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ALGEBRAIC METHODS FOR THE CONSTRUCTION OF ALGEBRAIC-DIFFERENCE EQUATIONS WITH DESIRED BEHAVIOR*

LAZAROS MOYSIS† AND NICHOLAS P. KARAMPETAKIS†

Abstract. For a given system of algebraic and difference equations, written as an Auto-Regressive (AR) representation $A(\sigma)\beta(k) = 0$, where σ denotes the shift forward operator and $A(\sigma)$ a regular polynomial matrix, the forward-backward behavior of this system can be constructed by using the finite and infinite elementary divisor structure of $A(\sigma)$. This work studies the inverse problem: Given a specific forward-backward behavior, find a family of regular or non-regular polynomial matrices $A(\sigma)$, such that the constructed system $A(\sigma)\beta(k) = 0$ has exactly the prescribed behavior. It is proved that this problem can be reduced either to a linear system of equations problem or to an interpolation problem and an algorithm is proposed for constructing a system satisfying a given forward and/or backward behavior.

Key words. Algebraic-difference equation, Behavior, Exact modeling, Auto-regressive representation, Discrete time system, Higher order system, descriptor system.

AMS subject classifications. 93A30, 93C55, 93C05, 93C35, 39A05, 15A29, 15A30.

1. Introduction. Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[\sigma]$ the ring of polynomials with coefficients from $\mathbb{R}$ and $\mathbb{R}(\sigma)$ the field of rational functions. By $\mathbb{R}[\sigma]^{p\times m}$, $\mathbb{R}(\sigma)^{p\times m}$, $\mathbb{R}_{pr}(\sigma)^{p\times m}$ we denote the sets of $p \times m$ polynomial, rational and proper rational matrices with real coefficients. We are going to study the behavior of systems of linear algebraic and difference equations that are described by the matrix equation

$$A_q\beta(k+q) + \cdots + A_1\beta(k+1) + A_0\beta(k) = 0,$$

or equivalently,

$$A(\sigma)\beta(k) = 0, \tag{1.1}$$

where $k = 0, 1, \ldots, N - q$, $\beta(k) \in \mathbb{R}^r$ is the state of the system, σ denotes the forward shift operator $\sigma^i\beta(k) = \beta(k+i)$, and

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \cdots + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{r\times r} \tag{1.2}$$

is a regular polynomial matrix with $\det A(\sigma) \neq 0$ and $A_q \neq 0$, $A_0 \neq 0$. Systems described by (1.1) are called *(Auto-Regressive) AR-representations*. The number q represents the maximum number of time shifts and is called the *lag* of the system [17].

Systems of the form (1.1) often appear in systems theory, since they accurately model many economic, biological and other discrete time phenomena. For example, the Leslie Population Growth Model in biology and the Leontief Model of a multisector economy in economics [5] are both examples of singular systems, which are easily seen to be a special case of AR systems (1.1).

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†School of Mathematical Sciences, Aristotle University of Thessaloniki, Thessaloniki, 54124, Greece (lazarosm@math.auth.gr, karampet@math.auth.gr).

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The solution space of system (1.1) consists of both forward and backward solutions and is denoted as

$$B := \{\beta(k) : [0, N] \to \mathbb{R}^r \,|\, (1.1) \text{ is satisfied } \forall k \in [0, N-q]\}\,.$$

The *forward solution space* is the vector space that consists of solutions $\beta(k)$ starting from given initial conditions $\beta(0), \ldots, \beta(q-1)$ and propagating forward in time, and is connected to the finite elementary divisor structure of $A(\sigma)$. The *backward solution space* is the vector space consisting of solutions $\beta(k)$ starting from given final conditions $\beta(N), \ldots, \beta(N-q+1)$ and propagating backward in time, and it is connected to the infinite elementary divisor structure of $A(\sigma)$, see also [2,15,16]. The algebraic structure of polynomial matrices has been studied in the early works of [6,9,10], and later in [1,4,7,8,11,12,22,25,30] and the references therein.

The construction of the solution space of such systems has been previously studied by various authors, initially in [10] and later in [2,15] whereas an extension of the method in [10] to non regular systems is given in [14]. In this paper, we study the inverse problem, that is: Given a certain forward/backward solution space, find a system of algebraic-difference equations with the prescribed solution space. A partial solution to this problem has been described in [10, Section 8.3], where only the smooth behavior for continuous time and the forward behavior for discrete time regular systems was studied. This method was later extended in [13] for continuous time systems, to include both the smooth and impulsive behavior and in [19] for discrete time systems to include both the forward and the backward behavior. Both these methods rely on the computation of the Jordan Pairs of $A(\sigma)$ and cannot be applied to non regular systems, that is, systems with $A(\sigma) \in \mathbb{R}[\sigma]^{r\times m}$ and $r \neq m$ or with $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ and $\det A(\sigma) = 0$. In addition, they are much less versatile in handling the free parameters of the matrices $A_0, \ldots, A_q$ and require a deep understanding of the structure of polynomial matrices. It should also be noted that the problem of constructing a system with prescribed solutions has been studied in the field of behavioral systems theory by [3,17,26–29,31–33], although the approach used was different.

In this paper, we shall further extend the results of [10,13,19] for the case where both a forward and backward behavior is under consideration, by using a novel methodology that can also be used to construct non regular systems, a case that was not addressed in [10,13]. The core of our proposed method lies in the fact that the vectors that formulate a solution of the system (forward or backward), actually satisfy a certain system of equations, which we are going to solve in terms of the unknown coefficients of $\mathrm{A}(\sigma)$, in order to obtain the original system. Thus, the problem of constructing a system of linear algebraic and difference equations is reduced to solving a linear system of equations.

The remaining of the paper is organized as follows. In Section 2, the necessary mathematical background from the theory of polynomial matrices is presented. In Section 3, the forward and backward solution of (1.1) is provided. In Section 4, an algorithm is proposed for the construction of a system with prescribed forward and backward behavior. In Section 5, examples are provided to illustrate the results. Section 6 concludes the paper.

2. Algebraic structure of polynomial matrices. In this section, some background on polynomial matrices is provided.

Definition 2.1. [23] A square polynomial matrix $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ is called *unimodular* if $\det A(\sigma) = c \in \mathbb{R}$, $c \neq 0$. A rational matrix $A(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$ is called *biproper* if $\lim_{\sigma\to\infty} A(\sigma) = E \in \mathbb{R}^{r\times r}$, with $rankE = r$.

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THEOREM 2.2. [9, 23] *Let $A(\sigma)$ be as in (1.2). There exist unimodular matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$, $U_R(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ such that*

$$S^{\mathbb{C}}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = diag\left(1,\dots,1, f_z(\sigma), f_{z+1}(\sigma),\dots,f_r(\sigma)\right), \tag{2.3}$$

with $1 \le z \le r$ and $f_j(\sigma)|f_{j+1}(\sigma)$, for $j = z, z+1,\dots,r$. $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ is called the Smith form *of $A(\sigma)$, where $f_j(\sigma) \in \mathbb{R}[\sigma]$ are the* invariant polynomials *of $A(\sigma)$. The zeros $\lambda_i \in \mathbb{C}$ of $f_j(\sigma)$, for $j = z, z+1,\dots,r$, are called* finite zeros *of $A(\sigma)$. Assume that $A(\sigma)$ has ℓ finite, distinct zeros. The* partial multiplicities *$n_{i,j}$ of each zero $\lambda_i \in \mathbb{C}$, for $i = 1,\dots,\ell$ satisfy*

$$0 \le n_{i,z} \le n_{i,z+1} \le \cdots \le n_{i,r},$$

with $f_j(\sigma) = (\sigma-\lambda_i)^{n_{i,j}}\hat{f}_j(\sigma)$, for $j = z,\dots,r$ and $\hat{f}_j(\lambda_i) \ne 0$. The terms $(\sigma-\lambda_i)^{n_{i,j}}$ are called finite elementary divisors *of $A(\sigma)$ at λ_i. The* multiplicity *of each zero is defined as $n_i = \sum_{j=z}^{r} n_{i,j}$. We denote by n the sum of the degrees of the finite elementary divisors of $A(\sigma)$,*

$$n := \deg\det A(\sigma) = \deg\left(\prod_{j=z}^{r} f_j(\sigma)\right) = \sum_{i=1}^{\ell}\sum_{j=z}^{r} n_{i,j}. \tag{2.4}$$

Similarly, we can find $U_L(\sigma), U_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ having no poles and zeros at $\sigma = \lambda_0$, such that

$$S^{\lambda_0}_{A(\sigma)}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = diag\left(1,\dots,1,(\sigma-\lambda_0)^{n_z},\dots,(\sigma-\lambda_0)^{n_r}\right).$$

$S^{\lambda_0}_{A(\sigma)}(\sigma)$ *is called the* local Smith form of $A(\sigma)$ at the point λ_0.

THEOREM 2.3. [24] *Let $A(\sigma)$ be as in (1.2). There exist biproper matrices $U_L(\sigma)$, $U_R(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$ such that*

$$U_L(\sigma)A(\sigma)U_R(\sigma) = S^{\infty}_{\mathrm{A}(\sigma)}(\sigma) = diag\left(\underbrace{\sigma^{q_1},\dots,\sigma^{q_k}}_{k}, \overbrace{\frac{1}{\sigma^{\hat{q}_{k+1}}}, \frac{1}{\sigma^{\hat{q}_{k+2}}},\dots,\frac{1}{\sigma^{\hat{q}_r}}}^{r-k}\right),$$

with

$$q_1 \ge \cdots \ge q_k \ge 0, \quad \hat{q}_r \ge \hat{q}_{r-1} \ge \cdots \ge \hat{q}_{k+1} > 0, \tag{2.5}$$

and $1 \le k \le r$. $S^{\infty}_{\mathrm{A}(\sigma)}(\sigma)$ is called the Smith form of $A(\sigma)$ at infinity*. If p_∞ is the number of q_i's in (2.5) with $q_i > 0$, then we say that $A(\sigma)$ has p_∞* poles at infinity*, each one of order $q_i > 0$. Also, if z_∞ is the number of $\hat{q}_i$'s in (2.5), then we say that $A(\sigma)$ has z_∞* zeros at infinity*, each one of order $\hat{q}_i > 0$. It is proved in [23] that $q_1 = q$.*

DEFINITION 2.4. [23] The *dual* polynomial matrix of $A(\sigma)$ is defined as

$$\tilde{A}(\sigma) := \sigma^q A\left(\frac{1}{\sigma}\right) = A_0\sigma^q + A_1\sigma^{q-1} + \cdots + A_q. \tag{2.6}$$

THEOREM 2.5. [23] *Let $\tilde{A}(\sigma)$ be as in (2.6). There exist matrices $\tilde{U}_L(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$, $\tilde{U}_R(\sigma) \in \mathbb{R}(\sigma)^{r\times r}$ having no poles or zeros at $\sigma = 0$, such that*

$$S^0_{\tilde{A}(\sigma)}(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = diag\left(\sigma^{\mu_1},\dots,\sigma^{\mu_r}\right).$$

$S^0_{\tilde{A}(\sigma)}(\sigma)$ is the local Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$. The terms σ^{μ_j} are the finite elementary divisors of $\tilde{A}(\sigma)$ at zero and are called the infinite elementary divisors (i.e.d.) *of $A(\sigma)$.*

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The connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix is given in [11,23]:

$$S^0_{\tilde{A}(\sigma)}(\sigma) = diag\left(1, \underbrace{\sigma^{q-q_2}, \ldots, \sigma^{q-q_k}}_{i.p.e.d.}, \underbrace{\sigma^{q+\hat{q}_{k+1}}, \ldots, \sigma^{q+\hat{q}_r}}_{i.z.e.d.}\right) \equiv diag\left(\sigma^{\mu_1}, \ldots, \sigma^{\mu_r}\right), \tag{2.7}$$

where by i.p.e.d. and i.z.e.d. we denote the infinite pole and infinite zero elementary divisors, respectively. From the above formula it is seen that the degrees of the infinite elementary divisors of $A(\sigma)$ are given by

$$\begin{aligned}
\mu_1 &= q - q_1 \overset{(q=q_1)}{=} 0, \\
\mu_j &= q - q_j, \qquad j = 2, 3, \ldots, k, \\
\mu_j &= q + \hat{q}_j, \qquad j = k+1, \ldots, r.
\end{aligned}$$

We denote by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$, that is

$$\mu := \sum_{j=1}^{r} \mu_j. \tag{2.8}$$

LEMMA 2.6. [2,10] *Let $A(\sigma)$ be as in (1.2). Let also n, μ be the sum of the degrees of the finite and infinite elementary divisors of $A(\sigma)$, as defined in (2.4) and (2.8). Then*

$$n + \mu = rq. \tag{2.9}$$

The above relation is of fundamental importance in the sequel, since it connects the dimension of the forward and backward behavior (n and μ respectively) of the AR-representation (1.1) with the lag (q) and the dimension (r) of $A(\sigma)$.

It should also be noted that in the case where the matrix $A(\sigma)$ is non-regular, the algebraic structure of $A(\sigma)$, and by extension the solution space of (1.1), is connected with additional invariants due to the left and right null space of $A(\sigma)$ (see [7,14,20]).

3. The behavior of a system described by an AR-Reresentation. In this section, we present the forward and backward behaviors of (1.1).

3.1. Finite elementary divisors and forward solution space. Let us assume that $A(\sigma)$ has ℓ finite, distinct zeros $\lambda_1, \ldots, \lambda_\ell$ where $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, \ell$, and let $S^{\mathbb{C}}_{A(\sigma)}(\sigma)$ be as in (2.3). Assume that the partial multiplicities of the zeros $\lambda_i \in \mathbb{C}$ are $0 \leq n_{i,z} \leq n_{i,z+1} \leq \cdots \leq n_{i,r}$. Let $u_j(\sigma) \in \mathbb{R}[\sigma]^r$ be the columns of $U_R(\sigma)$ and $u_j^{(\phi)}(\sigma) := (\partial^\phi / \partial\sigma^\phi) u_j(\sigma)$. Let also

$$\beta^i_{j,\phi} := \tfrac{1}{\phi!} u_j^{(\phi)}(\lambda_i), \quad i = 1, 2, \ldots, \ell, \quad j = z, z+1, \ldots, r, \quad \phi = 0, 1, \ldots, n_{i,j} - 1.$$

Define the vector valued functions

$$\beta^F_{i,j,\phi}(k) := \lambda_i^k \beta^i_{j,\phi} + k\lambda_i^{k-1}\beta^i_{j,\phi-1} + \cdots + \binom{k}{\phi}\lambda_i^{k-\phi}\beta^i_{j,0}, \qquad \text{for } \lambda_i \neq 0, \tag{3.10}$$

$$\beta^F_{i,j,\phi}(k) := \delta(k)\beta^i_{j,\phi} + \delta(k-1)\beta^i_{j,\phi-1} + \cdots + \delta(k-\phi)\beta^i_{j,0}, \qquad \text{for } \lambda_i = 0, \tag{3.11}$$

where $i = 1, 2, \ldots, \ell$, $j = z, z+1, \ldots, r$, $\phi = 0, 1, \ldots, n_{i,j} - 1$, $k \geq \phi$, and $\delta(k)$ denotes the Kronecker delta function.

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THEOREM 3.1. [14] *The vector valued functions $\beta^F_{i,j,\phi}(k)$, as defined in (3.10)–(3.11), are solutions of (1.1).*

THEOREM 3.2. *The vector valued functions $\beta^F_{i,j,\phi}(k)$ defined in (3.10)–(3.11) are solutions of (1.1) if and only if the vectors $\beta^i_{j,0},\dots,\beta^i_{j,n_{i,j}-1}$ satisfy the following system of equations:*

$$\begin{pmatrix} \frac{A^{(n_{i,j}-1)}(\lambda_i)}{(n_{i,j}-1)!} & \cdots & A(\lambda_i) \end{pmatrix} \underbrace{\begin{pmatrix} \beta^i_{j,0} & \cdots & 0_{r\times 1} \\ \vdots & \ddots & \vdots \\ \beta^i_{j,n_{i,j}-1} & \cdots & \beta^i_{j,0} \end{pmatrix}}_{W_{i,j}} = 0_{r\times n_{i,j}}, \tag{3.12}$$

with $W_{i,j}\in\mathbb{C}^{rn_{i,j}\times n_{i,j}}$.

Proof. By substituting $\beta^F_{i,j,\phi}(k)$ in (1.1) for $\phi=0,\dots,n_{i,j}-1$, the above system of equations is straightforwardly derived. ∎

3.2. Infinite elementary divisors and backward solution space. Let

$$S^0_{\tilde{A}(\sigma)}(\sigma)=\tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma)=diag\left(\sigma^{\mu_1},\dots,\sigma^{\mu_r}\right)$$

be the Smith form of $\tilde{A}(\sigma)$ at $\lambda=0$, with $\tilde{A}(\sigma)=A_0\sigma^q+A_1\sigma^{q-1}+\cdots+A_q$, as defined in (2.6). Let also $\tilde{U}_R(\sigma)=\begin{pmatrix}\tilde{u}_1(\sigma) & \cdots & \tilde{u}_r(\sigma)\end{pmatrix}$, where $\tilde{u}_j(\sigma)\in R(\sigma)^{r\times 1}$ and $\tilde{u}^{(i)}_j(\sigma)$, $\tilde{A}^{(i)}(\sigma)$ are the ith derivatives of $\tilde{u}_j(\sigma)$ and $\tilde{A}(\sigma)$ respectively, for $i=0,1,\dots,\mu_j-1$ and $j=2,\dots,r$ (since $\mu_1=0$). Let

$$x_{j,i}:=\tfrac{1}{i!}\tilde{u}^{(i)}_j(0),\quad i=0,1,\dots,\mu_j-1,\quad j=2,\dots,r, \tag{3.13}$$

and define the vector valued functions

$$\beta^B_{j,\phi}(k):=x_{j,\phi}\delta(N-k)+\cdots+x_{j,0}\delta(N-(k+\phi)), \tag{3.14}$$

where $j=2,\dots,r$, $\phi=0,\dots,\mu_j-1$.

THEOREM 3.3. [14] *The vector valued functions $\beta^B_{j,\phi}(k)$ defined in (3.14) are solutions of (1.1).*

THEOREM 3.4. *The vector valued functions $\beta^B_{j,\phi}(k)$ defined in (3.14), are solutions of (1.1) if and only if the vectors $x_{j,0},\dots,x_{j,\mu_j-1}$ in (3.13) satisfy the system of equations:*

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q} & \cdots & x_{j,q+\hat{q}_j-1} \\ 0 & x_{j,0} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} & \cdots & x_{j,\hat{q}_j-1} \end{pmatrix}}_{Q^{B_z}_j} = 0_{r\times(q+\hat{q}_j)}, \tag{3.15a}$$

with $Q^{B_z}_j\in\mathbb{R}^{r(q+1)\times(q+\hat{q}_j)}$, for the case of infinite zero elementary divisors (i.z.e.d.), μ_j, $j=k+1,\dots,r$, or

$$\begin{pmatrix} A_q & \cdots & A_{q_j+1} \end{pmatrix} \underbrace{\begin{pmatrix} x_{j,0} & x_{j,1} & \cdots & x_{j,q-q_j-1} \\ 0 & x_{j,0} & \cdots & \vdots \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & x_{j,0} \end{pmatrix}}_{Q^{B_p}_j} = 0_{r\times(q-q_j)}, \tag{3.15b}$$

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with $Q_j^{B_p} \in \mathbb{R}^{r(q+1)\times(q-q_j)}$, *for the case of infinite pole elementary divisors (i.p.e.d),* μ_j, $j=2,\dots,k$ *(see* (2.7)*).*

Proof. By substituting $\beta^B_{j,\phi}(k)$ in (1.1) for $\phi = 0,\dots,\mu_j-1$, the above system of equations is straightforwardly derived. ◻

4. Construction of a system with given forward and/or backward behavior. Theorem 3.2 states that in order for the vector valued function $\beta^F_{i,j,\phi}(k)$ in (3.10)–(3.11) to be a solution of $A(\sigma)\beta(k)=0$, the vectors $\beta^i_{j,0},\dots,\beta^i_{j,n_{i,j}-1}$ need to satisfy (3.12). Solving the above system of equations, we can obtain the matrices, $A^{(n_{i,j}-1)}(\lambda_i)\ \dots,\ A'(\lambda_i),\ A(\lambda_i)$, that represent the values of $A(\sigma)$ and its derivatives at λ_i. Thus, the evaluation of $A(\sigma)$ is reduced to a Hermite interpolation problem. Alternatively, using the relation

$$\frac{A^{(\varepsilon)}(\lambda_i)}{\varepsilon!} = \binom{q}{\varepsilon} A_q {\lambda_i}^{q-\varepsilon} + \dots + \binom{\varepsilon+1}{\varepsilon} A_{\varepsilon+1}\lambda_i + \binom{\varepsilon}{\varepsilon} A_\varepsilon$$

$$\Leftrightarrow \frac{A^{(\varepsilon)}(\lambda_i)}{\varepsilon!} = \begin{pmatrix} A_q & \cdots & A_\varepsilon \end{pmatrix} \begin{pmatrix} \binom{q}{\varepsilon}{\lambda_i}^{q-\varepsilon} I_r \\ \vdots \\ I_r \end{pmatrix},$$

for $\varepsilon = 0,\dots,n_{i,j}-1$, we rewrite (3.12) as follows:

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q_{i,j} W_{i,j} = 0_{r\times n_{i,j}}, \tag{4.16}$$

where

$$Q_{i,j} = \begin{pmatrix} \binom{q}{n_{i,j}-1}{\lambda_i}^{q-(n_{i,j}-1)} I_r & \cdots & q\lambda_i^{q-1} I_r & {\lambda_i}^q I_r \\ \vdots & \ddots & \vdots & \vdots \\ \binom{n_{i,j}}{n_{i,j}-1}\lambda_i I_r & \ddots & \vdots & \vdots \\ I_r & & 2\lambda_i I_r & \lambda_i^2 I_r \\ \vdots & \ddots & I_r & \lambda_i I_r \\ 0_r & \cdots & 0_r & I_r \end{pmatrix} \in \mathbb{C}^{r(q+1)\times rn_{i,j}}, \tag{4.17}$$

with $i = 1,2,\dots,\ell$, $j = z, z+1,\dots,r$.

In the case where $n_{i,j} > q$, the derivatives of $A(\sigma)$ of order higher than q in (3.12) will be equal to zero. In this case, the matrices $Q_{i,j}, W_{i,j}$ in (4.16) take the following simplified form

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} \begin{pmatrix} I_r & \cdots & \lambda_i^q I_r \\ \vdots & \ddots & \vdots \\ 0_r & \cdots & I_r \end{pmatrix} \begin{pmatrix} \beta^i_{j,n_{i,j}-q-1} & \cdots & \beta^i_{j,0} & \\ \vdots & & \ddots & \\ \beta^i_{j,n_{i,j}-1} & \cdots & \cdots & \beta^i_{j,0} \end{pmatrix} = 0_{r\times n_{i,j}}.$$

This system of equations can be used to solve the inverse problem. That is, given a time sequence in the form of $\beta^F_{i,j,\phi}(k)$, we can always solve the system of linear equations (4.16) in terms of the unknown matrices $A_0, A_1,\dots,A_q$ in order to construct $A(\sigma)$. Thus, the modeling problem has been reduced to solving a linear system of equations over $\mathbb{R}$.

Similarly to Theorem 3.2, Theorem 3.4 states that in order for a vector valued function $\beta^B_{j,\phi}(k)$ in (3.14) to be a solution of $A(\sigma)\beta(k)=0$, the vectors $x_{j,0},\dots,x_{j,\mu_j-1}$ need to satisfy (3.15). Solving this system of equations in terms of the unknown matrices $A_0, A_1,\dots,A_q$ we can construct the matrix $A(\sigma)$ and thus

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the corresponding AR-Representation (1.1) with the prescribed behavior. Note that the solution of (3.15) is actually based on the computation of the left kernel of a certain matrix or similarly from the right kernel of its transpose. Matlab is using two methods for the computation of the right kernel of the transpose of this matrix : a) finding an orthonormal basis of the null space of this matrix by using the singular value decomposition method, b) finding a "rational" basis for the null space by using the the reduced row echelon form of the matrix.

Combining Theorems 3.2 and 3.4, in order to construct a system of algebraic and difference equations that satisfies a desired forward and a backward behavior, we can solve both systems (4.16) and (3.15) and find a solution that satisfies both. As a result, the system produced will have a solution space spanned by the given vector valued functions. These results give rise to Algorithm 1.

Remark 4.1. In the case where, in Step 1, there exists no q such that $n + \mu = rq$, the resulting matrix $A(\sigma)$ will describe a system of algebraic-difference equations with $\beta^F_{i,j,\phi}(k)$ and $\beta^B_{\tilde{j},\tilde{\phi}}(k)$ as part of its solution space, which will include additional vector valued functions linearly independent from the ones that are given. In this case, the choice of the free parameters of the matrices A_i will determine the value of the additional zeros and thus, this choice will determine whether the constructed system will have certain properties, like stability.

Remark 4.2. In Step 5, the resulting matrices may have a large number of independent entries a_{ij}. If there are no other requirements on the system's structure, then the only constraint in choosing the free parameters is that the resulting matrix will have nonzero determinant. Nonetheless, several other structural properties may be required for the constructed system, like the polynomial matrix having symmetric ($A_i = A_i^T$), skew-symmetric ($A_i = -A_i^T$), or alternating ($A_i = (-1)^i A_i^T$ or $A_i = (-1)^{i+1} A_i^T$) coefficients. Systems with such structure often appear in continuous time, in the modelling of mechanical systems, see for example [18, 21].

Remark 4.3. Every matrix that is *left unimodularly equivalent* to the polynomial matrix $A(\sigma)$ constructed in Algorithm 1 gives rise to a model with exactly the same forward behavior with (1.1). That is, all matrices

$$A_1(\sigma) = U(\sigma)A(\sigma),$$

where $U(\sigma)$ is unimodular, satisfy $A_1(\sigma)\beta^F_{i,j,\phi}(k) = 0$. This is because multiplication by $U(\sigma)$ does not alter the finite zero structure of $A(\sigma)$ and thus, the forward behavior of the corresponding system remains the same (see [29]).

From the connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix $\tilde{A}(\sigma)$ in (2.7), it can be seen that the infinite elementary divisors of $A(\sigma)$, that generate the backward solutions of (1.1), are connected to the finite elementary divisors of the dual matrix at $\lambda = 0$, that in turn generate forward solutions for the dual system $\tilde{A}(\sigma)\beta(k) = 0$. More specifically, in [19] the connection between these two behaviors was explicitly given by the following theorem.

Theorem 4.4. [19] *The vector valued functions $\beta^B_{j,\phi}(k)$ defined in (3.14) are solutions of (1.1) if and only if the vector valued functions*

$$\tilde{\beta}_{j,\phi}(k) = x_{j,0}\delta(k - \phi) + \cdots + x_{j,\phi}\delta(k),$$

where $j = 2, \ldots, r$, $\phi = 0, \ldots, \mu_j - 1$, are solutions of the dual system $\tilde{A}(\sigma)\beta(k) = 0$.

Under the above consideration that the backward solutions of (1.1) give rise to forward solutions of its dual system, Remark 4.3 can also be applied here, as follows.

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Algorithm 1 Construction of a system with a prescribed forward and/or backward behavior.

Suppose that a finite number of functions of the form

$$\beta^F_{i,j,\phi}(k) := \lambda_i^k \beta^i_{j,\phi} + k\lambda_i^{k-1}\beta^i_{j,\phi-1} + \cdots + \binom{k}{\phi}\lambda_i^{k-\phi}\beta^i_{j,0}, \tag{4.18a}$$

$$\beta^F_{i,j,\phi}(k) := \delta(k)\beta^i_{j,\phi} + \delta(k-1)\beta^i_{j,\phi-1} + \cdots + \delta(k-\phi)\beta^i_{j,0}, \tag{4.18b}$$

$$\beta^B_{\tilde{j},\tilde{\phi}}(k) := x_{\tilde{j},\tilde{\phi}}\delta(N-k) + \cdots + x_{\tilde{j},0}\delta(N-(k+\tilde{\phi})) \tag{4.18c}$$

are given, with $i = 1, 2, \ldots, \ell$, $j = z, z+1, \ldots, r$, $\phi = 0, 1, \ldots, n_{i,j}-1$ and $k \geq \phi$, $\tilde{j} = 2, \ldots, r$, $\tilde{\phi} = 0, \ldots, \mu_{\tilde{j}}-1$.

Step 1 Define $n = \sum\limits_{i=1}^{\ell}\sum\limits_{j=z}^{r} n_{i,j}$ and $\mu := \sum\limits_{\tilde{j}=2}^{r}\mu_{\tilde{j}}$. If $r|(n+\mu)$, then

$$q = \frac{n+\mu}{r},$$

else

$$q = [\frac{n+\mu}{r}] + 1,$$

where $[\cdot]$ denotes the integer part of the given argument.
end if.

Step 2 Construct the matrices $Q_{i,j}, W_{i,j}$, defined in (3.12) and (4.17) and combine them as

$$Q_i = \begin{pmatrix} Q_{i,z} & \cdots & Q_{i,r} \end{pmatrix} \in \mathbb{C}^{r(q+1)\times rn_i},\ \ W_i = \begin{pmatrix} W_{i,z} & \cdots & W_{i,r} \end{pmatrix} \in \mathbb{C}^{rn_i\times n_i}, \qquad i = 1, \ldots, \ell,$$

where $n_i = \sum_{j=z}^{r} n_{i,j}$ and

$$Q = \begin{pmatrix} Q_1 & \cdots & Q_\ell \end{pmatrix} \in \mathbb{C}^{(q+1)r\times nr}, \quad W = \mathit{blockdiag}\,(W_1, \ldots, W_\ell) \in \mathbb{C}^{nr\times n}.$$

Step 3 Construct the matrices $Q^{B_z}_{\tilde{j}}$ and/or $Q^{B_p}_{\tilde{j}}$ defined in (3.15) and combine them as

$$Q^B = \begin{pmatrix} Q^B_2 & \cdots & Q^B_r \end{pmatrix} \in \mathbb{R}^{r(q+1)\times\mu},$$

that can be a combination of the matrices $Q^{B_z}_{\tilde{j}}$ and $Q^{B_p}_{\tilde{j}}$, depending on the form of $\beta^B_{\tilde{j},\tilde{\phi}}(k)$ that are given.

Step 4 Solve the system of equations

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} QW = 0_{r\times n},$$

and

$$\begin{pmatrix} A_q & \cdots & A_0 \end{pmatrix} Q^B = 0_{r\times\mu} \quad \text{or} \quad \begin{pmatrix} A_q & \cdots & A_{q_k+1} \end{pmatrix} Q^B = 0_{r\times\mu},$$

over $\mathbb{R}$ in terms of the unknown matrices A_i.

Step 5 Choose the free entries a_{ij} of each matrix A_i so that $\det A(\sigma) \neq 0$. The output matrix $A(\sigma)$ will correspond to a system of the form $A(\sigma)\beta(k) = 0$, with (4.18) as its solutions.

REMARK 4.5. Every polynomial matrix $A_1(\sigma)$ whose dual is left unimodularly equivalent to the dual of the polynomial matrix $A(\sigma)$ constructed in Algorithm 1 gives rise to a model with exactly the same backward

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behavior with (1.1). That is, all matrices $A_1(\sigma)$ such that:

$$\tilde{A}_1(\sigma) = U(\sigma)\tilde{A}(\sigma),$$

where $U(\sigma)$ is unimodular, satisfy $A_1(\sigma)\beta^B_{j,\phi}(k) = 0$.

Remark 4.6. In the case where backward propagating solutions due to the finite elementary divisors of $A(\sigma)$ are given, which are in the form

$$\beta^B_{i,j,\phi}(k) := \lambda_i^{N-k}\beta^i_{j,\phi} + k\lambda_i^{N-(k-1)}\beta^i_{j,\phi-1} + \cdots + \binom{k}{\phi}\lambda_i^{N-(k-\phi)}\beta^i_{j,0}, \quad \lambda_i \neq 0, \tag{4.19}$$

doing a simple reformulation, (4.19) can be rewritten as

$$\beta^B_{i,j,\phi}(k) := \left(\frac{1}{\lambda_i}\right)^k \lambda_i^N \beta^i_{j,\phi} + k\left(\frac{1}{\lambda_i}\right)^{k-1} \lambda_i^N \beta^i_{j,\phi-1} + \cdots + \binom{k}{\phi}\left(\frac{1}{\lambda_i}\right)^{k-\phi} \lambda_i^N \beta^i_{j,0},$$

which is their equivalent forward form.

Combining the results of Remarks 4.3 and 4.5, we conclude to the following.

Remark 4.7. Every polynomial matrix $A_1(\sigma)$ which is left unimodularly equivalent to the polynomial matrix $A(\sigma)$ constructed in Algorithm 1 and its dual matrix $\tilde{A}_1(\sigma)$ is left unimodularly equivalent to $\tilde{A}(\sigma)$, gives rise to a model with exactly the same forward and backward behavior.

Let $A_i(\sigma)\beta(k) = 0$, $i = 1, 2$ be two systems having the same forward and backward behavior. This means that the two matrices will have the same number of finite and infinite elementary divisors n and μ. Since these systems have also the same dimension r, from (2.9) it is derived that the two systems will also have the same lag q. In the following theorem, we argue that these systems are connected by a nonsingular transformation matrix $U(\sigma) = U \in \mathbb{R}^{r\times r}$.

Theorem 4.8. *Two systems of the form*

$$A_1(\sigma)\beta(k) = 0, \tag{4.20}$$
$$A_2(\sigma)\beta(k) = 0, \tag{4.21}$$

with the same lag q give rise to the same forward and backward behavior, if and only if their respective polynomial matrices $A_1(\sigma), A_2(\sigma)$ are connected by a left constant transformation matrix $U \in \mathbb{R}^{r\times r}$, with U invertible.

Proof. First, assume that the systems (4.20) and (4.21) give rise to the same forward and backward behavior. Then, according to Remark 4.7, it holds that

$$A_1(\sigma) = U(\sigma)A_2(\sigma), \tag{4.22}$$
$$\tilde{A}_1(\sigma) = V(\sigma)\tilde{A}_2(\sigma), \tag{4.23}$$

where $U(\sigma), V(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ are unimodular matrices. From (4.22), it holds that

$$A_1(\sigma) = U(\sigma)A_2(\sigma) \overset{\sigma\to\frac{1}{\sigma}}{\Rightarrow} A_1\left(\frac{1}{\sigma}\right) = U\left(\frac{1}{\sigma}\right)A_2\left(\frac{1}{\sigma}\right)$$
$$\overset{\times\sigma^q}{\Rightarrow} \sigma^q A_1\left(\frac{1}{\sigma}\right) = \sigma^q U\left(\frac{1}{\sigma}\right)A_2\left(\frac{1}{\sigma}\right)$$

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$$\Rightarrow \; \tilde{A}_1(\sigma) = U\left(\frac{1}{\sigma}\right)\tilde{A}_2(\sigma). \tag{4.24}$$

Combining (4.23) with (4.24), we have

$$\left(V(\sigma) - U\left(\frac{1}{\sigma}\right)\right)\tilde{A}_2(\sigma) = 0.$$

Thus, the matrix $\left(V(\sigma) - U\left(\frac{1}{\sigma}\right)\right)$ must belong to the left kernel of $\tilde{A}_2(\sigma)$, which since $\tilde{A}_2(\sigma)$ is nonsingular, is equal to the zero vector. So

$$V(\sigma) = U\left(\frac{1}{\sigma}\right),$$

and since $V(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ and $U\left(\frac{1}{\sigma}\right) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$, it holds that $V(\sigma) = U\left(\frac{1}{\sigma}\right) = U = V \in \mathbb{R}^{r\times r}$.

To prove the converse, assume that $A_1(\sigma), A_2(\sigma)$ are left unimodularly equivalent and $A_1(\sigma) = UA_2(\sigma)$. Let $\beta_1(k)$ be any solution (forward or backward) of $A_1(\sigma)\beta(k) = 0$. It holds that

$$A_1(\sigma)\beta_1(k) = 0 \;\Rightarrow\; UA_2(\sigma)\beta_1(k) = 0 \;\Rightarrow\; A_2(\sigma)\beta_1(k) = 0.$$

So every solution of (4.20) is a solution of (4.21). In the same fashion, let $\beta_2(k)$ be any solution (forward or backward) of $A_2(\sigma)\beta(k) = 0$. It holds that

$$A_2(\sigma)\beta_2(k) = 0 \;\Rightarrow\; U^{-1}A_1(\sigma)\beta_2(k) = 0 \;\Rightarrow\; A_1(\sigma)\beta_2(k) = 0.$$

So, every solution of (4.21) is a solution of (4.20) and thus, systems (4.20) and (4.21) have exactly the same solutions. □

5. Examples. In this section, several examples are presented to illustrate the use of the proposed algorithm.

Example 5.1. Let $\beta^F_{1,2,2}(k)$, $\beta^F_{2,2,0}(k)$ be the following vector valued functions

$$\beta^F_{1,2,2}(k) = \underbrace{\begin{pmatrix}2\\3\end{pmatrix}}_{\beta^1_{2,2}} 2^k + \underbrace{\begin{pmatrix}4\\1\end{pmatrix}}_{\beta^1_{2,1}} k2^{k-1} + \underbrace{\begin{pmatrix}2\\0\end{pmatrix}}_{\beta^1_{2,0}} \frac{k(k-1)}{2} 2^{k-2},\; \lambda_1 = 2, \quad \beta^F_{2,2,0}(k) = \underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\beta^2_{2,0}} 3^k,\; \lambda_2 = 3. \tag{5.25}$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ that has the prescribed functions in its solution space.

Step 1 These vector valued functions correspond to the zeros $\lambda_1 = 2$, $\lambda_2 = 3$, with multiplicities $n_1 = 3$, $n_2 = 1$. We have $n = n_1 + n_2 = 3 + 1 = 4$, $\mu = 0$ and $r = 2$. From (2.9), it holds that $n + 0 = 2q \Rightarrow q = 2$. So the matrix $A(\sigma)$ is $A(\sigma) = A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{2\times 2}$, with expected Smith form $S^{\mathbb{C}}_{A(\sigma)}(\sigma) = diag\left(1, (\sigma-3)(\sigma-2)^3\right)$.

Of course, since we desire the vector $\beta^F_{1,2,2}(k)$ to be a solution of (1.1), the vectors

$$\beta^F_{1,2,0}(k) = \underbrace{\begin{pmatrix}2\\0\end{pmatrix}}_{\beta^1_{2,0}} 2^k, \quad \beta^F_{1,2,1}(k) = \underbrace{\begin{pmatrix}2\\0\end{pmatrix}}_{\beta^1_{2,0}} k2^{k-1} + \underbrace{\begin{pmatrix}4\\1\end{pmatrix}}_{\beta^1_{2,1}} 2^k,$$

will also be solutions of the system. Yet, only the vector function (5.25) is required to construct the matrices Q_1, W_1.

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Steps 2 & 3 Construct the matrices

$$Q_1 = \begin{pmatrix} I_2 & 2\cdot 2I_2 & 2^2 I_2 \\ 0_2 & I_2 & 2I_2 \\ 0_2 & 0_2 & I_2 \end{pmatrix}, \; W_1 = \begin{pmatrix} \beta^1_{2,0} & 0_{2\times 1} & 0_{2\times 1} \\ \beta^1_{2,1} & \beta^1_{2,0} & 0_{2\times 1} \\ \beta^1_{2,2} & \beta^1_{2,1} & \beta^1_{2,0} \end{pmatrix}, \; Q_2 = \begin{pmatrix} 3^2 I_2 \\ 3I_2 \\ I_2 \end{pmatrix}, \; W_2 = \beta^2_{2,0},$$

and combine them as

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}, \quad W = \mathit{blockdiag}\,(W_1, W_2).$$

Step 4 Solve the system

$$\begin{pmatrix} A_2 & A_1 & A_0 \end{pmatrix} QW = 0_{2\times 4},$$

where

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{pmatrix}, \quad i = 0, 1, 2. \tag{5.26}$$

The resulting matrices are

$$A_0 = \begin{pmatrix} -\frac{5a_{12}}{4} - \frac{9a_{22}}{2} & -\frac{11a_{12}}{4} - \frac{15a_{22}}{2} \\ -\frac{5a_{14}}{4} - \frac{9a_{24}}{2} & -\frac{11a_{14}}{4} - \frac{15a_{24}}{2} \end{pmatrix}, \; A_1 = \begin{pmatrix} \frac{7a_{12}}{8} + \frac{11a_{22}}{4} & a_{12} \\ \frac{7a_{14}}{8} + \frac{11a_{24}}{4} & a_{14} \end{pmatrix}, \; A_2 = \begin{pmatrix} -\frac{a_{12}}{8} - \frac{a_{22}}{4} & a_{22} \\ -\frac{a_{14}}{8} - \frac{a_{24}}{4} & a_{24} \end{pmatrix}.$$

Step 5 Now we can choose values for a_{12}, a_{14}, a_{22}, a_{24}, such that $A_i \in \mathbb{R}^{r\times r}$ and $\det A(\sigma) \not\equiv 0$, since the wrong choice of the free variables may lead to linear dependence of the columns of $A(\sigma)$. The determinant of $A(\sigma)$ is given by

$$\det A(\sigma) = \frac{1}{8}(a_{14}a_{22} - a_{12}a_{24})(\sigma - 3)(\sigma - 2)^3,$$

so by Remark 4.3, different choices of the free parameters, such that $\det A(\sigma) \neq 0$, will lead to left unimodularly equivalent matrices that satisfy $A(\sigma)\beta(k) = 0$.
For example, by choosing $a_{12} = -\frac{14a_{22}}{3}$, $a_{14} = -\frac{38a_{22}}{3}$, $a_{24} = \frac{7a_{22}}{3}$, $a_{22} \neq 0$, the constructed polynomial matrix will have symmetric coefficients, $A_i = A_i^T$, for $i = 0, 1, 2$.

In the following example, the case where there is no lag q that satisfies (2.9) is studied.

Example 5.2. Let $\beta^F_{1,2,2}(k)$ be the following vector valued function

$$\beta^F_{1,2,2}(k) = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\beta^1_{2,2}} \left(\frac{1}{2}\right)^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta^1_{2,1}} k \left(\frac{1}{2}\right)^{k-1} + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta^1_{2,0}} \frac{k(k-1)}{2} \left(\frac{1}{2}\right)^{k-2}, \quad \lambda_1 = \frac{1}{2}$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ that has the prescribed function in its solution space.

Step 1 Since $n = n_1 = 3$, $\mu = 0$ and $r = 2$, from (2.9) we have $n + 0 = 2q \Rightarrow q = 3/2$. So set $q = [\frac{3}{2}] + 1 = 2$ and the matrix $A(\sigma)$ is $A(\sigma) = A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{2\times 2}$.

Steps 2 & 3 Construct the matrices Q_1 and W_1, which are given by

$$Q_1 = \begin{pmatrix} I_2 & 2\cdot\left(\frac{1}{2}\right) I_2 & \left(\frac{1}{2}\right)^2 I_2 \\ 0_2 & I_2 & \left(\frac{1}{2}\right) I_2 \\ 0_2 & 0_2 & I_2 \end{pmatrix}, \quad W_1 = \begin{pmatrix} \beta^1_{2,0} & 0_{2\times 1} & 0_{2\times 1} \\ \beta^1_{2,1} & \beta^1_{2,0} & 0_{2\times 1} \\ \beta^1_{2,2} & \beta^1_{2,1} & \beta^1_{2,0} \end{pmatrix}.$$

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Step 4 Solve the system

$$\begin{pmatrix} A_2 & A_1 & A_0 \end{pmatrix} Q_1 W_1 = 0_{2\times 3},$$

where A_0, A_1, A_2 as in (5.26).

Step 5 The resulting matrices are

$$A_0 = \begin{pmatrix} \frac{a_{02}}{8} - \frac{a_{12}}{16} - \frac{3a_{22}}{32} & a_{02} \\ \frac{a_{04}}{8} - \frac{a_{14}}{16} - \frac{3a_{24}}{32} & a_{04} \end{pmatrix},\ A_1 = \begin{pmatrix} \frac{a_{12}}{2} + \frac{a_{22}}{2} & a_{12} \\ \frac{a_{14}}{2} + \frac{a_{24}}{2} & a_{14} \end{pmatrix},\ A_2 = \begin{pmatrix} -\frac{a_{02}}{2} - \frac{3a_{12}}{4} - \frac{5a_{22}}{8} & a_{22} \\ -\frac{a_{04}}{2} - \frac{3a_{14}}{4} - \frac{5a_{24}}{8} & a_{24} \end{pmatrix},$$

and the matrix $A(\sigma)$ has determinant

$$\det A(\sigma) = \frac{1}{32}(-1+2\sigma)^3 g(\sigma),$$

where

$$g(\sigma) = 2a_{04}a_{12} - 2a_{02}a_{14} + 3a_{04}a_{22} - 3a_{02}a_{24} + (2a_{04}a_{22} + 3a_{14}a_{22} - 2a_{02}a_{24} - 3a_{12}a_{24})\sigma,$$

so the determinant is a polynomial of degree equal to 4. This means that the matrix has an extra zero and thus an extra solution, as it was expected. The value of this additional zero of $A(\sigma)$ will determine whether the constructed system will be stable or not, since λ_1 satisfies $|\lambda_1| < 1$. The zero of $g(\sigma)$ is given by

$$\lambda_2 = \frac{-2a_{04}a_{12} + 2a_{02}a_{14} - 3a_{04}a_{22} + 3a_{02}a_{24}}{2a_{04}a_{22} + 3a_{14}a_{22} - 2a_{02}a_{24} - 3a_{12}a_{24}},$$

so if the parameters of A_0, A_1, A_2 are chosen so that $|\lambda_2| < 1$, the constructed system will be stable.

On the other hand, one may assume that by choosing the appropriate values of the free parameters in order to eliminate the coefficient of σ in the extra polynomial $g(\sigma)$ of the determinant, while still keeping $\det A(\sigma) \neq 0$, will give a simple solution to the problem of undesired behavior. This is not the case, since this will lead to undesired backward behavior. For example, by choosing $a_{24} = a_{22} = a_{04} = a_{12} = 0$, $a_{02} = a_{14} = 1$, we obtain

$$A(\sigma) = \begin{pmatrix} \frac{1}{8} - \frac{\sigma^2}{2} & 1 \\ -\frac{1}{16} + \frac{\sigma}{2} - \frac{3\sigma^2}{4} & \sigma \end{pmatrix},$$

with $S^{\mathbb{C}}_{A(\sigma)}(\sigma) = diag\left(1, (2\sigma-1)^3\right)$ and $S^{0}_{\tilde{A}(\sigma)}(\sigma) = diag\left(1, \sigma\right)$.

The Smith form of the dual matrix at $\sigma = 0$ implies that in this case the matrix $A(\sigma)$ has an additional infinite elementary divisor. As shown in the previous section, the existence of an infinite elementary divisor implies the existence of additional backward behavior for the above system. Thus, we see that no matter what the values of the free variables a_{ij} will be, the system will exhibit additional behavior.

As an alternative though, one may proceed to construct a *non-regular* system that satisfies the prescribed behavior, that is, a system with $A(\sigma) \in \mathbb{R}[\sigma]^{r\times m}$ and $r \neq m$ or with $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ and $\det A(\sigma) = 0$.

Step 1 Under the assumption that the constructed system can be non-square, taking r=1 and n=3, we find $n = rq \Rightarrow q = 3$ (see also [20]). So $A(\sigma) = A_3\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{1\times 2}$.

Steps 2 & 3 For the above system, the matrix W_1 remains the same, while Q_1 is

$$Q_1 = \begin{pmatrix} 3\cdot\left(\frac{1}{2}\right) I_2 & 3\cdot\left(\frac{1}{2}\right)^2 I_2 & \left(\frac{1}{2}\right)^3 I_2 \\ I_2 & 2\cdot\left(\frac{1}{2}\right) I_2 & \left(\frac{1}{2}\right)^2 I_2 \\ 0_2 & I_2 & \left(\frac{1}{2}\right) I_2 \\ 0_2 & 0_2 & I_2 \end{pmatrix}.$$

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Step 4 Solve the system

$$\begin{pmatrix} A_3 & A_2 & A_1 & A_0 \end{pmatrix} Q_1 W_1 = 0_{1\times 3},$$

where $A_i = \begin{pmatrix} a_{i1} & a_{i2} \end{pmatrix}$.

Step 5 The resulting matrices are

$$A_3 = \begin{pmatrix} a_{31} & a_{32} \end{pmatrix},\ A_2 = \begin{pmatrix} -\frac{a_{02}}{2} - \frac{3a_{12}}{4} - \frac{5a_{22}}{8} - \frac{3a_{31}}{2} - \frac{7a_{32}}{16} & a_{22} \end{pmatrix},$$
$$A_1 = \begin{pmatrix} \frac{a_{12}}{2} + \frac{a_{22}}{2} + \frac{3a_{31}}{4} + \frac{3a_{32}}{8} & a_{12} \end{pmatrix},\ A_0 = \begin{pmatrix} \frac{a_{02}}{8} - \frac{a_{12}}{16} - \frac{3a_{22}}{32} - \frac{a_{31}}{8} - \frac{5a_{32}}{64} & a_{02} \end{pmatrix},$$

and the constructed matrix $A(\sigma)$ satisfies $A(\sigma)\beta(k) = 0$. What must be noted though is that since this procedure has led to the construction of a non-regular system, the solutions $\beta_i(k)$ of the system could be attributed to either its f.e.d. structure or its right null space. That is, as [14] demonstrates, non-regular systems exhibit an infinite number of forward and backward solutions due to the right null space of $A(\sigma)$. So in this case, the constructed system will include additional behavior that is undesired.

In the following example, we study the case where the matrix $A(\sigma)$ has complex zeros.

EXAMPLE 5.3. Let $\beta(k)$ be the following vector valued function

$$\beta(k) = \begin{pmatrix} \sqrt{2}\cos(\frac{2\pi}{3}k + \frac{\pi}{4}) \\ 2\cos(\frac{2\pi}{3}k + \frac{\pi}{6}) \end{pmatrix}.$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ that has the prescribed functions in its solution space. This vector valued function can be equivalently written as $\beta(k) = \beta^F_{1,2,0}(k) + \beta^F_{2,2,0}(k)$, where

$$\beta^F_{1,2,0}(k) = \underbrace{\begin{pmatrix} \frac{1}{2} + \frac{1}{2}i \\ \frac{\sqrt{3}}{2} + \frac{1}{2}i \end{pmatrix}}_{\beta^1_{2,0}} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^k, \quad \lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$\beta^F_{2,2,0}(k) = \underbrace{\begin{pmatrix} \frac{1}{2} - \frac{1}{2}i \\ \frac{\sqrt{3}}{2} - \frac{1}{2}i \end{pmatrix}}_{\beta^2_{2,0}} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^k, \quad \lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

So instead of $\beta(k)$, we should equivalently consider the following two complex vector valued functions $\beta^F_{1,2,0}(k)$ and $\beta^F_{2,2,0}(k)$.

Step 1 These vector valued functions correspond to the zeros $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, with multiplicities $n_1 = 1$, $n_2 = 1$. Since $n = n_1 + n_2 = 2$, $\mu = 0$ and $r = 2$, it holds that $n + 0 = 2q \Rightarrow q = 1$. So $A(\sigma) = A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{2\times 2}$, with expected Smith form $S^{\mathbb{C}}_{A(\sigma)}(\sigma) = diag\left(1, 1 + \sigma + \sigma^2\right)$.

Steps 2 & 3 Construct the matrices

$$Q_1 = \begin{pmatrix} \lambda_1 I_2 \\ I_2 \end{pmatrix},\ W_1 = \beta^1_{2,0},\ Q_2 = \begin{pmatrix} \lambda_2 I_2 \\ I_2 \end{pmatrix},\ W_2 = \beta^2_{2,0},$$

and combine them as

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}, \quad W = blockdiag(W_1, W_2). \tag{5.27}$$

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Step 4 Solve the system

$$\begin{pmatrix} A_1 & A_0 \end{pmatrix} QW = 0_{2\times 2}, \tag{5.28}$$

where A_0, A_1, A_2 as in (5.26). Since in (5.28), the matrices Q, W are complex, (5.27) is equivalent to solving the two following systems

$$\Re\left[\begin{pmatrix} A_1 & A_0 \end{pmatrix} QW\right] = 0_{2\times 2}, \quad \Im\left[\begin{pmatrix} A_1 & A_0 \end{pmatrix} QW\right] = 0_{2\times 2},$$

where $\Re, \Im$ denote the real and imaginary parts of the expression respectively.

Step 5 The resulting matrices are

$$A_0 = \begin{pmatrix} -(1+\sqrt{3})\left(a_{11}+\sqrt{3}a_{12}\right) & \frac{3+\sqrt{3}}{2}a_{11} + (2+\sqrt{3})a_{12} \\ -(1+\sqrt{3})\left(a_{13}+\sqrt{3}a_{14}\right) & \frac{3+\sqrt{3}}{2}a_{13} + (2+\sqrt{3})a_{14} \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix},$$

and the matrix $A(\sigma)$ has determinant $\det A(\sigma) = -(a_{12}a_{13} - a_{11}a_{14})(1+\sigma+\sigma^2)$.

We can easily verify that the given vector valued functions are solutions of the system. As an example, by choosing $\mathrm{a}_{11} = -\frac{2\left(3+2\sqrt{3}\right)a_{13}}{3+\sqrt{3}} - 2a_{14}$, $a_{12} = a_{13}$, with $\left(3+\sqrt{3}\right)a_{13}^2 + 2\left(3+2\sqrt{3}\right)a_{13}a_{14} + 2\left(3+\sqrt{3}\right)a_{14}^2 \neq 0$, the constructed system will have symmetric coefficients.

Example 5.4. Let $\beta^F_{1,2,1}(k)$, $\beta^F_{2,2,1}(k)$, $\beta^B_{2,3}(k)$ be the following vector valued functions

$$\beta^F_{1,2,1}(k) = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\beta^1_{2,1}} + \underbrace{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}_{\beta^1_{2,0}} k,\ \lambda_1 = 1,\ \ \beta^F_{2,2,1}(k) = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\beta^2_{2,1}} 2^k + \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{\beta^2_{2,0}} k2^{k-1},\ \lambda_2 = 2,$$

$$\beta^B_{2,3}(k) = \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{x_{2,3}} \delta(N-k) + \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{x_{2,2}} \delta(N-k-1) + \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_{2,1}} \delta(N-k-2) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{x_{2,0}} \delta(N-k-3).$$

We want to construct an AR-representation $A(\sigma)\beta(k) = 0$ that has the prescribed functions in its solution space.

Step 1 These vector valued functions correspond to the zeros $\lambda_1 = 1$, $\lambda_2 = 2$ with multiplicities $n_1 = 2$, $n_2 = 2$ and to an infinite elementary divisor of order $\mu_2 = 4$ (since $\mu_1 = 0$). Overall, $\mu = \mu_1 + \mu_2 = 4$, $n = n_1 + n_2 = 2 + 2 = 4$ and $r = 2$, so from (2.9) we have $n + \mu = 4 + 4 = 8 = rq \Rightarrow q = 4$. So $A(\sigma) = A_4\sigma^4 + A_3\sigma^3 + A_2\sigma^2 + A_1\sigma + A_0 \in \mathbb{R}[\sigma]^{2\times 2}$, with expected Smith forms $S^{\mathbb{C}}_{A(\sigma)}(\sigma) = diag\left(1, (\sigma-1)^2(\sigma-2)^2\right)$ and $S^0_{\tilde{A}(\sigma)}(\sigma) = diag\left(1, \sigma^4\right)$.

Steps 2 & 3 From the coefficients of $\beta_1(k)$ and $\beta_2(k)$, construct the matrices

$$Q_i = \begin{pmatrix} 4{\lambda_i}^3\ I_2 & {\lambda_i}^4\ I_2 \\ 3{\lambda_i}^2\ I_2 & {\lambda_i}^3\ I_2 \\ 2\lambda_i\ I_2 & {\lambda_i}^2\ I_2 \\ I_2 & \lambda_i\ I_2 \\ 0_2 & I_2 \end{pmatrix}, \quad W_i = \begin{pmatrix} \beta^i_{2,0} & 0_{2\times 1} \\ \beta^i_{2,1} & \beta^i_{2,0} \end{pmatrix}, \qquad i = 1, 2.$$

and combine them as

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix},\ W = blockdiag(W_1, W_2).$$

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From the coefficients of $\beta^B_{2,3}(k)$, since $q=4$ and $\mu_1=0$, we have that

$$\mu_2 = q - q_2 = 4 \Rightarrow 4 - q_2 = 4 \Rightarrow q_2 = 0,$$

so μ_2 corresponds to an infinite pole elementary divisor. Thus, we will use (3.15b),

$$Q^B = \begin{pmatrix} x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} \\ 0_{2\times 1} & x_{2,0} & x_{2,1} & x_{2,2} \\ 0_{2\times 1} & 0_{2\times 1} & x_{2,0} & x_{2,1} \\ 0_{2\times 1} & 0_{2\times 1} & 0_{2\times 1} & x_{2,0} \end{pmatrix}.$$

Steps 4 & 5 Solving the systems

$$\begin{pmatrix} A_4 & A_3 & A_2 & A_1 & A_0 \end{pmatrix} QW = 0_{2\times 4}, \quad \begin{pmatrix} A_4 & A_3 & A_2 & A_1 \end{pmatrix} Q^B = 0_{2\times 4},$$

the resulting matrices are

$$A_0 = \begin{pmatrix} -\frac{a_{02}}{5} + \frac{3a_{12}}{10} & a_{02} \\ -\frac{a_{04}}{5} + \frac{3a_{14}}{10} & a_{04} \end{pmatrix}, \; A_1 = \begin{pmatrix} -\frac{7a_{02}}{20} - \frac{11a_{12}}{10} & a_{12} \\ -\frac{7a_{04}}{20} - \frac{11a_{14}}{10} & a_{14} \end{pmatrix}, \; A_2 = \begin{pmatrix} \frac{3a_{02}}{4} + a_{12} & -\frac{17a_{02}}{20} - \frac{3a_{12}}{5} \\ \frac{3a_{04}}{4} + a_{14} & -\frac{17a_{04}}{20} - \frac{3a_{14}}{5} \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -\frac{3a_{02}}{10} - \frac{3a_{12}}{10} & -\frac{3a_{02}}{20} - \frac{2a_{12}}{5} \\ -\frac{3a_{04}}{10} - \frac{3a_{14}}{10} & -\frac{3a_{04}}{20} - \frac{2a_{14}}{5} \end{pmatrix}, \; A_4 = \begin{pmatrix} 0 & \frac{3a_{02}}{10} + \frac{3a_{12}}{10} \\ 0 & \frac{3a_{04}}{10} + \frac{3a_{14}}{10} \end{pmatrix},$$

and the determinant of $A(\sigma)$ is $\det A(\sigma) = \frac{3}{40}(a_{04}a_{12} - a_{02}a_{14})(\sigma-2)^2(\sigma-1)^2$.

It is easily checked that the given vector functions are solutions of the system $A(\sigma)\beta(k)=0$. The values for the free parameters of the matrices A_i can be chosen so that $\det A(\sigma) \neq 0$. An example of a resulting matrix is

$$A(\sigma) = \begin{pmatrix} \frac{3}{5} - \frac{11\sigma}{5} + 2\sigma^2 - \frac{3\sigma^3}{5} & 2\sigma - \frac{6\sigma^2}{5} - \frac{4\sigma^3}{5} + \frac{3\sigma^4}{5} \\ \frac{1}{10} - \frac{29\sigma}{20} + \frac{7\sigma^2}{4} - \frac{3\sigma^3}{5} & 1 + \sigma - \frac{29\sigma^2}{20} - \frac{11\sigma^3}{20} + \frac{3\sigma^4}{5} \end{pmatrix}.$$

In addition, we can find a polynomial matrix $A_1(\sigma)$ and unimodular matrices $U(\sigma)$, $V(\sigma)$ such that

$$A_1(\sigma) = U(\sigma)A(\sigma), \quad \tilde{A}_1(\sigma) = V(\sigma)\tilde{A}(\sigma),$$

so that $A_1(\sigma)$ satisfies $A(\sigma)\beta_i(k)=0$. An example is the matrix

$$A_1(\sigma) = \begin{pmatrix} \frac{7}{10} - \frac{73\sigma}{20} + \frac{15\sigma^2}{4} - \frac{6\sigma^3}{5} & 1 + 3\sigma - \frac{53\sigma^2}{20} - \frac{27\sigma^3}{20} + \frac{6\sigma^4}{5} \\ \frac{13}{10} - \frac{117\sigma}{20} + \frac{23\sigma^2}{4} - \frac{9\sigma^3}{5} & 1 + 5\sigma - \frac{77\sigma^2}{20} - \frac{43\sigma^3}{20} + \frac{9\sigma^4}{5} \end{pmatrix},$$

with

$$U(\sigma) = U = V(\sigma) = V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Moreover, there is no set of values for $a_{04}, a_{12}, a_{02}, a_{14}$ such that the constructed system has symmetric or skew symmetric coefficients and nonzero determinant.

6. Conclusions. A novel method has been proposed for constructing an AR-Representation that satisfies a prescribed forward and backward behavior, given in the form of vector valued functions. It was shown (see Example 5.2) that this method can also be used to construct non-regular systems. Thus, the proposed method is more versatile than previous ones for continuous or discrete time systems (see [10, 13, 19]) that only functioned for square matrices. The results presented in this work can also be extended with minor

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adjustments to the case of continuous time systems, where smooth and impulsive behaviors are of interest and the constructed matrices may need to satisfy certain properties like having symmetric, skew-symmetric or alternating coefficients.

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