \quad \gamma(G_+) = \frac{\sigma_+^{k-1} - \sigma_+^{-k+1}}{\sigma_+ - \sigma_+^{-1}}.$$

Recall that in Lemma 3.4 we determined the change from  $\gamma(G_+)$  to  $\gamma(G)$  by considering  $\lambda_1^+ - \lambda_1$  and  $x_{k-1}^+ - x_{k-1}$ . In this case, by (3.6), we must consider  $\lambda_1^+ - \lambda_1$  and  $1 - x_{k-1}$ . Now if  $x_{k-1}^+ > x_k^+$ , then vertex  $x_{k-1}$  in  $G$  is connected to all of  $C$  except perhaps a single vertex. Hence, in  $G$ , the vertex  $x_{k-1}$  is connected to all of  $C$  except at most two vertices. This gives the bound

$$1 - x_{k-1} \leq 3/\lambda_1,$$

and so as in the previous case,  $\gamma(G_+) > \gamma(G)$ .

So in all cases,  $x_{k-1}$  is connected to all vertices in  $C$  that have eigenvector entry larger than  $1/2$ . If all vertices in  $C$  have eigenvector entry larger than  $1/2$ , then  $x_{k-1}$  is connected to all of  $C$ , and this implies that  $x_{k-1} > x_k$ , which is a contradiction. At most one vertex in  $C$  is smaller than  $1/2$ , and so there is a single vertex  $z \in C$  with  $z < 1/2$ . We will quickly check that adding the edge  $\{x_{k-1}, z\}$  increases the principal ratio. As before let  $G_+$  be the graph obtained by adding this edge. The largest eigenvector entry in  $G_+$  is attained by  $x_{k-1}$ , as its neighborhood strictly contains the neighborhood of  $x_k$ . As above, adding the edge  $\{z, x_k\}$  increases the spectral radius at least

$$\lambda_1^+ > \left(1 + \frac{z}{2\lambda_1^2}\right) \lambda_1,$$

and we have  $1 - x_{k-1} < 1 - z/\lambda_1$ . Applying Lemma 3.4 we see that  $\gamma(G_+) > \gamma(G)$ , which is a contradiction. Finally we conclude that the degree of  $x_{k-1}$  must be smaller than  $11|C|/\sqrt{\log n}$ .  $\square$

We note that this lemma gives that  $x_{k-1} < 1/2$  which implies that any vertex in  $C$  has eigenvector entry larger than  $1/2$ .

LEMMA 3.8. *The vertex  $x_{k-1}$  has degree exactly 2 in  $G$ . Moreover,  $x_{k-1} < 2/\lambda_1$ .*

*Proof.* Let  $U = N(x_{k-1}) \cap C$ ,  $c = |U|$ . If  $c = 0$  then we are done. Otherwise let  $G_-$  be the graph obtained from  $G$  by deleting these  $C$  edges. We will show that  $\gamma(G_-) > \gamma(G)$ .

(1) *Change in  $\lambda_1$ :* We have by (2.1),

$$\lambda_1 - \lambda_1^- \leq 2c \frac{x_{k-1}}{\|v\|_2^2}.$$

By Cauchy–Schwarz,

$$\|v\|_2^2 > \sum_{x \in N(x_k)} x^2 \geq \frac{\left(\sum_{x \in N(x_k)} x\right)^2}{|C| + 1} \geq \frac{(n-k)^2}{n-k+1}.$$

We also have

$$x_{k-1} \leq \frac{c+2}{\lambda_1}.$$

Combining these we get

$$\lambda_1 - \lambda_1^- < \frac{9c^2}{\lambda_1(n-k+1)} \Rightarrow \lambda_1 < \left(1 + \frac{9c^2}{\lambda_1 \lambda_1^- (n-k+1)}\right) \lambda_1^-.$$

We have  $\lambda_1 \lambda_1^- > (n-k)^2$ , so

$$\lambda_1 < \left(1 + \frac{10c^2}{(n-k)^3}\right) \lambda_1^-.$$

(2) *Change in  $x_{k-1}$ :* At this point, we know that in  $G_-$  the vertices  $x_1, \dots, x_k$  form a pendant path, and so by the proof of Lemma 2.1, we have  $x_{k-1}^- = (1 + o(1))/\lambda_1$ . By the eigenvector equation and using that the vertices in  $C$  have eigenvector entry at least  $1/2$ , we have  $x_{k-1} > (1 + c/2)/\lambda_1$ . So

$$x_{k-1} - x_{k-1}^- > \frac{1}{\lambda_1} \left(\frac{c}{2} + o(1)\right).$$

In particular,

$$x_{k-1} > \left(1 + \frac{c}{3x_{k-1}^- \lambda_1}\right) x_{k-1}^-.$$

Applying Lemma 3.4, it suffices now to show that

$$(3.7) \quad \frac{10c^2}{(n-k)^3} \exp\left(2 \frac{10c^2}{(n-k)^3} \lambda_1^- \log n\right) < \frac{c}{9x_{k-1}^- \lambda_1 n}.$$

Now

$$\frac{10c^2}{(n-k)^3} < 10 \frac{11^2}{\log(n)} \frac{|C|^2}{(n-k)^3} < \frac{11^3 \log n}{\log n} \frac{1}{n} = \frac{11^3}{n}.$$

Similarly  $2 \frac{10c^2}{(n-k)^3} \lambda_1^- \log n < 2 \cdot 11^3$ , so the lefthand side of (3.7) is smaller than  $C_0/n$ , where  $C_0$  is an absolute constant. For the righthand side, recall that  $x_{k-1}^- \lambda_1 = 1 + o(1)$ , and also that

$$c > \frac{11}{\sqrt{\log n}} \left(\frac{n}{\log n} + o(1)\right) > \frac{10n}{\log^{3/2} n}.$$

So the righthand side is larger than  $1/\log^{3/2} n$ . Hence, for large enough  $n$ , the righthand side is larger than the lefthand side.  $\square$

We are now ready to prove the main theorem.

**THEOREM 1.** *For sufficiently large  $n$ , the connected graph  $G$  on  $n$  vertices with largest principal ratio is a kite graph.*

*Proof.* It remains to show that  $C$  induces a clique. Assume it does not, and let  $H$  be the graph  $P_k \cdot K_{n-k+1}$ . We will show that  $\gamma(H) > \gamma(G)$ , and this contradiction tells us that  $C$  is a clique. As before, Lemma 2.1 gives that

$$\gamma(H) = \frac{\sigma_H^k - \sigma_H^{-k}}{\sigma_H - \sigma_H^{-1}},$$

where

$$\sigma(H) = \frac{\lambda_1(H) - \sqrt{\lambda_1(H)^2 - 4}}{2}.$$

Since  $x_1, \dots, x_k$  form a pendant path we also know that

$$\gamma(G) = \frac{\sigma^k - \sigma^{-k}}{\sigma - \sigma^{-1}}.$$

Now,  $\lambda_1(H) > \lambda_1(G)$  because  $E(G) \subsetneq E(H)$ . Since the functions  $g(x) = x + \sqrt{x^2 - 4}$  and  $f(x) = (x^k - x^{-k})/(x - x^{-1})$  are increasing when  $x \geq 1$ , we have  $\gamma(H) > \gamma(G)$ .  $\square$

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