



ON THE NOTION OF SCALAR PRODUCT FOR FINITE-DIMENSIONAL DIFFEOLOGICAL VECTOR SPACES*

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Abstract. It is known that the only finite-dimensional diffeological vector space that admits a diffeologically smooth scalar product is the standard space of appropriate dimension. In this note, a way to dispense with this issue is considered, by introducing a notion of pseudo-metric, which, said informally, is the least-degenerate symmetric bilinear form on a given space. This notion is applied to make some observations on subspaces which split off as smooth direct summands (providing examples which illustrate that not all subspaces do), and then to show that the diffeological dual of a finite-dimensional diffeological vector space always has the standard diffeology and in particular, any pseudo-metric on the initial space induces, in the obvious way, a smooth scalar product on the dual.

Key words. Diffeological space, Diffeological vector space, Diffeologically smooth bilinear forms, Diffeological pseudo-metric.

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1. Introduction. One of the first surprising findings which one makes when encountering diffeology for the first time is the inclusion $L^\infty(V, W) < L(V, W)$, which says that the space of all (diffeologically) smooth linear maps between two diffeological vector spaces, V and W , can be strictly smaller than that of all maps that are simply linear¹. Stemming from that, an even more surprising situation presents itself: A (finite-dimensional) diffeological vector space that does not admit a smooth scalar product. This is a known fact (see [1, p. 74, Ex. 70]), and easily established at that, yet, it is still surprising, in and of itself, and also for how easily this can be illustrated, using the presence of just one non-differentiable (in the usual sense) plot.

Thus, as mentioned in the above-given reference, any finite-dimensional diffeological vector space that admits a smooth scalar product is necessarily the usual $\mathbb{R}^n$, with its usual smooth structure (the diffeology that consists of all usual smooth maps). The choices at this point are, to abandon the whole affair (meaning to concentrate on infinite-dimensional spaces, where the similar situation does not seem to occur), to consider a kind of pseudo-metrics (meaning the sort of least degenerate symmetric bilinear form that exists on a given space), or, finally, to re-define scalar product as ones taking values in $\mathbb{R}$ endowed, not with the standard diffeology, but with the piecewise-smooth diffeology. We concentrate on the second of the above approaches, and it does lead to a few interesting conclusions, mainly regarding the diffeological dual and the fact that a pseudo-metric induces an isomorphism of it with a specific subspace, which is a smooth summand. Furthermore, it gives rise to a smooth scalar product on the dual, reflecting as much as possible of the usual duality for the standard vector spaces.

Finally, a disclaimer: A lot of what is written here might be of a kind of implicit knowledge for people

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¹Say what? It happens in functional analysis already? Well, whatever. I prefer to keep it simple (jokes apart, in diffeology it happens in finite dimension already).

working in the area. Part of the aim of this paper is to collect these facts in one place, and to make explicit what is implicit elsewhere².

2. Diffeology and diffeological vector spaces. We briefly recall the main definitions regarding diffeological spaces and diffeological vector spaces.

Diffeological spaces and smooth maps. We first recall the notion of a diffeological space and that of a smooth map between such spaces.

DEFINITION 2.1. ([5]) A *diffeological space* is a pair $(X, \mathcal{D}_X)$ where X is a set and $\mathcal{D}_X$ is a specified collection of maps $U \rightarrow X$ (called *plots*) for each open set U in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ the following three conditions are satisfied:

1. (The covering condition) Every constant map $U \rightarrow X$ is a plot;
2. (The smooth compatibility condition) If $U \rightarrow X$ is a plot and $V \rightarrow U$ is a smooth map (in the usual sense), then the composition $V \rightarrow U \rightarrow X$ is also a plot;
3. (The sheaf condition) If $U = \cup_i U_i$ is an open cover and $U \rightarrow X$ is a set map such that each restriction $U_i \rightarrow X$ is a plot, then the entire map $U \rightarrow X$ is a plot as well.

Usually, instead of $(X, \mathcal{D}_X)$ one writes simply X to denote a diffeological space.

DEFINITION 2.2. ([5]) Let X and Y be two diffeological spaces, and let $f : X \rightarrow Y$ be any map. The map f is said to be *smooth* if for every plot $p : U \rightarrow X$ of X the composition $f \circ p$ is a plot of Y .

The standard examples of diffeological spaces are smooth manifolds, with diffeology consisting of all usual smooth maps $\mathbb{R}^k \supset U \rightarrow M$, for all $k \in \mathbb{N}$ and for all domains U in $\mathbb{R}^k$. A less standard example is, for instance, any Euclidean space $\mathbb{R}^n$ with the so-called *wire diffeology*, namely, the diffeology *generated* (see the next paragraph) by the set $C^\infty(\mathbb{R}, \mathbb{R}^n)$.

Generated diffeology. Let X be a set, and let $A = \{U_i \rightarrow X\}_{i \in I}$ be a set of maps with values in X . The *diffeology generated by A* is the smallest, with respect to inclusion, diffeology on X that contains A . It consists of all maps $f : V \rightarrow X$ such that there exists an open cover $\{V_j\}$ of V such that f restricted to each V_j either is constant or factors through some element $U_i \rightarrow X$ in A via a smooth map $V_j \rightarrow U_i$. Note that this construction illustrates the abundance of diffeologies on a given set: We can build one starting from *any* set of maps; in particular, given any map $p : U \rightarrow X$, there is (usually more than one) diffeology containing it.

Fine diffeology and coarse diffeology. Given a set X , the set of all possible diffeologies on X is partially ordered by inclusion, namely, a diffeology $\mathcal{D}$ on X is said to be *finer* than another diffeology $\mathcal{D}'$ if $\mathcal{D} \subset \mathcal{D}'$ (whereas $\mathcal{D}'$ is said to be *coarser* than $\mathcal{D}$). Among all diffeologies, there is the finest one, which turns out to be the natural *discrete diffeology* and which consists of all locally constant maps $U \rightarrow X$; and there is also the coarsest one, which consists of *all* possible maps $U \rightarrow X$, for all $U \subseteq \mathbb{R}^n$ and for all $n \in \mathbb{N}$. It is called *the coarse diffeology*.

²Putting an interesting statement as an exercise in a book is just an invitation for somebody to re-discover it in good faith. Just sayin'.

Subset diffeology. Let X be a diffeological space, and let $Y \subset X$ be a subset of it. The *subset diffeology* on Y is the *coarsest* diffeology on Y such that the inclusion map $Y \hookrightarrow X$ is smooth. What this means, specifically, is that $p : \mathbb{R}^k \supset U \rightarrow Y$ is a plot for the subset diffeology on Y if and only if its composition with the inclusion map is a plot of X .

Products and sums of diffeological spaces. Let $\{X_i\}_{i \in I}$ be a collection of diffeological spaces, where I is a set of indices. The *product diffeology* $\mathcal{D}$ on the product $\prod_{i \in I} X_i$ is the *coarsest* diffeology such that for each index $i \in I$ the natural projection $\pi_i : \prod_{i \in I} X_i \rightarrow X_i$ is smooth. Note, in particular, that for the product of two spaces $X_1 \times X_2$ every plot is locally of form (p_1, p_2) , where p_i is a plot of X_i for $i = 1, 2$.

If we now consider the disjoint union $\coprod_{i \in I} X_i$ of the spaces X_i , this can be endowed with the *sum diffeology*, which by definition is the *finest* diffeology such that all the natural injections $X_i \hookrightarrow \coprod_{i \in I} X_i$ are smooth. The plots for this diffeology are those maps that are plots of one of the components of the sum.

Diffeological vector space. Let V be a vector space over $\mathbb{R}$. The *vector space diffeology* on V is any diffeology of V such that the addition and the scalar multiplication are smooth, that is,

$$[(u, v) \mapsto u + v] \in C^\infty(V \times V, V) \quad \text{and} \quad [(\lambda, v) \mapsto \lambda v] \in C^\infty(\mathbb{R} \times V, V),$$

where $V \times V$ and $\mathbb{R} \times V$ are equipped with the product diffeology. A *diffeological vector space* over $\mathbb{R}$ is any vector space V over $\mathbb{R}$ equipped with a vector space diffeology.

Fine diffeology on vector spaces. The *fine diffeology* on a vector space $\mathbb{R}$ is the *finest* vector space diffeology on it; endowed with such, V is called a *fine vector space*. As an example, $\mathbb{R}^n$ with the standard diffeology is a fine vector space.

The fine diffeology admits a more or less explicit description of the following form: Its plots are maps $f : U \rightarrow V$ such that for all $x_0 \in U$ there exist an open neighbourhood U_0 of x_0 , a family of smooth maps $\lambda_\alpha : U_0 \rightarrow \mathbb{R}$, and a family of vectors $v_\alpha \in V$, both indexed by the same finite set of indices A , such that $f|_{U_0}$ sends each $x \in U_0$ into $\sum_{\alpha \in A} \lambda_\alpha(x) v_\alpha$:

$$f(x) = \sum_{\alpha \in A} \lambda_\alpha(x) v_\alpha \quad \text{for } x \in U_0.$$

A finite family $(\lambda_\alpha, v_\alpha)_{\alpha \in A}$, with $\lambda_\alpha \in C^\infty(U_0, \mathbb{R})$ and $v_\alpha \in V$, defined on some domain U_0 and satisfying the condition just stated, is called a *local family* for the plot f .

Fine vector spaces possess the following property [1, 3.9], which in general is not true: If V is a fine diffeological vector space, and W is any other diffeological vector space, then every linear map $V \rightarrow W$ is smooth, i.e., $L^\infty(V, W) = L(V, W)$.

Smooth scalar products. The existing notion of a *Euclidean diffeological vector space* does not differ much from the usual notion of the Euclidean vector space. A diffeological space V is Euclidean if it is endowed with a scalar product that is smooth with respect to the diffeology of V and the standard diffeology of $\mathbb{R}$; that is, if there is a fixed map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ that has the usual properties of bilinearity, symmetricity, and definite-positiveness and that is smooth with respect to the diffeological product structure on $V \times V$ and the standard diffeology on $\mathbb{R}$. As has already been mentioned in the introduction, for finite-dimensional

spaces this implies that the space in question is just the usual $\mathbb{R}^n$, for appropriate n (see [1]). For reasons of completeness we recall the precise statement and give a detailed proof of this fact³.

PROPOSITION 2.3. *Let V be $\mathbb{R}^n$ endowed with a vector space diffeology $\mathcal{D}$ such that there exists a smooth scalar product. Then every plot p of $\mathcal{D}$ is a smooth map in the usual sense.*

Proof. Let A be an $n \times n$ non-degenerate symmetric matrix such that the associated bilinear form on V is smooth with respect to the diffeology $\mathcal{D}$, and let $\{v_1, \dots, v_n\}$ be its eigenvector basis. Let λ_i be the eigenvalue relative to the eigenvector v_i .

Let $p : U \rightarrow V$ be a plot of $\mathcal{D}$; we wish to show that it is smooth as a map $U \rightarrow \mathbb{R}^n$. Recall that $\langle \cdot | \cdot \rangle_A$ being smooth implies that for any two plots $p_1, p_2 : U \rightarrow V$ the composition $\langle \cdot | \cdot \rangle_A \circ (p_1, p_2)$ is smooth as a map $U \rightarrow \mathbb{R}$. Let $c_i : U \rightarrow V$ be the constant map $c_i(x) = v_i$; this is of course a plot of $\mathcal{D}$. Set $p_1 = p$ and $p_2 = c_i$; then the above composition map writes as $\lambda_i \langle p(x) | v_i \rangle$, where $\langle \cdot | \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^n$.

Since A is non-degenerate, all λ_i are non-zero; this implies that each function $\langle p(x) | v_i \rangle$ is a smooth map. And since $v_1, \dots, v_n$ form a basis of V , this implies that for any $v \in V$ the function $\langle p(x) | v \rangle$ is a smooth one. In particular, this is true for any e_j in the canonical basis of $\mathbb{R}^n$; and in the case $v = e_j$ the scalar product $\langle p(x) | e_j \rangle$ is just the j -th component of $p(x)$. Thus, we obtain that all the components of p are smooth functions, therefore p is a smooth map. $\square$

3. The degeneracy of smooth forms on non-standard spaces. As mentioned in the introduction, no finite-dimensional diffeological vector space except for the standard one of a given dimension admits smooth non-degenerate bilinear forms; so there is only one space for each dimension that does. For exposition purposes, we start with a detailed illustration of how this happens, via an example (this part is however easily deduced from the proof of Proposition 2.3 above). We then consider the degree of degeneracy of smooth bilinear forms on a given vector space (what does it mean, for a given V , to be the least-degenerate bilinear form on it?), going on to the question of subspaces which do, or do not, split off as smooth direct summands, and finally considering the diffeological dual.

3.1. Pseudo-metrics. A *pseudo-metric* is, roughly speaking, the best possible substitute for the notion of a smooth scalar product in the case of a finite-dimensional vector space whose diffeology is not the standard one.

The first example. The following example, already considered in [2], is presented for illustrative purposes.

EXAMPLE 3.1. Let $V = \mathbb{R}^n$, and let $v_0 \in V$ be any non-zero vector. Let $p : \mathbb{R} \rightarrow V$ be defined as $p(x) = |x|v_0$; let $\mathcal{D}$ be any vector space diffeology on V that contains p as a plot⁴. Suppose that A is a symmetric $n \times n$ matrix, and assume that the bilinear form $\langle v | w \rangle_A = v^t A w$ associated to A is smooth with respect to $\mathcal{D}$ and the standard diffeology on $\mathbb{R}$. We claim that A is degenerate.

Indeed, $\langle v | w \rangle_A$ being smooth implies, in particular, that for any two plots $p_1, p_2 : \mathbb{R} \rightarrow V$ of V the composition map $\langle \cdot | \cdot \rangle_A \circ (p_1, p_2) : \mathbb{R} \rightarrow \mathbb{R}$ is smooth in the usual sense; this map acts as $\mathbb{R} \ni x \mapsto (p_1(x))^t A p_2(x)$. Let $w \in V$ be an arbitrary vector; denote by $c_w : \mathbb{R} \rightarrow V$ the constant map that sends everything to w ,

³Special thanks go to Patrick Iglesias-Zemmour and Yael Karshon who made me first think of it; I would have missed it otherwise.

⁴Such diffeology does certainly exist; for instance, the coarse diffeology would do.

$c_w(x) = w$ for all $x \in \mathbb{R}$. Such a map is a plot for any diffeology on V . But then $(\langle \cdot | \cdot \rangle_A \circ (p, c_w))(x) = |x|v_0^t A w$; the only way for this to be smooth is to have $v_0^t A w = 0$, and since there was no assumption on w , this implies that $\langle v_0 | \cdot \rangle_A$ is identically zero on the whole of V , i.e., that A is degenerate. In other words, V does not admit a smooth scalar product.

Note that in the above example we could have taken $p(x) = f(x)v_0$ with $f(x)$ any function $\mathbb{R} \rightarrow \mathbb{R}$ that is not differentiable in at least one point; this suggests that there are numerous diffeological vector spaces that do not admit diffeologically smooth scalar products. In fact, Proposition 2.3 shows that the phenomenon is much more general: In order to have a smooth scalar product, we must ensure that all plots are smooth maps.

The signature of a smooth bilinear form. Suppose that V is a finite-dimensional diffeological vector space; let A be a symmetric $n \times n$ matrix (with $n = \dim V$) such that the associated bilinear form $\langle \cdot | \cdot \rangle_A$ on V is smooth. Let $(\lambda_+, \lambda_-, \lambda_0)$ be the signature of this form; recall that V^* stands for the *diffeological dual* of V (see [6] and [7]), i.e., the set of all smooth linear functionals on V .

LEMMA 3.2. *Let V^* be the diffeological dual of V . Then*

$$\lambda_0 \geq n - \dim(V^*).$$

Proof. Choose a basis $\{v_1, \dots, v_n\}$ of eigenvectors of A such that the last λ_0 vectors are those relative to the eigenvalue 0. For $i = 1, \dots, n - \lambda_0$ let $v^i \in V^*$ be the dual function, that is, $v^i(w) = \langle v_i | w \rangle_A$ (it is obviously smooth since $\langle \cdot | \cdot \rangle_A$ is smooth, so is an element of V^*). It remains to notice that by standard reasoning the elements $v^1, \dots, v^{n-\lambda_0}$ are linearly independent (they belong to non-zero eigenvalues), so $n - \lambda_0 \leq \dim(V^*)$, hence the conclusion. $\square$

Notice that we can always find a smooth bilinear (symmetric) form for which we have $\lambda_0 = n - \dim(V^*)$. It suffices to take any basis $\{f_1, \dots, f_k\}$ of V^* (with $k = \dim(V^*)$) and consider $\sum_i f_i \otimes f_i$; this is obviously a symmetric bilinear form on V , and it being smooth follows from Theorem 2.3.5 of [6]. By an obvious argument, we can see that all the eigenvalues are non-negative; indeed, $(\sum_i f_i \otimes f_i)(v \otimes v) = \sum_i (f_i(v))^2 \geq 0$. Finally, the multiplicity of the eigenvalue 0 is precisely $n - k$, since $f_1, \dots, f_k$ form a basis of V^* .

Thus, the following definition makes sense.

DEFINITION 3.3. Let V be a diffeological vector space of finite dimension n . A *diffeological pseudo-metric* on V is a smooth bilinear symmetric form on V such that the eigenvalue 0 has multiplicity $n - \dim(V^*)$, and all the other eigenvalues are positive.

A pseudo-metric is the best notion of a smooth⁵ metric that can exist for an arbitrary finite-dimensional diffeological vector space; note that if a given space is standard, i.e., it is of the only type which admits a smooth metric, then obviously a pseudo-metric is a usual Euclidean metric. Of course, one can look for true metrics which are not smooth but are at least piecewise-differentiable, or simply continuous, something that we will do in the section that follows; meanwhile, we explore whatever applications pseudo-metrics might have.

REMARK 3.4. While it is more usual to use scalar products for various constructions of (multi)linear algebra, we do note that a number of them hold in part for symmetric bilinear forms. A specific example

⁵Which means a bilinear form that takes values in $\mathbb{R}$ with standard diffeology and is smooth with respect to the latter.

would be a Clifford algebra, which is defined for a vector space endowed with a symmetric bilinear form that *a priori* does not have to be a scalar product (an extension of this notion into the diffeological context was considered in [3]). Obviously, if it is not then one cannot speak of unitary action on the exterior algebra, for example; yet, various constructions hold.

3.2. Smooth direct sums and pseudo-metrics. One easily observed fact that distinguishes diffeological vector spaces from their usual linear counterpart is that, in the diffeological case, there are two types of direct sum: One that is smooth (in the sense described just below) and the other that is not.

Smooth splittings as direct sums. In general, if we have a diffeological vector space V (of finite dimension) that, as a usual vector space, decomposes into a direct sum $V = V_1 \oplus V_2$, then *a priori* the following two diffeologies are different:

- the given diffeology of V , and
- the vector space sum diffeology on $V_1 \oplus V_2$ obtained from the subset diffeologies on V_1 and V_2 ⁶.

EXAMPLE 3.5. Let us consider an example where the two might actually be different. Take $V = \mathbb{R}^3$ endowed with the vector space diffeology that is generated by the map $p : \mathbb{R} \rightarrow V$ acting by $p(x) = |x|(e_2 + e_3)$. Set $V_1 = \text{Span}(e_1, e_2)$ and $V_2 = \text{Span}(e_3)$. Observe that the subset diffeology on each of these subspaces is just the standard diffeology of $\mathbb{R}^2$ and $\mathbb{R}$ respectively⁷. This means that the vector space sum diffeology on their direct sum is again the standard one, which the diffeology on V is not; indeed, the projection on the second coordinate is smooth (in the usual sense) for the standard diffeology, which does not happen for the diffeology of V .

Standard subspaces that split off as direct summands. The example given in the previous paragraph easily leads to some more general observations, such as the following.

LEMMA 3.6. *Let V be a finite-dimensional diffeological vector space, let $V_1 \leq V$ be a subspace such that the subset diffeology is standard, and let $V = V_1 \oplus V_2$ be its decomposition into a direct sum for some subspaces $V_1, V_2 \leq V$. This decomposition is smooth if and only if there exists a basis $\{v_1, \dots, v_k\}$ of V_1 such that the projection⁸ on each vector v_i is a smooth linear functional on V .*

Proof. This is a consequence of Theorem 3.16 of [7]. Indeed, suppose that $V = V_1 \oplus V_2$ is a smooth decomposition. The projection on V_1 is then smooth by definition, and by assumption that V_1 is standard, so is the projection of V_1 onto any $v \in V_1$. The projection of V onto the vector v , being the composition of the two projections, is therefore smooth as well.

Suppose now that there exists a basis $\{v_1, \dots, v_k\}$ of V_1 as in the statement. Since V_1 is standard and all projections V on v_i are smooth, the projection of V onto V_1 is smooth as well, and it suffices to use the

⁶The vector space sum diffeology on a direct sum of diffeological vector spaces is the finest vector space diffeology that contains the product diffeology; it is formed of all maps that locally are (formal) sums of plots of the summands.

⁷Let us prove this claim. Let $q : U \rightarrow V_1$ be a plot for the subset diffeology on V_1 ; let $q_i : U \rightarrow \mathbb{R}$ be the projection on $\text{Span}(e_i) \leq V_1$ for $i = 1, 2$. It suffices to show that q_i must necessarily be a smooth map (in the usual sense). Note that by definition of the subset diffeology $Q(u) = (q_1(u), q_2(u), 0)$ must be a plot for the diffeology of V . By definition of the latter diffeology in the neighbourhood of zero it writes as $Q(u) = \sum_{i=1}^3 f_i(u)e_i + \sum_{j=1}^k h_j(u) \cdot (p \circ F_j)(u)$, where $f_1, f_2, f_3, h_1, \dots, h_k$, and $F_1, \dots, F_k$ are usual smooth maps $U \rightarrow \mathbb{R}$; therefore up to smooth summands $Q(u)$ writes as $(0, \sum_j h_j(u)|F_j(u)|, \sum_j h_j(u)|F_j(u)|)$, from which we conclude that $\sum_j h_j(u)|F_j(u)| \equiv 0$ in the neighbourhood of $u = 0$. This implies that q_1, q_2 are usual smooth maps, and the claim is proved for V_1 ; the reasoning is analogous in the case of V_2 .

⁸By this we mean the map $V \rightarrow \mathbb{R}$ acting by $\pi_i(\sum_i \alpha_i v_i + w) = \alpha_i$, where $w \in V_2$.

equivalence of conditions (1) and (3) of Theorem 3.16 of [7] to obtain the desired claim. $\square$

The maximal standard subspace relative to a pseudo-metric. As follows from the above discussion, a pseudo-metric exists on any diffeological vector space (and we cannot do any better). On the other hand, given a pseudo-metric $\langle \cdot | \cdot \rangle_A$ on a fixed diffeological vector space $(V, \mathcal{D})$ of finite dimension, the subspace V_0 of V spanned by all the eigenvectors belonging to the positive eigenvalues of $\langle \cdot | \cdot \rangle_A$ and considered with the subset diffeology $\mathcal{D}_0$ induced by $\mathcal{D}$ is endowed, via the restriction of $\langle \cdot | \cdot \rangle_A$, with a true metric; it follows then from [1, p. 74, Ex. 70] that $(V_0, \mathcal{D}_0)$ is the standard Euclidean space of dimension $n - \lambda_0 = n - \dim(V^*)$.

LEMMA 3.7. *The subspace V_0 splits off as a smooth direct summand.*

Proof. Denote by V_1 the subspace generated by all the eigenvectors relative to the eigenvalue 0. Obviously, V splits as the direct sum $V = V_0 \oplus V_1$ in the usual sense; we need to check that this decomposition is smooth, that is, that the diffeology on V coincides with the direct sum diffeology on $V_0 \oplus V_1$. Now, by Lemma 3.6, it suffices to check that the projection on each eigenvector in some basis of V_0 is a smooth linear functional on the whole of V .

The matrix A being symmetric, there exists an orthonormal (with respect to the usual scalar product on $\mathbb{R}^n$ underlying V) basis $\{v_1, \dots, v_n\}$ of V composed of eigenvectors of A , where we can obviously assume that the eigenvectors relative to all the positive eigenvalues (and so forming a basis of V_0) are the first k vectors, $v_1, \dots, v_k$. Let v_i be one of these, relative to the eigenvalue λ_i , let $\pi : V \rightarrow \mathbb{R}$ be the corresponding projection, and let $p : U \rightarrow V$ be any plot of V ; we need to show that $\pi \circ p : U \rightarrow \mathbb{R}$ is smooth in the usual sense. Write $p(x) = \sum_{j=1}^n p_j(x)v_j$; recall that $p_i(x) = \pi \circ p$.

Since A defines a smooth bilinear form on V , and all the constant maps are plots, the assignment $x \mapsto \langle v_i | p(x) \rangle_A$ defines a map $U \rightarrow \mathbb{R}$ which is smooth in the usual sense. However, we have $\langle v | p(x) \rangle_A = \sum_{j=1}^n \lambda_j p_j(x) \langle v_i | v_j \rangle = \lambda_i \|v_i\|^2 (\pi \circ p)(x)$, where $\|v_i\|$ is the usual Euclidean norm of v_i . Since neither it nor λ_i are zero, we conclude that $\pi \circ p$ is smooth, as wanted. $\square$

Furthermore, the following is true.

PROPOSITION 3.8. *The subspace V_0 is a maximal subspace of V with the subset diffeology that is standard and which splits off as a smooth direct summand.*

Proof. Suppose that $V_1 \supset V_0$ is a bigger subspace of V such that its subset diffeology is standard and V_1 also splits off as a smooth direct summand; let $V = V_1 \oplus V'_1$ be the corresponding smooth direct sum decomposition. Define a new bilinear form φ on V by setting it to be the zero form on V'_1 , a scalar product on V_1 , and V_1 and V'_1 to be orthogonal. Since by assumption the decomposition is smooth and V_1 has the standard diffeology, φ is smooth, and by construction it is symmetric and positive semi-definite. Furthermore, it has rank $\dim(V_1) > \dim(V_0)$, which contradicts Lemma 3.2, and we obtain the claim. $\square$

In other words, a pseudo-metric on V allows to extract, in a sense, from V its biggest standard part, on which the diffeology includes essentially the usual smooth maps. Let us be more precise.

COROLLARY 3.9. *Let $V = W_0 \oplus W_1$ be a smooth decomposition such that $\dim(W_0) \geq \dim(V^*)$, and the subset diffeology on W_0 is standard. Then there exists a pseudo-metric $\langle \cdot | \cdot \rangle_A$ on V such that W_0 is the space generated by the eigenvectors of A relative to the positive eigenvalues, and W_1 is the space generated by the eigenvectors relative to the eigenvalue 0.*

In particular, $\dim(W_0) = \dim(V^*)$. To summarize, all subspaces of V with standard sub-diffeology that

in addition split off as smooth direct summands are associated to some pseudo-metric, provided they have a sufficiently large dimension. We can actually show more: There is only one of them.

PROPOSITION 3.10. *The subspace V_0 is an invariant of the finite-dimensional diffeological space V , i.e., it does not depend on the choice of a pseudo-metric $\langle \cdot | \cdot \rangle_A$.*

Proof. Let $W_0 \neq V_0$ be another subspace of V of dimension $k = \dim(V^*)$ such that the subset diffeology on W_0 is standard and W_0 splits off as a smooth summand, $V = W_0 \oplus W_1$. Let $w_0 \in W_0 \setminus V_0$; it writes uniquely as $w_0 = v_0 + v_1$. Note that by Lemma 3.6 the projections on both w_0 and v_0 are smooth linear functionals; therefore so is the projection on v_1 . That would imply that $\text{Span}(V_0, v_1)$ has standard diffeology and splits off as a smooth summand; since its dimension is strictly greater than $\dim(V_0) = \dim(V^*)$, this is a contradiction. $\square$

3.3. The metric on V^* induced by a pseudo-metric on V . Here we consider a finite-dimensional diffeological vector space V (we assume that its underlying vector space is identified with $\mathbb{R}^n$), endowed with a pseudo-metric $\langle \cdot | \cdot \rangle_A$ given by a symmetric matrix A . As for usual vector spaces, the pseudo-metric induces a symmetric bilinear form on the diffeological dual V^* of V ; in this section we deal with this induced form.

The functional diffeology on V^ .* Recall [7] that in general, the diffeological dual V^* of V is *not* even isomorphic to V , so not diffeomorphic to it; this occurs already in the finite-dimensional case (in addition to [7], see Example 3.1 in [2]). Thus, it might well occur that while the diffeology on V is not standard, the corresponding functional diffeology on its dual is. The following statement makes it precise.

THEOREM 3.11. *Let V be a finite-dimensional diffeological vector space, and let V^* be its diffeological dual. Then the functional diffeology on V^* is standard (i.e., V^* is diffeomorphic to some $\mathbb{R}^k$ with standard diffeology, for the appropriate k).*

Proof. This is an easy consequence of Remark 3.6 and Property 8 (Section 5) of [7]. Indeed, let V_0 be the maximal standard subspace of V that splits off smoothly, and let $V = V_0 \oplus V_1$ be the corresponding smooth decomposition. It follows from Remark 3.6 of [7] that

$$V^* = (V_0 \oplus V_1)^* \cong V_0^* \times V_1^* = V_0^*.$$

By Property (8) in [7], since V_0 is fine and finite-dimensional, so is V_0^* ; in particular, it is a standard space. $\square$

The subspace V_0 and the diffeological dual. The proofs of Lemma 3.7 and of Theorem 3.11 suggest that each pseudo-metric on V gives a natural identification between the corresponding V_0 and V^* (that the two spaces are *a priori* diffeomorphic is now obvious, since they are both standard spaces of the same dimension). More precisely, the following is true.

THEOREM 3.12. *Let V be a finite-dimensional diffeological vector space, and let $\langle \cdot | \cdot \rangle_A$ be a pseudo-metric on V . The restriction on the subspace V_0 of the induced map $\Psi : V \rightarrow V^*$ (i.e., the map given by $v \mapsto [w \mapsto \langle w | v \rangle_A]$) is a diffeomorphism.*

Proof. As has been already observed in the proof of Theorem 3.11, the subspace V_0 admits an orthonormal (with respect to the canonical scalar product on $\mathbb{R}^n$ underlying V) basis $\{v_1, \dots, v_k\}$ composed of eigenvectors of A ; furthermore, if $f_i = \Psi(v_i)$ for $i = 1, \dots, k$, then f_i form a basis of V^* . In particular, the restriction of Ψ to V_0 is a bijection with V^* , hence an isomorphism.

Now, applying the reasoning from the proof of Lemma 3.7 we find that each plot p of V_0 is of form $p(u) = p_1(u)v_1 + \dots + p_k(u)v_k$ for some smooth real-valued maps $p_1, \dots, p_k$. Hence, the composition $\Psi \circ p$

writes as $(\Psi \circ p)(u) = p_1(u)f_1 + \cdots + p_k(u)f_k$, which is obviously a plot of V^* (the maps $p_1, \dots, p_k$ being smooth, this is a plot for any vector space diffeology on V^*), therefore Ψ is smooth. *Vice versa*, arguing as in the proof of Theorem 3.11, every plot q of V^* writes as $q(u) = q_1(u)f_1 + \cdots + q_k(u)f_k$ for some smooth real-valued maps $q_1, \dots, q_k$. Thus, the composition $\Psi^{-1} \circ q$ is of form $(\Psi^{-1} \circ q)(u) = q_1(u)v_1 + \cdots + q_k(u)v_k$, which is a plot of V_0 . We conclude that Ψ is smooth with smooth inverse, hence the conclusion. $\square$

The induced metric on V^ .* It is now easy to see that any pseudo-metric on V induces a true metric on V^* , via the smooth surjection Ψ described in Theorem 3.12. Let us describe this precisely; notice first of all that if $f \in V^*$, then there is a single element $v_0 \in \Psi^{-1}(f)$ such that $v_0 \in V_0$. Furthermore, every other element in $\Psi^{-1}(f)$ is of form $v_0 + v_1$, where $v_1 \in V^\perp$, the orthogonal of the whole V with respect to the pseudo-metric $\langle \cdot | \cdot \rangle_A$ (equivalently, the subspace V_1 spanned by all the eigenvectors relative to the eigenvalue 0). Let now $g \in V^*$ be another element of the dual V^* , and let $w_0 + w_1 \in \Psi^{-1}(g)$. Then obviously, $\langle v_0 + v_1 | w_0 + w_1 \rangle_A = \langle v_0 | w_0 \rangle_A = \langle (\Psi|_{V_0})^{-1}(f) | (\Psi|_{V_0})^{-1}(g) \rangle_A$. The first equality implies that the pushforward of the pseudo-metric $\langle \cdot | \cdot \rangle_A$ to V^* is well-defined (it could have been replaced by a reference to [7], Proposition 3.12 (1), of which it is a partial instance); the second equality means that this pushforward is a true metric. We summarize this discussion in the following statement.

COROLLARY 3.13. *Any pseudo-metric $\langle \cdot | \cdot \rangle_A$ on V induces a true metric on the diffeological dual V^* of V , via the natural pairing that assigns to each $v \in V$ the smooth linear functional $\langle \cdot | v \rangle_A$.*

Using the reasoning employed so far, we can also establish the inverse of this statement, namely, that every smooth metric (scalar product) on V^* is induced by a pseudo-metric on V . Indeed, let $\langle \cdot | \cdot \rangle_B$ be smooth metric on V^* , given by a $k \times k$ matrix B . Let $\{f_1, \dots, f_k\}$ be an orthonormal basis of V^* composed of eigenvectors of B^9 . As has already been noted, the form $\sum_{i=1}^k f_i \otimes f_i$ gives a certain pseudo-metric $\langle \cdot | \cdot \rangle_A$ on V ; it remains to check that this pseudo-metric does induce $\langle \cdot | \cdot \rangle_B$ on V^* , and this follows from the fact that $\{f_1, \dots, f_k\}$ is an orthonormal basis.

3.4. A simple example of a pseudo-metric. Let us return to an above example, considering $V = \mathbb{R}^3$ endowed with the vector space diffeology generated by the plot $p : \mathbb{R} \rightarrow V$ given by $p(x) = |x|(e_2 + e_3)$. Observe first that the diffeological dual of V is generated by maps e^1 and $e^2 - e^3$ (with $\{e^1, e^2, e^3\}$ being the canonical basis of the usual dual of $\mathbb{R}^3$). In particular, $\dim(V^*) = 2$.

It is also easy to see that any smooth symmetric bilinear form on V is given by a matrix of form $\begin{pmatrix} c & a & -a \\ a & b & -b \\ -a & -b & b \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$ (the vector $(0, |x|, |x|)$ must be orthogonal to any other vector of $\mathbb{R}^3$ with respect to the associated bilinear form). For this matrix to define a pseudo-metric, we must have $b, c > 0$ and $a^2 < bc$.

To give a specific example, we can take $a = 1$, $b = c = 2$, obtaining $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}$. The two positive eigenvalues of A are $\lambda_1 = 3 + \sqrt{3}$ and $\lambda_2 = 3 - \sqrt{3}$; the corresponding (non unitary) eigenvectors

⁹Note that the basis $\{f_1, \dots, f_k\}$ generates the standard diffeology on V^* , in the sense that each plot $q : U \rightarrow V^*$ is locally of form $q(u) = q_1(u)f_1 + \cdots + q_k(u)f_k$ for some smooth functions $q_1, \dots, q_k : U \rightarrow \mathbb{R}$. Indeed, each plot does write in this manner, simply by virtue of $\{f_1, \dots, f_k\}$ being a basis; furthermore, $\langle \cdot | \cdot \rangle_B$ being smooth, and each constant map being a plot, implies that $\langle q(u) | f_i \rangle_B = q_i(u)$ is smooth in the usual sense as a map $U \rightarrow \mathbb{R}$.

are $v'_1 = \begin{pmatrix} 3 + \sqrt{3} \\ 3 + 2\sqrt{3} \\ -3 - 2\sqrt{3} \end{pmatrix}$ and $v'_2 = \begin{pmatrix} 3 - \sqrt{3} \\ 3 - 2\sqrt{3} \\ 2\sqrt{3} - 3 \end{pmatrix}$. We can however easily find that the subspace V_0 can be (better) described as $V_0 = \text{Span}(e_1, e_2 - e_3)$; the restriction of $\langle \cdot | \cdot \rangle_A$ to V_0 , with respect to the basis $\{e_1, e_2 - e_3\}$ has the matrix $\begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix}$. We also note that the subspace V_1 (generated by eigenvectors relative to 0) is $\text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$.

Finally, let us calculate the matrix of the induced metric on V^* with respect to its basis $\{e^1, e^2 - e^3\}$. For this, we need to find their pre-images with respect to $\Psi|_{V_0}$ (we just write Ψ for brevity). By an easy calculation $\Psi^{-1}(e^1) = \frac{2}{3}e_1 - \frac{1}{6}(e_2 - e_3)$ and $\Psi^{-1}(e^2 - e^3) = -\frac{1}{3}e_1 + \frac{1}{3}(e_2 - e_3)$. Calculating their pairwise products with respect to the pseudo-metric $\langle \cdot | \cdot \rangle_A$, we find the matrix $\frac{1}{9} \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$.

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