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NEW CONTRIBUTIONS TO SEMIPOSITIVE AND MINIMALLY SEMIPOSITIVE MATRICES*

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Abstract. Semipositive matrices (matrices that map at least one nonnegative vector to a positive vector) and minimally semipositive matrices (semipositive matrices whose no column-deleted submatrix is semipositive) are well studied in matrix theory. In this article, this notion is revisited and new results are presented. It is shown that the set of all $m \times n$ minimally semipositive matrices contains a basis for the linear space of all $m \times n$ matrices. Apart from considerations involving principal pivot transforms and the Schur complement, results on semipositivity and/or minimal semipositivity for the following classes of matrices are presented: intervals of rectangular matrices, skew-symmetric and almost skew-symmetric matrices, copositive matrices, N-matrices, almost N-matrices and almost P-matrices.

Key words. Semipositive matrix, Minimally semipositive matrix, Principal pivot transform, Moore-Penrose inverse, Left inverse, Interval of matrices.

AMS subject classifications. 15A09, 15B48.

1. Introduction. Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the real numbers. We denote $\mathbb{R}^{n\times 1}$ by $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we denote $x \geq 0$ to signify the fact that all the components of x are nonnegative. $x > 0$ means all the entries of x are positive. A similar definition is adopted for matrices, too. Matrix $A \in \mathbb{R}^{m\times n}$ is said to be *semipositive*, if there exists a vector $x \geq 0$ such that $Ax > 0$. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be *minimally semipositive* if it is semipositive and no column-deleted submatrix of A is semipositive. In some places, this may be referred to as *minimal semipositivity*. By the continuity of a matrix as a linear map, it follows that A is semipositive if and only if there exists a vector $x \in \mathbb{R}^n$ with $x > 0$ such that $Ax > 0$ [9, Lemma 2.1]. Such a vector x is called a *semipositivity vector* of A. Let us also recall a result that characterizes minimal semipostivity of square matrices. Let $A \in \mathbb{R}^{n\times n}$. Then A is minimally semipositive if and only if A^{-1} exists and $A^{-1} \geq 0$ [9, Theorem 3.4]. This latter statement is sometimes referred to as inverse positivity of A. We thus have the following reformulation: A square matrix A is minimally semipositive if and only if it is inverse positive. More generally, one has: Let $A \in \mathbb{R}^{m\times n}$. If A is semipositive and has a nonnegative left inverse, then A is minimally semipositive. If A is minimally semipositive, then A has a nonnegative left inverse (see [9, Theorem 3.6] and [22, Theorem 2.3]). The notion of a semipositive matrix was considered by Stiemke [17] in connection with the problem of existence of positive solutions of linear systems. Semipositive matrices are also known as Steimke matrices or S-matrices [7]. These matrices play an important role in the field of linear complementarity problems [5], for instance. For more details, we refer to [2, 7, 9, 22].

Let us briefly review some recent results on semipositive matrices. In [6], the authors, among other results show that any matrix with at least two columns is a sum of two semipositive matrices (Theorem 2.1)

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and any matrix with at least two rows is a product of two semipositive matrices (Theorem 2.4). It is shown there (Corollary 5.3) that any spectrum (of a real matrix) with at least two elements is the spectrum of a semipositive matrix (see also Proposition 5.1 of [19]) and that any real matrix which is not a negative scalar matrix, is similar to a semipositive matrix (Theorem 5.2). A matrix $A \in \mathbb{R}^{n\times n}$ is semipositive if and only if there exist positive matrices X and Y such that X is invertible and $A = YX^{-1}$ [19, Theorem 3.1]. Let us also point to some of the recent results that were proved in [16]. The semipositive cone $K_A = \{x \geq 0 : Ax \geq 0\}$ of a matrix A, is considered under the assumption that A is a semipositive matrix. The duality of K_A is studied and it is shown that K_A is a proper polyhedral cone. The relation among semipositivity cones of two matrices is examined via generalized inverse positivity. Perturbations and intervals of semipositive matrices are discussed. Connections to certain matrix classes pertinent to linear complementarity theory are also studied.

Here is an outline of the article. In Section 2, we collect needed definitions and results. Section 3 presents two results: Theorem 3.1, where we give a basis consisting of minimally semipositive matrices for the linear space $\mathbb{R}^{m\times n}$, and Theorem 3.2, which shows that any matrix with at least as many rows as there are columns, is the difference of two minimally semipositive matrices. It is natural ask whether a product of semipositive or a product of minimally semipositive matrices is again semipositive or minimally semipositive, respectively. We discuss these problems in Section 4, from Theorem 4.1 to Theorem 4.6. In Section 5, we consider some problems involving interval matrices and semipositivity. For two matrices $A, B \in \mathbb{R}^{m\times n}$ such that $A \leq B$ (entrywise), consider the interval of matrices defined by $[A, B] = \{C : A \leq C \leq B\}$. We establish the following result: If A and B are minimally semipositive matrices, then any $C \in [A, B]$ is also minimally semipositive. In Section 6, we establish semipositivity and/or minimal semipositivity for the Schur complement and the principal pivot transform in Theorem 6.1. In Section 7, we study a number of matrix classes primarily arising from the theory of linear complementarity problem, in the context of semipositivity/minimal semipositivity. Theorem 7.4 deals with skew-symmetric and almost skew-symmetric matrices. A result for two classes of copositive matrices is presented in Theorem 7.7, while Theorem 7.14 presents a sufficient condition for an N-matrix of the first category to be minimally semipositive. Theorem 7.19 shows that almost N-matrices are not minimally semipositive, whereas Theorem 7.20 gives a characterization for an almost P-matrix to be semipositive. A generalization of semipositivity is recalled and connections to game theory are presented in the penultimate section. Two open questions are posed in the final section.

2. Preliminaries. For $A \in \mathbb{R}^{n\times n}$, let $\sigma(A)$ denote the set of all eigenvalues of the matrix A and let $\rho(A)$ denote the *spectral radius* of A, viz., the maximum of the moduli of the eigenvalues of A. If $\rho(A) < 1$, then $I - A$ is invertible. The next result says something about the nonnegativity of $(I - A)^{-1}$.

THEOREM 2.1. [21, Theorem 3.16] *Let $A \in \mathbb{R}^{n\times n}$. Then $\rho(A) < 1$ if and only if $(I - A)^{-1}$ exists and*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

If, in addition $A \geq 0$, then $(I - A)^{-1} \geq 0$.

The next result concerns an upper bound for the spectral radius.

THEOREM 2.2. [10, Theorem 16.2] *For $A \in \mathbb{R}^{n\times n}$, suppose that the inequality $Ax \leq \delta x$ holds, for some $x > 0$. Then $\rho(A) \leq \delta$.*

Next, we turn our attention to the notion of the principal pivot transform.

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Definition 2.3. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ be such that A is invertible. The *principal pivot transform of M with respect to A*, denoted by $ppt(M,A)$, is defined as $ppt(M,A) = \begin{pmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & M/A \end{pmatrix}$, where $M/A = D - CA^{-1}B$ is the *Schur complement of M in A*.

The following theorem is known as the domain-range exchange property of the principal pivot transform.

Theorem 2.4. [18, Theorem 3.1] *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ be a partitioned matrix such that A is invertible. Given a pair of vectors $x, y \in \mathbb{R}^n$ partitioned as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ conformally with partition of M, define vectors $u, v \in \mathbb{R}^n$ such that $u_1 = y_1$, $u_2 = x_2$, $v_1 = x_1$ and $v_2 = y_2$. Then $H = ppt(M,A)$ is the unique matrix with the property that for every such x, y, one has $y = Ax$ if and only if $Hu = v$.*

For more details about the principal pivot transforms we refer to [18]. The following result connecting principal pivot transform and semipositive matrices was established in [19].

Theorem 2.5. [19, Theorem 3.2] *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ be a partitioned matrix such that A is invertible. Then M is semipositive if and only if the principal pivot transform $ppt(M,A)$ is semipositive.*

Next, we discuss the notion of a certain generalized inverse. For $A \in \mathbb{R}^{m\times n}$, let A^T, $R(A)$ and $N(A)$ denote the *transpose* of A, the *range space* of A, and the *null space* of A, respectively. The *Moore-Penrose inverse* of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying (1) $A = AXA$, (2) $X = XAX$, (3) $(AX)^T = AX$, and (4) $(XA)^T = XA$, and is denoted by $A^\dagger$. Given any matrix $A \in \mathbb{R}^{m\times n}$ a matrix $X \in \mathbb{R}^{n\times m}$ satisfying $XA = I$ is called a *left inverse* and is denoted by A^L. If A is nonsingular, then $A^{-1} = A^\dagger = A^L$. For more details, we refer the reader to the books [1, 3].

3. Some general results. The purpose of this section is to present two results of general interest. In the first result, we construct a basis consists of minimally semipositive matrices for the linear space of $m \times n$ real matrices. This might be useful in studying linear preserver problems, where it is important to know whether a collection of $m \times n$ matrices contains a basis for $\mathbb{R}^{m\times n}$. The second result states that any square matrix is a difference of two minimally semipositive matrices.

Theorem 3.1. *There is a basis of minimally semipositive matrices for $\mathbb{R}^{m\times n}$, $m \geq n$.*

Proof. Define the set $\mathcal{B}$ of matrices $\{A^{ij} : i \in \{1,\ldots,m\},\ j \in \{1,\ldots,n\}\}$ as follows: For $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$, if $i = sn + j$ for some integer $s \geq 0$, define

$$(A^{ij})_{kl} = \begin{cases} 1 & \text{if } i \neq k \text{ and } k = tn + l \text{ for some integer } t \geq 0, \\ 2 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise} \end{cases}$$

and if $i \neq sn + j$, define

$$(A^{ij})_{kl} = \begin{cases} 1 & \text{if } k = tn + l \text{ for some integer } t \geq 0, \\ -2 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

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Now, let us explicitly write down the elements of the set $\mathcal{B}$:

$$A^{11} = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \quad A^{22} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \ \ldots,$$

$$A^{n-1n-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \quad A^{nn} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 2 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix},$$

$$A^{n+1n} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \ \ldots, \ A^{12} = \begin{pmatrix} 1 & -2 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \ \ldots,$$

$$A^{13} = \begin{pmatrix} 1 & 0 & -2 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix}, \ \ldots$$

Then for each i, j, the matrix A^{ij} is minimally semipositive. For, each row of A^{ij} contains exactly one positive entry, and if we remove any column, then the column deleted submatrix contains a nonpositive row. Also, it is easy to verify that $\mathcal{B} = \{A^{11}, \ldots, A^{mn}\}$ spans $\mathbb{R}^{m\times n}$. ◻

Next, we show a rather curious result that any matrix (with at least as many rows as there are columns) is the difference of two minimally semipositive matrices. Clearly, this is a stronger result than Theorem 3.1.

THEOREM 3.2. *Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$. Then there exist minimal semipositive matrices $B, C \in \mathbb{R}^{m\times n}$ such that $A = B - C$.*

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Proof. For $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$, if $i \neq sn+j$ for some integer $s \geq 0$, define

$$b_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \leq 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_{ij} = \begin{cases} -a_{ij} & \text{if } a_{ij} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

When $i = sn+j$, choose b_{ij} and c_{ij} such that

$$b_{ij} > \sum_{\substack{k=1 \\ k\neq j}}^{n} |b_{ik}|, \quad c_{ij} > \sum_{\substack{k=1 \\ k\neq j}}^{n} |c_{ik}| \quad \text{and} \quad a_{ij} = b_{ij} - c_{ij}.$$

It follows that $A = B - C$ and that B and C are semipositive with a common semipositivity vector $e^T = (1,\dots,1)^T$. Note that each row of B, C contains exactly one positive entry. So, any column deleted submatrix contains a nonpositive row. Thus, B and C are minimally semipositive. ∎

Let us illustrate the construction of the previous result, by an example.

Example 3.3. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 0 & -2 \\ -5 & -2 & -1 \\ 1 & 7 & 4 \end{pmatrix} \in \mathbb{R}^{4\times 3}$. Take

$$B = \begin{pmatrix} 5 & -2 & 0 \\ 0 & 5 & -2 \\ -5 & -2 & 8 \\ 13 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 4 & 0 & -3 \\ -4 & 5 & 0 \\ 0 & 0 & 9 \\ 12 & -7 & -4 \end{pmatrix}.$$

Then B, C are minimally semipositive and $A = B - C$.

4. Considerations involving products. In this section, we study products of semipositive matrices. First, we show that semipositivity and minimal semipositivity are invariant under multiplication by permutation and positive diagonal matrices (Theorem 4.1). Two results that characterize the semipositivity of a product of two matrices, in terms of the semipositivity/minimal semipositivity of one of the factors (Theorem 4.2 and Theorem 4.6) follow.

Theorem 4.1. *Let $A \in \mathbb{R}^{m\times n}$. Suppose $D_1 \in \mathbb{R}^{m\times m}$ and $D_2 \in \mathbb{R}^{n\times n}$ are diagonal matrices with positive diagonal entries and $P_1 \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{n\times n}$ be permutation matrices. We have:*

(a) If A is semipositive, then $P_1D_1AD_2P_2$ is semipositive.

(b) If A is minimally semipositive, then $P_1D_1AD_2P_2$ is minimally semipositive.

Proof. (a) Suppose that A is semipositive. Then there exist positive vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that $Ax = y$. Let $z = P_2^T D_2^{-1} x$. Then $z > 0$ and $P_1D_1AD_2P_2z = P_1D_1y > 0$. Thus, $P_1D_1AD_2P_2$ is a semipositive matrix.

(b) If A is minimally semipositive, then, A has a nonnegative left inverse, say, X. Then $P_2^T D_2^{-1} X D_1^{-1} P_1^T$ is a nonnegative left inverse of the matrix $P_1D_1AD_2P_2$ and hence, $P_1D_1AD_2P_2$ is minimally semipositive. ∎

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THEOREM 4.2. *Let $A \in \mathbb{R}^{n\times n}$. We have:*

(a) *If A is minimally semipositive, then BA is semipositive for every semipositive matrix $B \in \mathbb{R}^{m\times n}$, for any $m \in \mathbb{N}$.*

(b) *If BA is semipositive for all semipositive matrices $B \in \mathbb{R}^{m\times n}$, for some $m \in \mathbb{N}$, then A is minimally semipositive.*

Proof. (a) Let A be a minimally semipositive matrix and B be a semipositive matrix with a semipositivity vector x. Then $A^{-1} \geq 0$ and so $A^{-1}x$ is a semipositivity vector for BA. Thus, BA is semipositive.

(b) Suppose that BA is semipositive for every semipositive matrix B. We first prove that A is invertible. Let $x = (x_1, \dots, x_n)^T$ be such that $x^TA = 0$. Suppose that $x \neq 0$. With out loss of generality we can assume $x_1 > 0$. Let B be the matrix whose first row is the row vector x^T and the remaining rows are equal to the row vector $e^T = (1, \dots, 1)$. Since the first column of B is positive, the matrix B is semipositive. However, by definition the first row of the matrix BA is zero so that BA is not semipositive. This is a contradiction. Hence, A is invertible. Next, let us show that $C = A^{-1} \geq 0$. On contrary, suppose that $c_{ij} < 0$ for some i, j. Let B be the matrix whose first row is the negative of the i^{th} row of C and all other rows equal to e^T. Since the j^{th} column of B is positive, the matrix B is semipositive. However the first row of the matrix BA is nonpositive, and hence, BA is not semipositive, contradiction to the assumption. Thus, $A^{-1} \geq 0$, and hence, A is minimally semipositive. □

REMARK 4.3. Let $A = \begin{pmatrix} -3 & -3 & 3 \\ -3 & -5 & 3 \\ 2 & 2 & -1 \end{pmatrix}$. Then A is semipositive. We claim that A is not minimally semipositive. By Theorem 4.2, it suffices to show that there exists a semipositive B such that BA is not semipositive. One may verify that a choice is $B = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

The proof of the next result is similar to part (b) of the theorem above and is skipped.

THEOREM 4.4. *Let $A \in \mathbb{R}^{m\times n}$ be a semipositive matrix. If BA is semipositive for all semipositive matrices $B \in \mathbb{R}^{k\times m}$, for some $k \in \mathbb{N}$, then A is minimally semipositive.*

REMARK 4.5. If $A \in \mathbb{R}^{m\times n}$ is minimally semipositive and $B \in \mathbb{R}^{n\times n}$ is semipositive then BA need not be semipositive. This is shown by the following choices: $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. The post-multiplication of a semipositive matrix by a positive matrix need not be semipositive. This is shown by the matrices $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ (which is semipositive) and (the positive matrix) $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In particular, it follows that if A and B are semipositive matrices, then their product need not be semipositive. In the next result, in item (a), we show that semipositivity is preserved under pre-multiplication by nonnegative irreducible matrices. The case of minimal semipositivity is considered next. In item (c), we establish a product result for rectangular matrices with the additional assumption that the product is semipositive.

THEOREM 4.6. (a) *Let $A \in \mathbb{R}^{m\times n}$ be a semipositive matrix. If $B \in \mathbb{R}^{m\times m}$ is a nonnegative irreducible matrix, then BA is a semipositive matrix.*

(b) *Let $A, B \in \mathbb{R}^{n\times n}$ be minimally semipositive matrices. Then AB is minimally semipositive.*

(c) *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ be two minimally semipositive matrices such that AB is semipositive. Then AB is a minimally semipositive matrix.*

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Proof. (a) Let A be a semipositive matrix with semipositivity vector u. Since B is a nonnegative irreducible matrix, we have $BAu > 0$. Thus, BA is semipositive.

(b) Since $B^{-1}A^{-1} \geq 0$ is the inverse of AB, by the remarks made in the introduction, it follows that AB is minimally semipositive.

(c) Suppose that A and B are minimally semipositive matrices. Then there exist left inverses A^L and B^L of A and B, respectively such that $A^L \geq 0$ and $B^L \geq 0$. Now B^LA^L is a left inverse of AB and $B^LA^L \geq 0$. Again, it follows that AB is minimally semipositive. ∎

Remark 4.7. In (c) of Theorem 4.6, the assumption that AB is semipositive cannot be dropped. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then A, B are minimally semipositive but $AB = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not semipositive.

5. Considerations involving intervals of matrices. This section concerns semipositivity or minimal semipositivity of certain intervals of matrices.

Recall that matrix $A \in \mathbb{R}^{m\times n}$ is said to be *rectangular monotone*, if $Ax \geq 0$ implies $x \geq 0$ for all $x \in \mathbb{R}^n$. It is known that, A is rectangular monotone if and only if A has a nonnegative left inverse [11]. From the discussion in the introductory section, it now follows that if A is semipositive and rectangular monotone, then A is minimally semipositive. Conversely, if A is minimally semipositive, then A is rectangular monotone.

Following [10], we define a *bilateral interval* as $[A, B] = \{C : A \leq C \leq B\}$ for $A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times n}$ and $A \leq B$. A *unilateral interval* is an interval of the type $(-\infty, B]$ so that $C \in (-\infty, B]$ if and only if $C \leq B$.

Let us recall the following rather well known result for a unilateral interval, which has been reformulated using the terminology adopted here. Let $int(\mathbb{R}^n_+)$ denote the set of interior points of $\mathbb{R}^n_+$.

Theorem 5.1. [10, Theorem 25.4] *Let $B \in \mathbb{R}^{n\times n}$ with $C \in (-\infty, B]$. Let B be inverse positive. Then C is inverse positive if and only if $int(\mathbb{R}^n_+) \cap C\mathbb{R}^n_+ \neq \emptyset$.*

It is interesting to note that the condition $int(\mathbb{R}^n_+) \cap C\mathbb{R}^n_+ \neq \emptyset$ is equivalent to C being semipositive. Thus, Theorem 5.1 may be rewritten as: Let C and B be real square matrices with $C \leq B$, where B is minimally semipositive. Then C is semipositive if and only if C is minimally semipositive. This bears a striking resemblence to a very well known result for Z-matrices (matrices whose off-diagonal entries are nonpositive), which states that a Z-matrix is semipositive if and only if it is minimally semipositive (see [2] for a proof). In Theorem 5.3 below, we extend the necessity part (of the result for the matrix C stated earlier) to rectangular matrices.

Let $A, B \in \mathbb{R}^{m\times n}$. If A is semipositive and $A \in (-\infty, B]$, then it is easy to see that B is a semipositive matrix. Next, in Corollary 5.5, we show that this property extends to rectangular matrices that are minimally semipositive.

Lemma 5.2. *Let $A \in \mathbb{R}^{m\times n}$ with $A = U - V$. Suppose that U has a nonnegative left inverse U^L which satisfies $U^LV \geq 0$ and $\rho(U^LV) < 1$. Then A is rectangular monotone.*

Proof. By Theorem 2.1, it follows that $(I - U^LV)$ is invertible and $(I - U^LV)^{-1} \geq 0$. Set $X = (I - U^LV)^{-1}U^L$. Then $X \geq 0$ and $XA = (I - U^LV)^{-1}U^L(U - V) = (I - U^LV)^{-1}(I - U^LV) = I$. ∎

Theorem 5.3. *Let $A, B \in \mathbb{R}^{m\times n}$ with $A \in (-\infty, B]$. Suppose that B is rectangular monotone. If A is semipositive, then A is rectangular monotone.*

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Proof. Since $A \leq B$, there exists $T \geq 0$ such that $A = B - T$. Since B is rectangular monotone, there exists $B^L \geq 0$ such that $B^L B = I$. Also, $B^L T \geq 0$. Let $x \in \mathbb{R}^n_+$ such that $Ax \in int(\mathbb{R}^m_+)$. Since $A \leq B$, it follows that $Bx \in int(\mathbb{R}^m_+)$. Since Bx and Ax are positive, there exists $\epsilon > 0$ such that $\epsilon Bx \leq Ax$. Thus, $(B - A)x \leq (1 - \epsilon)Bx$. Now $TB^L Bx = Tx = (B - A)x$. Hence, $TB^L Bx \leq (1 - \epsilon)Bx$. By Theorem 2.2, $\rho(B^L T) = \rho(TB^L) \leq (1 - \epsilon) < 1$. Hence, by Lemma 5.2, A is rectangular monotone. ∎

Remark 5.4. Suppose that there exist matrices A and B such that $A \in (-\infty, B]$, with A and B both being rectangular monotone. This does not guarantee that A is semipositive (even if B is semipositive). Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then A and B are rectangular monotone, with B being semipositive. However, since A has a zero row, it is not semipositive.

Corollary 5.5. *Let $A, B \in \mathbb{R}^{m\times n}$ with B being minimally semipositive. If $A \in (-\infty, B]$ and A is semipositive, then A is minimally semipositive.*

Proof. Since B is minimally semipositive, B is rectangular monotone. By Theorem 5.3, A is rectangular monotone, and hence, A is minimally semipositive. ∎

Corollary 5.6. *Let $A, B \in \mathbb{R}^{m\times n}$ be minimally semipositive matrices. If $A \in (-\infty, B]$, then C is minimally semipositive for every $C \in [A, B]$.*

For the next result, we need the following notation. Let $diag(t_1, t_2, \ldots, t_n)$ denote the diagonal matrix whose entries are $t_1, t_2, \ldots, t_n$. For any $A, B \in \mathbb{R}^{n\times n}$, we define the following sets:

$$h(A, B) = \{C : C = tA + (1 - t)B, t \in [0, 1]\},$$

$$r(A, B) = \{C : C = TA + (I - T)B, T = diag(t_1, t_2, \ldots, t_n), t_i \in [0, 1], 1 \leq i \leq n\},$$

and

$$c(A, B) = \{C : C = AT + B(I - T), T = diag(t_1, t_2, \ldots, t_n), t_i \in [0, 1], 1 \leq i \leq n\}.$$

The first set is the set of all convex linear combinations of the matrices A and B; the second is the set of all matrices each of whose rows is a convex linear combination of the corresponding rows of A and B. The third set is the same as the second in whose definition, columns replace rows.

In what follows, we present semipositivity results for the first two sets and show that such an analogue does not hold for the third.

Theorem 5.7. *Let $A, B \in \mathbb{R}^{m\times n}$ be semipositive matrices with a common semipositivity vector u. The following hold:*

(*a*) *If $C \in h(A, B)$, then C is semipositive with semipositivity vector u.*

(*b*) *If $C \in r(A, B)$, then C is semipositive with semipositivity vector u.*

Proof. It is enough to prove (b). Let $u \in \mathbb{R}^n$ be such that $u > 0$, $Au > 0$ and $Bu > 0$. Let $C \in r(A, B)$ so that $C = TA + (I - T)B$, where $T = diag(t_1, t_2, \ldots, t_m), t_i \in [0, 1], 1 \leq i \leq m$. Now, $(Cu)_i = t_i(Au)_i + (1 - t_i)(Bu)_i > 0$. Thus, C is a semipositive matrix (with the semipositive vector u). ∎

Remark 5.8. Theorem 5.7 does not hold for the subset $c(A, B)$. Let $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. It is easy to see that A and B are semipositive with the common semipositive vector $(1, 1)^T$. We have $C = \begin{pmatrix} \frac{3}{2} & 0 \\ \frac{-1}{4} & 0 \end{pmatrix} \in c(A, B)$, which is not semipositive.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 35-53, February 2018.

6. Considerations involving block matrices. In this section, we study the semipositivity notions for the principal pivot transform. Let us observe that if M is semipositive, then it does not imply that M^T or any principal submatrix of M is semipositive. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ be a semipositive matrix such that A is invertible. Then the Schur complement $M/A = D - CA^{-1}B$ need not be semipositive. For example, consider the semipositive matrix $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then $M^T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $D = 0$ and $M/A = 0$, are not semipositive. Note however, that if M is a minimally semipositive matrix, then $(M^T)^{-1} = (M^{-1})^T \geq 0$ and so M^T is minimally semipositive. It may be shown by examples that if M is minimally semipositive, then a principal submatrix of M need not be minimally semipositive. The principal pivot transform of a minimally semipositive matrix need not be minimally semipositive. Consider the minimally semipositive matrix $M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. The principal pivot transform $ppt(M, A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, with respect the sub matrix $A = (1)$, is not minimally semipositive.

In the first result to follow, we show that if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is minimally semipositive, then the Schur complement M/A is minimally semipositive.

THEOREM 6.1. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ *be a minimally semipositive matrix such that* $A \in \mathbb{R}^{r\times r}$ *is invertible. We then have the following:*

(a) *The Schur complement* M/A *is minimally semipositive.*

(b) $H = ppt(M, A)$ *is semipositive. For an* $n \times n$ *matrix* X*, let* $X_{.j}$ *denote the submatrix of* X *obtained by deleting the* j^{th} *column. Let* $j \in \{r + 1, \dots, n\}$*. Then the matrix obtained from* H*, by deleting its* j^{th} *column, is not semipositive.*
(c) *Let* $C = 0$*. Then* A *and* D *are minimally semipositive.*

Proof. (a) Since M is minimally semipositive, M^{-1} exists and $M^{-1} \geq 0$. By the Banachiewicz-Schur formula, M^{-1} can be written as

$$\begin{pmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{pmatrix}.$$

Thus, $(M/A)^{-1} \geq 0$. Hence, M/A is minimally semipositive.

(b) The first part is known, for instance Theorem 3.2 of [19]. By the domain-range exchange property, we have $M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ if and only if $H \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}$. Let $j \in \{r + 1, \dots, n\}$. Suppose that $H_{.j}$ is semipositive. Let $u \in \mathbb{R}^{n-1}$ be such that $u > 0$ and $H_{.j}u > 0$. Define the vector $w \in \mathbb{R}^n$ such that $w_i = u_i$ for all $i \neq j$ and $w_j = 0$. Then, w is a semipositive vector for the matrix H. Let $Hw = z$. Now, by the domain-range exchange property, we get $M \begin{pmatrix} z_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ z_2 \end{pmatrix}$. Since $w_j = 0$, the submatrix of M, obtained by deleting the column j, is semipositive. This contradicts the minimal semipositivity of M.
(c) Since M is a minimally semipositive matrix, M^{-1} exists and $M^{-1} \geq 0$. Thus, A and D are also invertible. So, M^{-1} can be written as

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 35-53, February 2018.

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

Thus, $A^{-1} \geq 0$ and $D^{-1} \geq 0$ so that A and D are minimally semipositive. □

Remark 6.2. It is possible to consider an extension of the idea of the principal pivot transform involving the Moore-Penrose inverse of the leading principal subblock. This leads to statements that are more general than the results of this section.

7. Considerations involving other classes of matrices. The purpose of this section is to establish semipositivity and/or minimal semipositivity of several classes of matrices. While the first two classes are skew-symmetric and almost skew-symmetric matrices, the other classes are primarily studied in the context of the linear complementarity problem.

We begin by showing how one could construct semipositive matrices by using skew-symmetric matrices or almost skew-symmetric matrices as its blocks. This provides a class of examples for semipositive matrices, other than the classes of positive definite matrices and irreducible nonnegative matrices.

First, we collect a couple of preliminary results that will be used in this section. The next result is referred to as Ville's theorem of the alternative and is an easy consequence of the Farkas' lemma [12]. Note that the second statement is precisely the semipositivity of A^T.

Theorem 7.1. *Let $A \in \mathbb{R}^{m\times n}$. Then exactly one of the following is true:*

(a) *There is an $x \geq 0$, $x \neq 0$, such that $Ax \leq 0$.*

(b) *There is a $y \geq 0$ such that $A^T y > 0$.*

Definition 7.2. $A \in \mathbb{R}^{n\times n}, n \geq 2$ is called an almost skew-symmetric matrix if its symmetric part $(\frac{A+A^T}{2})$ has rank one.

If $A \in \mathbb{R}^{n\times n}, n \geq 2$ is almost skew-symmetric, then the symmetric part of A has exactly one nonzero eigenvalue. Let $\delta(A)$ denote the nonzero eigenvalue of the symmetric part of A.

Theorem 7.3. [4, Theorem 3.2] *Let $A \in \mathbb{R}^{n\times n}, n \geq 2$ be an almost skew-symmetric matrix with $\delta(A) > 0$. Then A is positive semidefinite.*

Theorem 7.4. *Let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix, or an almost skew-symmetric matrix with $\delta(A) > 0$. Define $M = \begin{pmatrix} I & A \\ A & I \end{pmatrix}$. Then M is semipositive.*

Proof. Let A be skew-symmetric. Suppose that there exists a vector $z = (x^T, y^T)^T \geq 0, z \neq 0$ such that $M^T z \leq 0$. Then

$$x - Ay \leq 0 \quad \text{and} \quad -Ax + y \leq 0.$$

Let $v = x + y$. Then $v \geq 0, v \neq 0$ and $v - Av \leq 0$. Thus, $v^T v - v^T Av \leq 0$, so that $v^T v \leq 0$, since, A is skew-symmetric and so $v^T Av = 0$. This is a contradiction. By Theorem 7.1, there exists $w \geq 0$ such that $Mw > 0$. Hence, M is semipositive.

Next, let A be almost skew-symmetric with $\delta(A) > 0$. Once again, suppose that there exists a vector $z = (x^T, y^T)^T \geq 0, z \neq 0$ such that $M^T z \leq 0$. Then

$$x + A^T y \leq 0 \quad \text{and} \quad A^T x + y \leq 0.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Let $v = x+y$. Then $v \geq 0, v \neq 0$ and $v + A^T v \leq 0$. Thus, $v^T v + v^T A^T v \leq 0$ and so one has $v^T Av < 0$. This however, is a contradiction, since by Theorem 7.3, A is positive semidefinite. By Theorem 7.1, there exists $w \geq 0$ such that $Mw > 0$. Hence, M is semipositive. □

Remark 7.5. Let us note that there is a simpler way of proving the first part of Theorem 7.4, making use of what is known as Tucker's theorem for skew-symmetric matrices [20, Theorem 5]. Let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix. Then there exists $x \geq 0$ such that $Ax \geq 0$ and $Ax + x > 0$.

Let A be skew-symmetric. By Tucker's theorem, there exists $x \geq 0$ such that $Ax \geq 0$ and $Ax + x > 0$. Let $y = \begin{pmatrix} x \\ x \end{pmatrix} \geq 0$. Then

$$My = \begin{pmatrix} I & A \\ A & I \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x + Ax \\ x + Ax \end{pmatrix} > 0.$$

Thus, M is semipositive.

In what follows, we present results for matrix classes that are mainly studied in the context of the linear complementarity problem. Let $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The linear complementarity problem $LCP(A, q)$ is to find if there exists $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad y = Ax + q \geq 0 \quad \text{and} \quad x^T y = 0.$$

A is called a Q-matrix, if $LCP(A, q)$ has a solution for all $q \in \mathbb{R}^n$. We refer the reader to the book [5] for more details on the properties of these matrices.

Let us recall that $A \in \mathbb{R}^{n\times n}$ is called an *N-matrix* if all the principal minors of A are negative. In particular, all the diagonal entries of an N-matrix are negative. An N-matrix A is said to be of the *first category*, if A has at least one positive entry. Otherwise, A is said to be of the *second category*. A real square matrix A is called an *almost N-matrix*, if all the proper principal minors of A are negative, while the determinant of A is positive. An almost N-matrix A is said to be of the *first category*, if A as well as A^{-1} has at least one positive entry. $A \in \mathbb{R}^{n\times n}$ is called a *P-matrix* if all the principal minors of A are positive. It is known that every P-matrix is a Q-matrix. $A \in \mathbb{R}^{n\times n}$ is called an *almost P-matrix* if all the proper principal minors of A are positive, whereas the determinant of A is negative. Note that all these classes of matrices are invertible. Further note that, if A is an almost P-matrix, then A^{-1} is an N-matrix. We shall also be interested in three classes of copositive matrices. A is called *copositive*, if one has $x^T Ax \geq 0$ whenever $x \geq 0$. A is referred to as a *strictly copositive matrix*, if $x^T Ax > 0$ whenever $0 \neq x \geq 0$. A copositive matrix A is called *copositive plus* if $x \geq 0,\ x^T Ax = 0$ imply $(A + A^T)x = 0$. Any strictly copositive matrix is copositive plus. A necessary and sufficient condition for a copositive plus matrix to be strictly copositive is that, either it is nonsingular or it is singular and has no nonnegative eigenvector associated with the zero eigenvalue. Any positive semidefinite matrix is copositive plus, while any positive definite matrix is strictly copositive. Finally, a copositive matrix A is referred to as a *copositive star* matrix if $x \geq 0,\ Ax \geq 0$ and $x^T Ax = 0$ imply $A^T x \leq 0$. Copositive star matrices were introduced and studied in [8]. It is known that if A is a copositive plus matrix, then it is a copositive star matrix. Also, a symmetric copositive star matrix is copositive plus. We refer the reader to [8] for applications of copositive star matrices to the linear complementarity problem.

Next, we present results that are pertinent to semipositivity and/or minimal semipositivity for these classes of matrices. The first result to follow, collects well known statements and also provides a proper perspective.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 35-53, February 2018.

THEOREM 7.6. *Let $A \in \mathbb{R}^{n\times n}$.*

(a) *If A is a symmetric positive definite matrix then A is semipositive [9, Theorem 2.10].*

(b) *Let A be a P-matrix or an N-matrix of the first category. Then A is semipositive [14, Theorem 2].*

(c) *Let A be an almost N-matrix, of order $n \geq 4$. Then A is semipositive if and only if A is (an almost N-matrix) of the first category [13, Corollary 3.2].*

In the background of this result, in this section, we show the following: Let $A \in \mathbb{R}^{n\times n}$ be an N-matrix of the first category, with $n \geq 3$. We characterize its minimal semipositivity (Theorem 7.14). On the other hand, we show that if A is an almost N-matrix, then A is never minimally semipositive (Theorem 7.19). A characterization for the semipositivity of almost P-matrices (Theorem 7.20) is given.

First, we show that item (a) of Theorem 7.6 is true for a bigger class of matrices, viz. copositive matrices.

THEOREM 7.7. *Let $A \in \mathbb{R}^{n\times n}$.*

(a) *If A is strictly copositive, then A is semipositive. (see also the paragraph before Theorem 7.21)*

(b) *If A is a symmetric copositive plus matrix, which is not strictly copositive, then A is not semipositive.*

Proof. (a) Since A is strictly copositive, so is A^T. Suppose that there exists a vector $x \geq 0$ such that $A^T x \leq 0$. Then $x^T Ax \leq 0$, a contradiction to the strict copositivity of A. By Theorem 7.1, there exists $y \geq 0$ such that $Ay > 0$. Hence, A is semipositive.

(b) Since A is not strictly copositive, there exists $x \geq 0$ such that $x^T Ax = 0$. Thus, $Ax = 0$, since A is copositive plus. By Theorem 7.1, there does not exist any $y \geq 0$ such that $Ay > 0$. Hence, A is not semipositive. ∎

REMARK 7.8. In (b), if we relax the symmetry assumption, then the conclusion does not hold. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Then A is a non-symmetric copositive plus matrix. A is semipositive with a semipositive vector $x = (1, 0)^T$. Let $B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Then B is copositive-plus matrix and B is not symmetric. B is also not semipositive.

REMARK 7.9. It appears that, unlike the classes of strictly copositive matrices and copositive plus matrices, there is nothing conclusive that one could say about the class of copositive star matrices. This is shown by the following examples. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then A is a copositive star matrix which is not semipositive. Let $B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Then B is a copositive star matrix which is also semipositive.

REMARK 7.10. Given $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, if the quadratic function $f(z) = z^T(Az + q),\ z \geq 0$ is bounded below then $LCP(A, q)$ has a solution [5, Corollary 3.7.12]. By Theorem 3.8.5 of [5], if A is strictly copositive, then A is a Q-matrix.

Let A be a Q-matrix. Consider $q = -e$. Then there exists $x \geq 0$ such that $y = Ax - e \geq 0$ (and $x^T y = 0$). In particular, one has $x \geq 0$ with $Ax > 0$, showing that A is semipositive. In view of the previous

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 35-53, February 2018.

paragraph, we have an alternative proof of (a) of Theorem 7.7. Note however, that the argument here leads to a stronger conclusion that a strictly copositive matrix is a Q-matrix.

REMARK 7.11. Recall that matrix $A \in \mathbb{R}^{n\times n}$ is said to be strictly semimonotone if the following implication holds [5, Definition 3.9.9]:

$$x \in \mathbb{R}^n_+ \setminus \{0\} \Rightarrow \text{ there exists } k \text{ such that } x_k > 0 \text{ and } (Ax)_k > 0.$$

Let $A \in \mathbb{R}^{n\times n}$ be a strictly copositive matrix. Then all its principal submatrices are also strictly copositive. By Theorem 7.7, all the principal submatrices of A are semipositive. By Corollary 3.9.13 of [5], A is strictly semimonotone.

Next, we turn our attention to N-matrices. First, we prove certain fundamental properties of these matrices. Though we expect that these must be known, we have not found a suitable reference for these results. This also makes the article as self-contained as possible. In what follows, all the pairs of the subscripts refer to distinct indices.

Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and let $1 \le k_1 < k_2 < \cdots < k_r \le n$. Then $A[k_1, k_2, \ldots, k_r]$ denote the $r \times r$ submatrix of A whose (i, j) entry is $a_{k_i k_j}$.

THEOREM 7.12. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an N-matrix of the first category. Then the following hold:*

(i) All the entries of A are nonzero.

(ii) Each row and column of A contains atleast one positive entry.

(iii) Both a_{ij} and a_{ji} have the same sign.

(iv) If $a_{ij}, a_{ik} > 0$, then $a_{jk}, a_{kj} < 0$.

(v) If $a_{ij}, a_{kj} > 0$, then $a_{ik}, a_{ki} < 0$.

Proof. (i) As mentioned earlier, all the diagonal entries are negative, since A is N-matrix. Suppose that $a_{ij} = 0$, where $i \neq j$. Consider the principal submatrix

$$A[i, j] = \begin{pmatrix} a_{ii} & 0 \\ a_{ji} & a_{jj} \end{pmatrix}.$$

Then $det(A[i, j])$ is positive, a contradiction. Hence, (i) holds.

(ii) As A is N-matrix of the first category, so is A^T. By (b) of Theorem 7.6, A and A^T are semipositive. Thus, each row and column of A contains at least one positive entry.

(iii) Suppose that $a_{ij} > 0$ and $a_{ji} < 0$ for some $i \in \{1, \ldots j-1, j+1, \ldots n\}$ and $j \in \{1, \ldots i-1, i+1, \ldots n\}$. Consider the principal submatrix

$$A[i, j] = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}.$$

Then $det(A[i, j])$ is positive, a contradiction. Hence, (iii) is true.

(iv) Suppose that $a_{ij}, a_{ik} > 0$, where $j \neq k$. Let $a_{jk} > 0$. From (ii), $a_{kj} > 0$. Consider the 3×3 principal submatrix

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 34, pp. 35-53, February 2018.

$$A[i,j,k] = \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix}.$$

Then $det(A[i,j,k]) = a_{ii}(a_{jj}a_{kk} - a_{jk}a_{kj}) - a_{ij}(a_{ji}a_{kk} - a_{ki}a_{jk}) + a_{ik}(a_{ji}a_{kj} - a_{ki}a_{jj}) > 0$, a contradiction. Hence, (iv) is true.

(v) Similar to (iv). □

REMARK 7.13. From items (iv) and (v) above, it follows that for an N-matrix of the first category $(n \geq 3)$, there is a row which contains a non-diagonal negative entry.

In the next result, we present a class of semipositive matrices, which turn out to be also minimally semipositive. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is an N-matrix of the first category. Then $a, d < 0$ and by item (ii) of Theorem 7.12, $b, c > 0$, so that $A^{-1} > 0$. Thus, A is minimally semipositive. In the next result, we prove that A is minimally semipositive for $n \geq 3$. Recall that, in the first place, by item (b) of Theorem 7.6, such matrices are semipositive.

THEOREM 7.14. *Let $b, c \in \mathbb{R}^{n-1}$ be positive vectors and $A = \begin{pmatrix} \alpha & b^T \\ c & D \end{pmatrix} \in \mathbb{R}^{n\times n}, n \geq 3$ be an N-matrix of the first category. Then, A/α is minimally semipositive if and only if A is minimally semipositive.*

Proof. The sufficiency part follows from Theorem 6.1. We prove the necessity part. Since A is N-matrix of the first category, all its principal minors are negative. So are all the diagonal entries of the matrix $adj(A)$. Thus, the diagonal entries of $A^{-1} = \frac{adj(A)}{det(A)}$ are positive. Suppose that the Schur complement (of α in A) $A/\alpha = D - \alpha^{-1}cb^T$ is minimally semipositive. Then $(A/\alpha)^{-1} \geq 0$. Since $\alpha < 0$ and $b, c > 0$, it follows that $-\alpha^{-1}b^T(A/\alpha)^{-1}$ and $-\alpha^{-1}(A/\alpha)^{-1}c$ are nonnegative. By the Banachiewicz-Schur formula,

$$A^{-1} = \begin{pmatrix} \alpha^{-1} + \alpha^{-2}b^T(A/\alpha)^{-1}c & -\alpha^{-1}b^T(A/\alpha)^{-1} \\ -\alpha^{-1}(A/\alpha)^{-1}c & (A/\alpha)^{-1} \end{pmatrix} \geq 0.$$

Hence, A is minimally semipositive. □

Let us give two examples.

EXAMPLE 7.15. Let $\alpha = -2$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & -3 \\ -3 & -2 \end{pmatrix}$. Then

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 3 & -1 & -3 \\ 5 & -3 & -2 \end{pmatrix}$$

is N-matrix of the first category. The Schur complement (of α in A) $A/\alpha = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 6 \end{pmatrix}$ is minimally semipositive. Now $A^{-1} = \frac{1}{3}\begin{pmatrix} 7 & 4 & 1 \\ 9 & 6 & 0 \\ 4 & 1 & 1 \end{pmatrix}$. Thus, A is minimally semipositive.

EXAMPLE 7.16. Let $A = \begin{pmatrix} -1 & 3 & 3 \\ 3 & -2 & -3 \\ 3 & -3 & -3 \end{pmatrix}$ and $\alpha = -1$. Then $A/\alpha = \begin{pmatrix} 7 & 6 \\ 6 & 6 \end{pmatrix}$ is not minimally

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semipositive. By the theorem above, it follows that A is not minimally semipositive.

Remark 7.17. By (b) of Theorem 4.1, minimal semipositivity is invariant under multiplication by permutation matrices. Thus, if an N-matrix of the first category contains a row in which all the non-diagonal entries are positive, and the Schur complement corresponding to that diagonal is minimally semipositive, then A is minimally semipositive.

Remark 7.18. Let us consider the possibility of extending Theorem 7.14 by proposing the following analogue: Let $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be an N-matrix of the first category such that A_{12}, A_{21} are positive matrices and the Schur complement $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is minimally semipositive. One may ask if M is minimally semipositive. We show that the answer is in the negative. Let $A_{11} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $A_{21} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$, and $A_{22} = \begin{pmatrix} -1 & -6 \\ -3 & -1 \end{pmatrix}$, so that

$$M = \begin{pmatrix} -1 & -2 & 1 & 2 \\ -1 & -1 & 2 & 1 \\ 2 & 1 & -1 & -6 \\ 2 & 3 & -3 & -1 \end{pmatrix}.$$

Then M is N-matrix of the first category and the Schur complement (of A in M) $M/A = \begin{pmatrix} 4 & -5 \\ 0 & 2 \end{pmatrix}$ is minimally semipositive. But

$$M^{-1} = \begin{pmatrix} 2.125 & 2.125 & 0.75 & 1.875 \\ -0.875 & 0.125 & -0.25 & -0.125 \\ 0.375 & 1.375 & 0.25 & 0.625 \\ 0.5 & 0.5 & 0 & 0.5 \end{pmatrix} \not\geq 0$$

and so M is not minimally semipositive.

In the next result, we show that almost N-matrices are not minimally semipositive. Again, recall that almost N-matrices of the first category are semipositive for $n \geq 4$ (item (c) of Theorem 7.6).

Theorem 7.19. *Let $A \in \mathbb{R}^{n\times n}$ be an almost N-matrix. Then A is not minimally semipositive.*

Proof. Let A be an almost N-matrix. Then $det(A) > 0$ and all its proper principal minors are negative. So are all the diagonal entries of the matrix $adj(A)$. Thus,

$$A^{-1} = \frac{adj(A)}{det(A)} \not\geq 0,$$

and so A is not minimally semipositive. ∎

We close this section with a result for almost P-matrices. As was mentioned earlier, P-matrices are semipositive (item (b), Theorem 7.6).

Theorem 7.20. *Let $A \in \mathbb{R}^{n\times n}$ be an almost P-matrix which is also semipositive. Then A^{-1} is N-matrix of the first category. Conversely, if A^{-1} is an N-matrix of the first category, then A is semipositive and an almost P-matrix.*

Proof. Suppose that A is an almost P-matrix which is also semipositive. Since A is an almost P-matrix,

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as mentioned earlier, A^{-1} is N-matrix. Now A^{-1} is also semipositive, since A semipositive. Thus, each row of A^{-1} contains a positive entry. Hence, A^{-1} is an N-matrix of the first category.
Conversely, suppose that A^{-1} is an N-matrix of the first category. Then A is an almost P-matrix. By Theorem 7.6, A^{-1} is semipositive. Thus, A is semipositive. ∎

7.1. A generalization of semipositivity and a game theory perspective. A *two-person zero-sum matrix game* consisting of two players, may be described as follows: Player I chooses an integer i, $1 \leq i \leq m$ and player II chooses an integer j, $1 \leq j \leq n$, simultaneously, after which player I pays an amount a_{ij} to player II. Here a_{ij} may be positive, negative or zero. A *strategy* for player I is a probability vector $(p_1, p_2, \ldots, p_m)^T$ (meaning that $p_i \geq 0$ for all i and $\sum\limits_{i=1}^{m} p_i = 1$). So, a strategy for player I is that he will choose integer i with probability p_i. A celebrated result of John von Neumann is the statement that there exist strategies $p = (p_1, p_2, \ldots, p_m)^T$ for player I and $q = (q_1, q_2, \ldots, q_n)^T$ for player II and a unique real number v (called the *value* of the game) such that the following inequalities hold:

$$\sum_{i=1}^{m} p_i a_{ij} \leq v \text{ for each } j = 1, 2, \ldots, n$$

and

$$\sum_{j=1}^{n} q_j a_{ij} \geq v \text{ for each } i = 1, 2, \ldots, m.$$

In such a case, we refer to the matrix $A = (a_{ij})$ as a *game* and the value of the game is denoted by $v(A)$. The corresponding strategies p and q are called *optimal strategies* for the two players. In the game described above, we may sometimes say that player I is the *minimizer* (i.e., he wants to give player II as least as possible) and player II is the *maximizer*. It is well known that if there exists $0 \neq x \geq 0$ such that $Ax \geq 0$, then $v(A) \geq 0$; if there exists $0 \neq y \geq 0$ with $y^T A \leq 0$, then $v(A) \leq 0$.

A strategy p of a player is referred to as a *mixed strategy* if $p > 0$. A game is called *completely mixed* if all the optimal strategies of both its players are mixed. This leads us to the relationship between the notion of semipositivity and game theory. A matrix A is semipositive if and only if its value is positive. It also follows that a semipositive matrix A is minimally semipositive, then (its value is positive and) the game A is completely mixed. Let us note that the converse is not true as illustrated by the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Here A is semipositive (so that its value is positive) but not minimally semipositive. However, the game A is completely mixed.

With this point of view, now we may reformulate some of the results of this section. (a) of Theorem 7.7 states that if A strictly copositive, then $v(A) > 0$, while (b) is the same as saying that for a symmetric copositive plus matrix A which is not strictly copositive, one has $v(A) \leq 0$. Theorem 7.14 could be paraphrased as: Let $b, c \in \mathbb{R}^{n-1}$ be positive vectors and $A = \begin{pmatrix} \alpha & b^T \\ c & D \end{pmatrix} \in \mathbb{R}^{n \times n}, n \geq 3$ be an N-matrix of the first category. If A/α is minimally semipositive then A is completely mixed and conversely, if A is minimally semipositive, then A/α is completely mixed. Further, we may give yet another proof of the first part of Theorem 7.4. Let B be skew-symmetric. Then it follows that $I + B$ is a positive definite matrix. If there exists $y \geq 0$ such that $y^T(I + B) \leq 0$, upon post-multiplication by y, a contradiction ensues. Thus, $v(I + B) > 0$, showing that the matrix $I + B$ is semipositive. Now, if A is skew-symmetric, then the matrix

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Volume 34, pp. 35-53, February 2018.

$M = \begin{pmatrix} I & A \\ A & I \end{pmatrix}$ could be written as $I + B$, where B is a skew-symmetric matrix. By what was proved above, it follows that M is a semipositive matrix.

In this regard, it is pertinent to point to the fact that a game theoretic proof was given [15, Theorem 3] to item (a) of Theorem 7.7 and more importantly, to the following result characterizing invertibility of M-matrices [15, Theorem 1]. This result is stated here to reinforce the fact that a rather distinguished class of semipositive matrices is given by the class of all invertible M-matrices.

THEOREM 7.21. *Let A be a matrix all of whose off-diagonal entries are nonpositive. Then the following statements are equivalent:*

(a) *There exists $x > 0$ such that $Ax > 0$.*

(b) *A is invertible and $A^{-1} \geq 0$.*

(c) *A is an invertible M-matrix.*

Next, let us recall a notion, more general than semipositivity [15, Definition 2]. Let S_0 be the set of real possibly rectangular matrices A for which one has $Ax \geq 0$, for some $0 \neq x \geq 0$. Let us simply refer to such a matrix as *generalized semipositive*. Matrix A in S_0 is referred to as *irreducible in* S_0, if no column deleted submatrix of A remains in S_0. It is shown in [15, Theorem 5] that a real nonsingular square matrix A is irreducible in S_0 if and only if $A^{-1} > 0$. Thus, it follows that a nonsingular A, irreducible in S_0, is minimally semipositive. That the converse is not true is shown by $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$. One may verify that A is minimally semipositive but A is not irreducible in S_0.

In this context, it must be mentioned that any singular irreducible M-matrix A is generalized semipositive [2, Chapter 6, Theorem 4.16]. Let us recall that a singular M-matrix A is of the form $A = \rho(B)I - B$, where $B \geq 0$. Irreducibility here is in the usual sense (and not as in the previous paragraph), that the underlying directed graph (associated with A) is strongly connected. A simple application of the Perron-Frobenius Theorem shows that such a matrix A satisfies the condition: there exists $x > 0$ such that $Ax = 0$. It also follows that one has the implication $Ax \geq 0 \Rightarrow Ax = 0$. We leave the details and point to [2], for the interested reader.

8. Concluding remarks. Let $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$. First, note that $\sigma(A) = \{-1\}$. By induction, it may be shown that $A^n = (-1)^n \begin{pmatrix} 1-2n & 2n \\ -2n & 1+2n \end{pmatrix}$. Now, it is clear that any positive integer power of the matrix A contains a positive column, and so all the matrices $A^k,\ k \geq 1$ are semipositive. In fact, it is easy to observe that all the odd powers of A have a common semipositivity vector $(1,0)^T$ and all the even powers of A have a common semipositivity vector $(0,1)^T$. This example serves two purposes. We detail this next.

It may be recalled that if all the positive integral powers of a matrix have a common semipositivity vector, then the matrix has a positive eigenvalue [19, Corollary 5.3]. The matrix A shows that this result does not hold when the assumption on the existence of a common semipositivity vector is relaxed. Now consider a square minimally semipositive matrix. By a standard argument involving the Perron-Frobenius theorem, it follows that such a matrix has a positive eigenvalue (and a nonnegative eigenvector, associated to it). The matrix A shows that a semipositive matrix need not have this property, even if all its positive

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integral powers are semipositive.

As was mentioned earlier, if a matrix is inverse positive, then it is semipositive; in fact, minimally semipositive. Let us note the following generalization of this result (whose proof is easy and hence skipped): Suppose that the Moore-Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is nonnegative. Let $A \in \mathbb{R}^{m\times n}$. If $A^{\dagger} \geq 0$ and $int(\mathbb{R}^m_+) \cap R(A) \neq \phi$, then A is semipositive.

Let us conclude with the following questions which have arisen from our study.

(1) Let A be an invertible M-matrix. Then A is minimally semipositive. Further, if A is irreducible, then A^{-1} is a positive matrix. Unlike invertible M-matrices, irreducible minimal semipositive matrices with positive diagonal entries need not have positive inverse. For example, let $A = \begin{pmatrix} 0.375 & 0.25 & -0.125 \\ -0.375 & 0.75 & 0.125 \\ 0.25 & -0.5 & 0.25 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$. Finding sufficient conditions under which the inverse of a minimal semipositive matrix is positive appears to be an interesting problem. This will also yield a sufficient condition under which a minimally semipositive matrix is irreducible in S_0.

(2) The next question concerns Remark 7.18. Let $M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be an N-matrix of the first category, where A_{12}, A_{21} are positive matrices and the Schur complement $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is minimally semipositive. When is M minimally semipositive?

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REFERENCES

[1] A. Ben-Israel and T.N.E. Greville. *Generalized inverses:Theory and applications.* Springer-Verlag, New York, 2003.
[2] A. Berman and R.J. Plemmons. *Nonnegative matrices in the mathematical sciences.* Society for Industrial and Applied Mathematics, Philadelphia, 1994.
[3] S.L. Campbell and C.D. Meyer Jr. *Generalized inverses of linear transformations.* Society for Industrial and Applied Mathematics, Philadelphia, 2009.
[4] Projesh Nath Choudhury and K.C. Sivakumar. Tucker's theorem for almost skew-symmetric matrices and a proof of Farkas' lemma. *Linear Algebra Appl.*, 482:55–69, 2015.
[5] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The linear complementarity problem.* Society for Industrial and Applied Mathematics, Philadelphia, 2009.
[6] J. Dorsey, T. Gannon, C.R. Johnson, and M. Turnansky. New results about semi-positive matrices. *Czechoslovak Math. J.*, 66:621–632, 2016.
[7] M. Fiedler and V. Pták. Some generalizations of positive definiteness and monotonicity. *Numer. Math.*, 9:163–172, 1966.
[8] M.S. Gowda. Pseudomonotone and copositive star matrices. *Linear Algebra Appl.*, 113:107–118, 1989.
[9] C.R. Johnson, M.K. Kerr, and D.P. Stanford. Semipositivity of matrices. *Linear Multilinear Algebra*, 37:265–271, 1994.
[10] M.A. Krasnosel'skij, J.A. Lifshits, and A.V. Sobolev. *Positive linear Systems: The Method of Positive Operators.* Fizmatgiz, Moscow, 1985 (in Russian). English translation by Jurgen Appell, Heldermann Verlag, Berlin, 1989.
[11] O.L. Mangasarian. Characterizations of real matrices of monotone kind. *SIAM Rev.*, 10:439–441, 1968.
[12] O.L. Mangasarian. *Nonlinear Programming.* McGraw-Hill Book Co., New York, 1969.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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[13] C. Olech, T. Parthasarathy, and G. Ravindran. Almost N-matrices and linear complementarity. *Linear Algebra Appl.*, 145:107–125, 1991.

[14] T. Parthasarathy. P-matrices and N-matrices. On Global Univalence Theorems, Lecture Notes in Mathematics, Springer, Berlin, 977:6–16, 1983.

[15] T.E.S. Raghavan. Completely mixed games and M-matrices. *Linear Algebra Appl.*, 21:35–45, 1978.

[16] K.C. Sivakumar and M.J. Tsatsomeros. Semipositive matrices and their semipositive cones. *Positivity*, 22(1):379-398, 2018.

[17] E. Stiemke. Über positive Lösungen homogener linearer Gleichungen. *Math. Ann.*, 76:340–342, 1915.

[18] M.J. Tsatsomeros. Principal pivot transforms: properties and applications. *Linear Algebra Appl.*, 307:151–165, 2000.

[19] M.J. Tsatsomeros. Geometric mapping properties of semipositive matrices. *Linear Algebra Appl.*, 498:349–359, 2016.

[20] A.W. Tucker. Dual systems of homogeneous linear relations. *Linear Inequalities and Related Systems*, Annals of Mathematics Studies, Vol. 38, Princeton University Press, Princeton, 3–18, 1956.

[21] R.S. Varga. *Matrix Iterative Analysis.* Springer-Verlag, Berlin, 2000.

[22] H.J. Werner. Characterizations of minimal semipositivity. *Linear Multilinear Algebra*, 37:273–278, 1994.