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JORDAN TRIPLE PRODUCT HOMOMORPHISMS ON TRIANGULAR MATRICES TO AND FROM DIMENSION ONE*

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Abstract. A map Φ is a Jordan triple product (JTP for short) homomorphism whenever $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all A, B. We study JTP homomorphisms on the set of upper triangular matrices $\mathcal{T}_n(\mathbb{F})$, where $\mathbb{F}$ is the field of real or complex numbers. We characterize JTP homomorphisms $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$ and JTP homomorphisms $\Phi : \mathbb{F} \to \mathcal{T}_n(\mathbb{F})$. In the latter case we consider continuous maps and the implications of omitting the assumption of continuity.

1. Introduction.

Denote by $\mathcal{M}_n(\mathbb{F})$ the set of all $n \times n$ matrices over a field $\mathbb{F}$. A *Jordan triple product homomorphism* (JTP for short) $\Phi : \mathcal{A} \to \mathcal{B}$ is a mapping from an $\mathcal{A} \subseteq \mathcal{M}_n(\mathbb{F})$ to a $\mathcal{B} \subseteq \mathcal{M}_m(\mathbb{F})$ such that

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$$

for every $A, B \in \mathcal{A}$.

If $\operatorname{char} \mathbb{F} \neq 2$, it is easy to see that an additive mapping $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is JTP if and only if it is a *Jordan homomorphism*, i.e.,

$$\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$$

for every $A, B \in \mathcal{M}_n(\mathbb{F})$. Jordan homomorphisms are widely studied in ring theory. Moreover, Jordan operator algebras are important in mathematical foundations of quantum mechanics [3].

In scope of these facts, it is natural to study generalizations of Jordan homomorphisms. Results to date include:

- In [8], Lu showed that every bijective JTP homomorphism is additive.
- In an excellent survey paper [9], Šemrl asked for a characterization of not necessarily bijective JTP homomorphisms.
- Kuzma [6] responded by characterizing nondegenerate JTP homomorphisms from $\mathcal{M}_n(\mathbb{F})$ to $\mathcal{M}_n(\mathbb{F})$.
- Lešnjak and Sze [7] studied injective JTP homomorhisms Φ from $\mathcal{M}_n(\mathbb{F})$ to $\mathcal{M}_n(\mathbb{F})$ and proved they are of the standard form, namely there exist $\sigma \in \mathbb{F}$, $\sigma = \pm 1$, an injective homomorphism φ of $\mathbb{F}$ and an invertible $S \in \mathcal{M}_n(\mathbb{F})$ such that either $\Phi(A) = \sigma S A_\varphi S^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{F})$ or $\Phi(A) = \sigma S A_\varphi^T S^{-1}$

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for all $A \in \mathcal{M}_n(\mathbb{F})$. Here, A_φ is the image of A under φ applied entrywise. In particular, every such Φ is linear.

- Dobovišek [1] characterized JTP homomorphisms from $\mathcal{M}_n(\mathbb{F})$ to $\mathbb{F}$ and showed they take the form $\Phi(\cdot) = \pm\varphi(\det \cdot)$ for some multiplicative φ.
- Later, he studied JTP homomorphisms from $\mathcal{M}_2(\mathbb{F})$ to $\mathcal{M}_3(\mathbb{F})$ in [2].
- The authors of this paper characterized JTP homomorphisms from $n \times n$ Hermitian matrices to scalars and vice versa [4], and from Hermitian matrices of dim 2 to themselves [5].

In the present paper, we study Jordan triple product homomorphisms from the set $\mathcal{A}$ to the set $\mathcal{B}$, where one of them is an algebra of $n \times n$ upper-triangular matrices $\mathcal{T}_n(\mathbb{F})$ over the field of real or complex numbers $\mathbb{F}$ and the other is the field $\mathbb{F}$.

In Section 2 we characterize all JTP homomorphisms from $\mathcal{T}_n(\mathbb{C})$ to $\mathbb{C}$. In Section 3 we study JTP homomorphisms from $\mathbb{F}$ to $\mathcal{T}_n(\mathbb{F})$. First we characterize all continuous mappings of this kind for the real case, then we omit the assumption of continuity. At the end we also consider continuous JTP homomorphisms from $\mathbb{C}$ to $\mathcal{M}_n(\mathbb{C})$ or $\mathcal{T}_n(\mathbb{C})$.

2. From $\mathcal{T}_n(\mathbb{C})$ to $\mathbb{C}$. In this section we characterize all JTP homomorphisms from the algebra of upper-triangular complex matrices $\mathcal{T}_n(\mathbb{C})$ to the field of complex numbers. We start with a more general but simple lemma.

LEMMA 2.1. *Let $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a JTP homomorphism with $\Phi(0) = 0$ and $\Phi(I) = I$. Then $\Phi(A^k) = \Phi(A)^k$ for every $A \in \mathcal{T}_n(\mathbb{C})$ and every $k \in \mathbb{N}$, and $\Phi(A^k) = \Phi(A)^k$ for every invertible $A \in \mathcal{T}_n(\mathbb{C})$ and $k \in \mathbb{Z}$.*

Proof. Take $A \in \mathcal{T}_n(\mathbb{C})$. Then $\Phi(A^2) = \Phi(AIA) = \Phi(A)\Phi(I)\Phi(A) = \Phi(A)^2$. Assume $k \geq 3$. If $k = 2j+1$ is odd, then

$$\Phi(A^k) = \Phi(A^j A A^j) = \Phi(A^j)\Phi(A)\Phi(A^j) = \Phi(A)^j \Phi(A)\Phi(A)^j = \Phi(A)^k$$

by induction. Otherwise, if $k = 2j$ is even, then $\Phi(A^k) = \Phi(A^{2j}) = \Phi(A^j)^2 = \Phi(A)^k$.

For the second part of lemma take A invertible. Then

$$I = \Phi(I) = \Phi(A^{-1}A^2A^{-1}) = \Phi(A^{-1})\Phi(A)^2\Phi(A^{-1}),$$

hence $\Phi(A)$ invertible. Now take $\Phi(A) = \Phi(AA^{-1}A) = \Phi(A)\Phi(A^{-1})\Phi(A)$. Since $\Phi(A)$ is invertible, we get $\Phi(A)\Phi(A^{-1}) = I$ which gives us $\Phi(A^{-1}) = \Phi(A)^{-1}$. ∎

We now consider a JTP homomorphism $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$. We first show, that the assumptions from Lemma 2.1 are not restrictive.

From $\Phi(I)^3 = \Phi(I)$ it follows that $\Phi(I) \in \{-1, 0, 1\}$. If $\Phi(I) = 0$, then $\Phi(A) = \Phi(I)\Phi(A)\Phi(I) = 0$ for every $A \in \mathcal{T}_n(\mathbb{C})$. If $\Phi(I) = -1$, define $\Phi'(A) = -\Phi(A)$. Then $\Phi'(I) = 1$ and Φ' is a JTP homomorphism. Hence we may assume that $\Phi(I) = 1$.

From $\Phi(0)^3 = \Phi(0)$ it again follows that $\Phi(0) \in \{-1, 0, 1\}$. If $\Phi(0) = 1$, then $\Phi(A) = \Phi(0)\Phi(A)\Phi(0) = \Phi(0) = 1$ for every $A \in \mathcal{T}_n(\mathbb{C})$. If $\Phi(0) = -1$, then $1 = \Phi(I) = \Phi(0)\Phi(I)\Phi(0) = \Phi(0) = -1$, which is a contradiction. Hence $\Phi(0) = 0$.

Definition 2.2. *A JTP homomorphism* $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$ *is regular when* $\Phi(0) = 0$ *and* $\Phi(I) = 1$.

For a regular JTP homomorphism $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$ we define functions $\varphi_i : \mathbb{C} \to \mathbb{C}$ as $\varphi_i(a) = \Phi(\operatorname{diag}(1, \dots, 1, a, 1, \dots, 1))$ with a at i-th position. Here, $\operatorname{diag}(a_1, \dots, a_n)$ denotes a diagonal matrix with diagonal entries $a_1, \dots, a_n$. Define also $\sqrt{a} := \sqrt{|a|}(\cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2})$ for every $a \in \mathbb{C}$ with $a = |a|(\cos\alpha + i\sin\alpha)$.

Lemma 2.3. *The functions* φ_i *are multiplicative unital maps.*

Proof. First we prove that $\varphi_i(a^2) = \varphi_i(a)^2$. Take $a \in \mathbb{C}$. Then

$$\begin{aligned}\varphi_i(a^2) &= \Phi(\operatorname{diag}(1, \dots, 1, a^2, 1 \dots, 1)) \\ &= \Phi(\operatorname{diag}(1, \dots, 1, a, 1, \dots, 1)) \cdot 1 \cdot \Phi(\operatorname{diag}(1, \dots, 1, a, 1, \dots, 1)) = \varphi_i(a)^2.\end{aligned}$$

Next, take $a, b \in \mathbb{C}$. Then

$$\begin{aligned}\varphi_i(ab) &= \Phi(\operatorname{diag}(1, \dots, 1, \sqrt{b}, 1, \dots, 1)) \cdot \Phi(\operatorname{diag}(1, \dots, 1, a, 1, \dots, 1)) \cdot \\ &\qquad \cdot \Phi(\operatorname{diag}(1, \dots, 1, \sqrt{b}, 1, \dots, 1)) \\ &= \varphi_i(\sqrt{b})^2 \varphi_i(a) = \varphi_i(a)\varphi_i(b),\end{aligned}$$

which concludes the proof. ∎

Corollary 2.4. *Take arbitrary* $a_1, \dots, a_n \in \mathbb{C}$. *Then*

$$\Phi(\operatorname{diag}(a_1, \dots, a_n)) = \varphi_1(a_1) \cdots \varphi_n(a_n).$$

Proof. Write

$$\begin{aligned}\Phi(\operatorname{diag}(a_{11}, \dots, a_{nn})) &= \varphi_1(\sqrt{a_{11}})\Phi(\operatorname{diag}(1, a_{22}, \dots, a_{nn}))\varphi_1(\sqrt{a_{11}}) \\ &= \varphi_2(\sqrt{a_{22}})\varphi_1(\sqrt{a_{11}})\Phi(\operatorname{diag}(1, 1, a_{33}, \dots, a_{nn}))\varphi_1(\sqrt{a_{11}})\varphi_2(\sqrt{a_{22}}) \\ &= \cdots \\ &= \varphi_1(\sqrt{a_{11}})^2 \cdots \varphi_n(\sqrt{a_{nn}})^2,\end{aligned}$$

which concludes the proof. ∎

Let $E_{ij} \in \mathcal{T}_n(\mathbb{C})$ denote the matrix whose (i, j)-entry is 1 and every other entry is 0.

Lemma 2.5. *Take* $A \in \mathcal{T}_n(\mathbb{C})$, *such that* $A = \operatorname{diag}(a_{11}, \dots, a_{nn}) + a_{ij}E_{ij}$, $a_{ii}, a_{jj} \neq 0$. *Then* $\Phi(A) = \varphi_1(a_{11}) \cdots \varphi_n(a_{nn})$.

Proof. Define $D = \operatorname{diag}(d_{11}, \dots, d_{nn})$ where $d_{kk} = -\frac{a_{ii}}{a_{jj}}$ if $k = j$ and $d_{kk} = 1$ otherwise. Then ADA is diagonal and we can calculate the image $\Phi(ADA)$:

$$\begin{aligned}\Phi(ADA) &= \Phi(\operatorname{diag}(a_{11}^2, \dots, a_{j-1,j-1}^2, -a_{ii}a_{jj}, a_{j+1,j+1}^2, \dots, a_{nn}^2)) \\ &= \varphi_1(a_{11})^2 \cdots \varphi_{j-1}(a_{j-1,j-1})^2 \cdot \varphi_j(-a_{ii}a_{jj}) \cdot \varphi_{j+1}(a_{j+1,j+1})^2 \cdots \varphi_n(a_{nn})^2.\end{aligned}$$

On the other hand,

$$\Phi(ADA) = \Phi(A)\Phi(D)\Phi(A) = \Phi(A)^2 \varphi_j\left(-\frac{a_{ii}}{a_{jj}}\right).$$

Using multiplicativity of φ_j, we get the desired result. ∎

We are now able to prove the main result of this section.

THEOREM 2.6. *Let $\Phi : \mathcal{T}_n(\mathbb{C}) \to \mathbb{C}$ be a regular JTP homomorphism. Then there exist multiplicative unital maps $\varphi_i : \mathbb{C} \to \mathbb{C}$ for $i = 1, 2, ..., n$ such that*

$$\Phi(A) = \Phi(\mathrm{diag}(a_{11}, \ldots, a_{nn})) = \varphi_1(a_{11}) \cdots \varphi_n(a_{nn})$$

for any matrix $A \in \mathcal{T}_n(\mathbb{C})$ with diagonal entries $a_{11}, ..., a_{nn}$.

Proof. We devise an algorithm that will diagonalise A or some power of A, whilst preserving the value of $\Phi(A)$, but first we derive necessary means. Choose a non-diagonal entry of A, namely a_{ij}. If $a_{ij} = 0$, there is nothing to do. Suppose $a_{ij} \neq 0$ and $a_{ii} + a_{jj} \neq 0$. Define

$$B_{ij} = I - \frac{a_{ij}}{a_{ii} + a_{jj}} E_{ij}.$$

Then by Lemma 2.5 we have that $\Phi(B_{ij}) = 1$, a simple matrix multiplication of $B_{ij}AB_{ij}$ annihilates a_{ij} whilst preserving diagonal entries, and $\Phi(B_{ij}AB_{ij}) = \Phi(B_{ij})\Phi(A)\Phi(B_{ij}) = \Phi(A)$.

If for some $a_{ij} \neq 0$ it holds that $a_{ii} + a_{jj} = 0$ and $a_{ii} \neq 0$, we replace B_{ij} with C_{ij}, defined as follows:

$$C_{ij} = I - 2E_{jj} + \frac{a_{ij}}{2a_{ii}} E_{ij}.$$

Then $\Phi(C_{ij}) = \varphi_j(-1)$ by Lemma 2.5, and

$$\Phi(C_{ij}AC_{ij}) = \Phi(C_{ij})\Phi(A)\Phi(C_{ij}) = \varphi_j(-1)^2\Phi(A) = \Phi(A),$$

where $C_{ij}AC_{ij}$ has zero (i, j)-th entry, whilst the diagonal entries are preserved.

Denote $F_{ij} = \begin{cases} I; & a_{ij} = 0 \quad \text{or} \quad a_{ii} = a_{jj} = 0 \\ B_{ij}; & a_{ij} \neq 0,\, a_{ii} + a_{jj} \neq 0 \\ C_{ij}; & a_{ij} \neq 0,\, a_{ii} + a_{jj} = 0. \end{cases}$

The algorithm goes as follows:

- Start with the first entry in the first diagonal above the main diagonal, a_{12}.
- Apply F_{12} to get: $A^{(1)} := F_{12}AF_{12}$.
- Proceed to the next entry in the same diagonal, $a_{23}^{(1)}$.
- Apply $F_{23}^{(1)}$ to $A^{(1)}$ to get: $A^{(2)} := F_{23}^{(1)}A^{(1)}F_{23}^{(1)}$.
- Repeat the process. When the end of diagonal, $a_{n-1,n}^{(n-1)}$, is reached, proceed to the first entry in the next diagonal, $a_{13}^{(n-1)}$.
- Follow the process down the second diagonal.
- Repeat the process for every diagonal.

The final matrix, $A^{(k)}$, has the same diagonal entries as A, the only possible non-zero off-diagonal entries $a_{ij}^{(k)}$ are those with $a_{ii} = a_{jj} = 0$, and $\Phi(A^{(k)}) = \Phi(A)$.

To prove the assertion, take $r \in \mathbb{N}$ such that $(A^{(k)})^r$ is diagonal, that is, $(A^{(k)})^r = \mathrm{diag}(a_{11}^r, \ldots, a_{nn}^r)$. Then, by Corollary 2.4,

$$\Phi(A)^r = \Phi(A^{(k)})^r = \Phi((A^{(k)})^r) = \prod_{i=1}^{n} \varphi_i(a_{ii})^r.$$

If $\Phi(A)^r = 0$, then so is $\Phi(A)$ and the assertion follows. If not, notice that $(A^{(k)})^{r+1}$ is also diagonal, hence

$$\Phi(A)^{r+1} = \Phi((A^{(k)})^{r+1}) = \prod_{i=1}^{n} \varphi_i(a_{ii})^{r+1}.$$

Dividing these two equations, we get $\Phi(A) = \prod_{i=1}^{n} \varphi_i(a_{ii}) = \Phi(\mathrm{diag}(a_{11}, \ldots, a_{nn}))$, which concludes the proof. ∎

Remark. The multiplication $F_{ij}^{(l)} A^{(l)} F_{ij}^{(l)}$ adds a multiple of i-th column of $A^{(l)}$ to its j-th column and a multiple of j-th row to its i-th row. If $a_{pr} \neq 0$ but $a_{pp} = a_{rr} = 0$, then the algorithm does not annihilate a_{pr}. Hence entries of this kind might interfere with the annihilation process of the algorithm. But that does not happen as the algorithm is designed in such manner that it pushes these entries to higher diagonals through each step. Therefore these entries have no effect on entries already annihilated.

3. From $\mathbb{F}$ to $\mathcal{T}_n(\mathbb{F})$. In this section we consider JTP homomorphisms from the field $\mathbb{F}$ of real or complex numbers to the algebra of upper-triangular real or complex matrices $\mathcal{T}_n(\mathbb{F})$. We again start with more general lemmas.

LEMMA 3.1. *Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ JTP homomorphism. Then there exists an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that*

$$\Phi(\lambda) = S \begin{bmatrix} \Psi(\lambda) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} S^{-1},$$

where Ψ is a JTP homomorphism with $\Psi(0) = 0$.

Proof. From $0^3 = 0$ we get $\Phi(0)^3 = \Phi(0)$, hence $\Phi(0)$ can be written as

$$\Phi(0) = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} S^{-1}.$$

With respect to this decomposition, we can write

$$\Phi(\lambda) = \begin{bmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{bmatrix}.$$

From $\Phi(0) = \Phi(0)\Phi(\lambda)\Phi(0)$ we derive that $B_5 = I$, $B_9 = -I$ and $B_6, B_8 = 0$.

On the other hand, from $\Phi(0) = \Phi(\lambda)\Phi(0)\Phi(\lambda)$ we get that $B_2, B_3, B_4, B_7 = 0$. Hence we get the desired form. ∎

Without the loss of generality we assume from now on that $\Phi(0) = 0$.

LEMMA 3.2. *Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ JTP homomorphism with $\Phi(0) = 0$. Then there exists an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that*

$$\Phi(\lambda) = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Psi_1(\lambda) & 0 \\ 0 & 0 & -\Psi_2(\lambda) \end{bmatrix} S^{-1},$$

where Ψ_1, Ψ_2 are JTP homomorphisms with $\Psi_1(1), \Psi_2(1) = I$.

Proof. From $1^3 = 1$ we get $\Phi(1)^3 = \Phi(1)$. Hence $\Phi(1)$ can be written as

$$\Phi(1) = S \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} S^{-1}.$$

Write

$$\Phi(\lambda) = \begin{bmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{bmatrix}$$

with respect to the previous decompostion. From $\Phi(\lambda) = \Phi(1)\Phi(\lambda)\Phi(1)$ we get that

$$B_1, B_2, B_3, B_4, B_6, B_7, B_8 = 0.$$ ∎

Without the loss of generality we assume that $\Phi(1) = I$. Then it follows from Lemma 2.1 that $\Phi(a^k) = \Phi(a)^k$ for every $k \in \mathbb{Z}$.

LEMMA 3.3. *Take $\Phi : \mathbb{F} \to \mathcal{M}_n(\mathbb{C})$ JTP homomorphism with $\Phi(0) = 0$ and $\Phi(1) = I$. Then there exist $S \in \mathcal{M}_n(\mathbb{C})$ invertible and Ψ_1, Ψ_2 JTP homomorphisms such that $\Psi_1(-1) = I$, $\Psi_2(-1) = -I$ and*

$$\Phi(\lambda) = S \begin{bmatrix} \Psi_1(\lambda) & 0 \\ 0 & \Psi_2(\lambda) \end{bmatrix} S^{-1}.$$

Proof. The idea is the same as in the previous two lemmas, so the details are omitted. ∎

Without the loss of generality we may and will assume that $\Phi(-1) = \pm I$. The same conclusions also hold if $\Phi : \mathbb{F} \to \mathcal{T}_n(\mathbb{C})$.

DEFINITION 3.4. *A JTP homomorphism $\Phi : \mathbb{F} \to \mathcal{T}_n(\mathbb{C})$ is regular, if $\Phi(0) = 0$ and $\Phi(1) = I$ and $\Phi(-1) = \pm I$.*

It turns out that continuous regular JTP homomorphisms $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{R})$ are easy to treat.

PROPOSITION 3.5. *If a regular JTP homomorphism $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{R})$ is continuous, then it is multiplicative. On $\mathbb{R}^+$ it is uniquely determined by the matrix $\Phi(2)$. If $a > 0$, then $\Phi(a) = \Phi(2)^{\log_2 a}$. If $a < 0$, then $\Phi(a) = \pm\Phi(|a|)$.*

Proof. We already know that $\Phi(2^k) = \Phi(2)^k$ for $k \in \mathbb{Z}$. Since $\Phi(2) = \Phi(\sqrt{2})^2$, the eigenvalues of $\Phi(2)$ are positive. The matrix $\Phi(2^{\frac{1}{m}}) = \Phi(2^{\frac{1}{2m}})^2$ also has positive eigenvalues and its m-th power is $\Phi(2)$. Hence $\Phi(2^{\frac{1}{m}})$ is the unique positive m-th root of $\Phi(2)$. Thus $\Phi(2^x) = \Phi(2)^x$ for every $x \in \mathbb{Q}$. By continuity of Φ we can extend that to every $x \in \mathbb{R}$.

If $a < 0$, then $\Phi(a) = \Phi(\sqrt{|a|})\Phi(-1)\Phi(\sqrt{|a|}) = \pm\Phi(|a|)$. ∎

In what follows we omit the assumption of continuity.

PROPOSITION 3.6. *Let $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{F})$ be a regular JTP homomorphism, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then there exist an invertible $S \in \mathcal{M}_n(\mathbb{C})$ and multiplicative maps $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that*

$$\Phi(a) = S \begin{bmatrix} \Phi_1(a) & & 0 \\ & \ddots & \\ 0 & & \Phi_k(a) \end{bmatrix} S^{-1} \qquad \text{with} \qquad \Phi_i(a) = \begin{bmatrix} \varphi_i(a) & & \bigstar \\ & \ddots & \\ 0 & & \varphi_i(a) \end{bmatrix}.$$

We split the proof into several steps.

Step 1: Prove proposition for the case $n = 2$.

Proof. If $\Phi(a) = \begin{bmatrix} \varphi(a) & \star \\ 0 & \varphi(a) \end{bmatrix}$ for every $a > 0$, then this also holds for every $a \in \mathbb{R}$. So, suppose there exists $a > 0$, such that $\Phi(a) = \begin{bmatrix} \varphi_1(a) & \star \\ 0 & \varphi_2(a) \end{bmatrix}$ with $\varphi_1(a) \neq \varphi_2(a)$. The matrix $\Phi(A)$ is diagonalizable, so we may assume without the loss of generality that $\Phi(a) = \begin{bmatrix} \varphi_1(a) & 0 \\ 0 & \varphi_2(a) \end{bmatrix}$. Then

$$\Phi(a) = \Phi(\sqrt{a})^2 = \begin{bmatrix} \varphi_1(\sqrt{a}) & x \\ 0 & \varphi_2(\sqrt{a}) \end{bmatrix}^2 = \begin{bmatrix} \varphi_1(a) & x(\varphi_1(\sqrt{a}) + \varphi_2(\sqrt{a})) \\ 0 & \varphi_2(a) \end{bmatrix},$$

hence we derive the equation $0 = x(\varphi_1(\sqrt{a}) + \varphi_2(\sqrt{a}))$. If $\varphi_1(\sqrt{a}) = -\varphi_2(\sqrt{a})$, then it must be that $\varphi_1(a) = \varphi_2(a)$, a contradiction. Thus $x = 0$.

If $\Phi(b) = \begin{bmatrix} \varphi_1(b) & 0 \\ 0 & \varphi_2(b) \end{bmatrix}$ for every $b > 0$, then this is true also for $b \in \mathbb{R}$. So, suppose there exists $b > 0$ such that $\Phi(b) = \begin{bmatrix} \varphi_1(b) & \psi(b) \\ 0 & \varphi_2(b) \end{bmatrix}$ with $\psi(b) \neq 0$.

By calculating $\Phi(\sqrt{a}\, b^2 \sqrt{a}) = \Phi(bab)$ and equating upper-right entries, we obtain equation

$$\varphi_1(\sqrt{a})\varphi_2(\sqrt{a})\psi(b)(\varphi_1(b) + \varphi_2(b)) = \varphi_1(a)\varphi_2(b)\psi(b) + \varphi_2(a)\varphi_2(b)\psi(b).$$

Write $\lambda = \frac{\varphi_1(\sqrt{a})}{\varphi_2(\sqrt{a})}$ and use λ in previous equation:

$$\varphi_1(b) + \varphi_2(b) = \lambda\varphi_1(b) + \frac{1}{\lambda}\varphi_2(b).$$

Next, replace $\sqrt{a}$ with $\frac{1}{\sqrt{a}}$ and calculate upper-right entries of equation $\Phi(\frac{1}{\sqrt{a}} b^2 \frac{1}{\sqrt{a}}) = \Phi(b\frac{1}{a}b)$. We get $\varphi_1(b) + \varphi_2(b) = \frac{1}{\lambda}\varphi_1(b) + \lambda\varphi_2(b)$.

Hence $(\lambda - \frac{1}{\lambda})(\varphi_1(b) - \varphi_2(b)) = 0$, so either $\lambda - \frac{1}{\lambda} = 0$ or $\varphi_1(b) - \varphi_2(b) = 0$. If $\lambda - \frac{1}{\lambda} = 0$, then $\lambda = \pm 1$. That is a contradiction since it must be that $\varphi_1(a) = \varphi_2(a)$. On the other hand, if $\varphi_1(b) - \varphi_2(b) = 0$, then $\lambda + \frac{1}{\lambda} = 2$, which again implies that $\lambda = 1$, a contradiction. ∎

We proceed for general n. Choose $a > 0$ and denote $A = \Phi(a)$.

Step 2: Using upper-triangular similarity, if necessary, we may assume without the loss of generality, that from $a_{ii} \neq a_{jj}$ it follows that $a_{ij} = 0$.

Proof. If A is upper triangular, then it is upper-triangularly similar to a generalized direct sum A' of upper-triangular matrices $A_1, \ldots, A_k$, where each A_r has a single eigenvalue and their spectra don't intersect, see [10, Proposition 1.2]. Hence $a_{ii} \neq a_{jj}$ implies $a_{ij} = 0$. ∎

Take λ an eigenvalue of A.

Step 3: Using permutation similarity, if necessary, we may assume without the loss of generality that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad \text{where} \qquad A_1 = \begin{bmatrix} \lambda & & \bigstar \\ & \ddots & \\ 0 & & \lambda \end{bmatrix},$$

A_2 is upper-triangular and λ is **not** an eigenvalue of A_2.

Proof. Look at the diagonal of A. Suppose there exists k such that $a_{kk} \neq \lambda$ and $j > k$ with $a_{jj} = \lambda$. Then there exists i such that $a_{ii} \neq \lambda$ and $a_{i+1,i+1} = \lambda$. Hence,

$$A = \begin{bmatrix} \bigstar & \bigstar & \bigstar \\ 0 & \begin{bmatrix} a_{ii} & 0 \\ 0 & \lambda \end{bmatrix} & \bigstar \\ 0 & 0 & \bigstar \end{bmatrix}.$$

With respect to that decomposition,

$$B = \Phi(b) = \begin{bmatrix} \bigstar & \bigstar & \bigstar \\ 0 & \begin{bmatrix} \star & \star \\ 0 & \star \end{bmatrix} & \bigstar \\ 0 & 0 & \bigstar \end{bmatrix}$$

for any $b > 0$. Restricting Φ to the central block, Step 1 proves that the central block B is diagonal.

We apply the similarity by

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\ 0 & 0 & I \end{bmatrix},$$

which trades places of a_{ii} and λ, but it preserves the upper-triangularity of B. To finish the proof, repeat inductively. $\square$

Step 4: Suppose

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad \text{with} \qquad A_1 = \begin{bmatrix} \lambda & & \bigstar \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}$$

and λ is **not** an eigenvalue of A_2. Take $B = \Phi(b)$ for $b > 0$ and write $B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$ with respect to the decomposition of A. Then $B_2 = 0$.

Proof. Suppose $B_2 \neq 0$. Denote

$$A_2 = \begin{bmatrix} \alpha_1 & & \bigstar \\ & \ddots & \\ 0 & & \alpha_p \end{bmatrix}, \quad B_1 = \begin{bmatrix} \beta_1 & & \bigstar \\ & \ddots & \\ 0 & & \beta_q \end{bmatrix}, \quad B_3 = \begin{bmatrix} \gamma_1 & & \bigstar \\ & \ddots & \\ 0 & & \gamma_p \end{bmatrix}$$

and $B_2 = [b_{ij}]_{i=1,\,j=1}^{q,\quad p}$.

Take the left-most and lower-most nonzero entry of B_2. So, suppose by induction that

- $b_{kl} \neq 0$,
- $b_{k1}, \ldots, b_{k,l-1} = 0$,
- $b_{k+1,l}, \ldots, b_{ql} = 0$, and
- $b_{ij} = 0$ for all $i > k$, $j < l$.

Since $\Phi(ba^2b) = \Phi(ab^2a)$ and $\Phi(b\frac{1}{a^2}b) = \Phi(\frac{1}{a}b^2\frac{1}{a})$, we get that

$$B_1A_1^2B_2 + B_2A_2^2B_3 = A_1(B_1B_2 + B_2B_3)A_2$$

and

$$B_1A_1^{-2}B_2 + B_2A_2^{-2}B_3 = A_1^{-1}(B_1B_2 + B_2B_3)A_2^{-1}.$$

Observe (k,l)-th entry of matrices on both-hand sides of the equations. We obtain

$$\beta_k\lambda^2 b_{kl} + b_{kl}\alpha_l^2\gamma_l = \lambda b_{kl}\gamma_l\alpha_l + \lambda\beta_k b_{kl}\alpha_l$$

and

$$\beta_k\frac{1}{\lambda^2}b_{kl} + b_{kl}\frac{1}{\alpha_l^2}\gamma_l = \lambda^{-1}b_{kl}\alpha_l^{-1}(\gamma_l + \beta_k).$$

Since $b_{kl}, \lambda, \alpha_l \neq 0$, divide both equations by $\lambda b_{kl}\alpha_l$ to get

$$\beta_k\frac{\lambda}{\alpha_l} + \gamma_l\frac{\alpha_l}{\lambda} = \gamma_l + \beta_k \qquad \text{and} \qquad \beta_k\frac{\alpha_l}{\lambda} + \gamma_l\frac{\lambda}{\alpha_l} = \gamma_l + \beta_k.$$

Subtract these two equalities to obtain

$$(\beta_k - \gamma_l)(\frac{\lambda}{\alpha_l} - \frac{\alpha_l}{\lambda}) = 0.$$

If $\frac{\lambda}{\alpha_l} = \frac{\alpha_l}{\lambda}$, then $\lambda = \alpha_l$, which is a contradiction. If $\beta_k = \gamma_l$, then $\frac{\lambda}{\alpha_l} + \frac{\alpha_l}{\lambda} = 2$, which again gives $\lambda = \alpha_l$, another contradiction. Hence $B_2 = 0$. $\square$

Proof of Proposition 3.6. So far we have proven that, as soon as we have $a > 0$ such that $\Phi(a)$ has at least two distinct eigenvalues, $\Phi(b)$ can be split into diagonal blocks for every $b > 0$. Now we use induction to finish the proof.

We have proven that a regular JTP homomorphism $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{F})$ can be split into diagonal blocks, where in each block diagonal entries are equal to each other. We now look for the form of off-diagonal entries. We start with the case $n = 2$.

PROPOSITION 3.7. *Let $\Phi : \mathbb{R} \to \mathcal{T}_2(\mathbb{R})$ be a regular JTP homomorphism. Then there exists an invertible $S \in \mathcal{M}_2(\mathbb{R})$ such that either*

$$\Phi(a) = S\begin{bmatrix}\varphi_1(a) & 0\\ 0 & \varphi_2(a)\end{bmatrix}S^{-1}$$

with $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ multiplicative, or

$$\Phi(a) = S\varphi(a)\begin{bmatrix}1 & \psi(\log_{a_0} a)\\ 0 & 1\end{bmatrix}S^{-1}$$

with $\varphi : \mathbb{R} \to \mathbb{R}$ multiplicative and $\psi : \mathbb{R} \to \mathbb{R}$ additive such that $\psi(a_0) = 1$ where $a_0 > 0, a_0 \neq 1$.

Proof. Suppose there exists $a > 0$ such that $\varphi_1(a) \neq \varphi_2(a)$. Then the assertion holds by proposition 3.6. So suppose that $\varphi(a) = \varphi_1(a) = \varphi_2(a)$ for every a. If $\Phi(a)$ is diagonal for every a, we have the first case. Choose $a_0 > 0$ such that $\Phi(a_0)$ is not diagonal. Then there exists an invertible matrix S such that $\Phi(a_0) = S\varphi(a_0)\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}S^{-1}$. For $b, c > 0$ write $x = \log_{a_0} b$, $y = \log_{a_0} c$, and

$$\Phi(b) = S\varphi(b)\begin{bmatrix}1 & \psi(x)\\0 & 1\end{bmatrix}S^{-1}, \qquad \Phi(c) = S\varphi(c)\begin{bmatrix}1 & \psi(y)\\0 & 1\end{bmatrix}S^{-1}.$$

From equating the upper-right entries of $\Phi(bcb) = \Phi(b)\Phi(c)\Phi(b)$ it follows that $\psi(2x+y) = \psi(x) + \psi(y) + \psi(x) = \psi(2x) + \psi(y)$ for every x, y. Hence $\psi : \mathbb{R} \to \mathbb{R}$ is additive and $\psi(0) = 0$, $\psi(1) = 1$. ◻

Proposition 3.8. *Let $\Phi : \mathbb{R} \to \mathcal{T}_n(\mathbb{R})$ a regular JTP homomorphism of the form*

$$\Phi(a) = \begin{bmatrix}\varphi(a) & & \bigstar\\ & \ddots & \\ 0 & & \varphi(a)\end{bmatrix}$$

with $\varphi : \mathbb{R} \to \mathbb{R}$ multiplicative. Then there exist an invertible $S \in \mathcal{T}_n(\mathbb{R})$, $a_0 > 0$, $a_0 \neq 1$, and $\delta_i \in \{0, 1\}$ such that

$$\Phi(a_0) = S\varphi(a_0)\begin{bmatrix}1 & \delta_1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1}\\ 0 & & & 1\end{bmatrix}S^{-1} \quad \text{and} \quad \Phi(b) = S\varphi(b)\begin{bmatrix}1 & & \psi_{ij}(x)\\ & \ddots & \\ 0 & & 1\end{bmatrix}S^{-1},$$

where $x = \log_{a_0}|b|$. Here, the functions $\psi_{ij} : \mathbb{R} \to \mathbb{R}$ satisfy the functional equations $\psi_{i,i+k}(0) = 0$ for $k \geq 1$, $\psi_{i,i+1}(1) = \delta_i$, $\psi_{i,i+k}(1) = 0$ for $k \geq 2$, and

$$\begin{aligned}\psi_{ij}(2x+y) =& 2\psi_{ij}(x) + \psi_{ij}(y) + \sum_{k=i+1}^{j-1}(\psi_{ik}(x)\psi_{kj}(y) + \psi_{ik}(y)\psi_{kj}(x)) + \\ &+ \sum_{k=i+1}^{j-2}\sum_{s=k+1}^{j-1}\psi_{ik}(x)\psi_{ks}(y)\psi_{sj}(x)\end{aligned}$$

for every $x, y \in \mathbb{R}$. In particular, $\psi_{i,i+1}$ are additive.

Proof. We choose $a_0 > 0$ such that $\Phi(a_0)$ is not diagonal. Using the invertible matrix S we put it into its Jordan form. Then the set of functional equations is a result of careful multiplication of right-hand side of JTP property of Φ. ◻

Example 3.9. *Consider case $n = 3$ of the previous proposition. Then ψ_{12} and ψ_{23} are (not necessarily equal) additive maps. For ψ_{13} it holds that*

$$\psi_{13}(2x+y) = 2\psi_{12}(x) + \psi_{13}(y) + \psi_{12}(x)\psi_{23}(y) + \psi_{12}(y)\psi_{23}(x).$$

*Suppose there exists a_0 such that $\operatorname{rank}(\Phi(a_0) - \varphi(a_0)I) = 1$ and there **doesn't** exist a_0 such that $\operatorname{rank}(\Phi(a_0) - \varphi(a_0)I) = 2$. Then if ψ_{ij} is nonzero, it must be additive for any combination of i, j.*

We conclude the paper by considering continuous JTP homomorphisms mapping from the field of complex numbers. With $\mathcal{S}^1$ we denote the unit circle in $\mathbb{C}$.

LEMMA 3.10. *Let $\Phi : \mathcal{S}^1 \to \mathcal{M}_n(\mathbb{C})$ be a continuous JTP homomorphism with $\Phi(1) = I$. Then there exists an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that*

$$\Phi(\omega) = S \begin{bmatrix} \omega^{\alpha_1} I & & 0 \\ & \ddots & \\ 0 & & \omega^{\alpha_k} I \end{bmatrix} S^{-1},$$

where $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$.

Proof. If $\omega \in \mathcal{S}^1$ is t-th root of unity ($\omega^t = 1$), Then $\Phi(\omega)^t = I$ and $\Phi(\omega)$ is diagonalizable. Take $\omega \in \mathcal{S}^1$ with $\omega^t = 1$ for some t such that $\Phi(\omega)$ has a maximal possible number of distinct eigenvalues. Without the loss of generality we may write

$$\Phi(\omega) = \begin{bmatrix} \omega^{\alpha_1} I & & 0 \\ & \ddots & \\ 0 & & \omega^{\alpha_k} I \end{bmatrix}.$$

Take $\omega_1 \in \mathcal{S}^1$ with $\omega_1^s = 1$. Define $m = \mathrm{lcm}(s,t)$. With ω_2 denote m-th root of unity. Then $\omega = \omega_2^{t'}$, $\omega_1 = \omega_2^{s'}$ for some integers t' and s'. It holds that $\Phi(\omega_2)^{t'} = \Phi(\omega)$, where $\Phi(\omega_2)$ has at most k distinct eigenvalues. Hence

$$\Phi(\omega_2) = \begin{bmatrix} \varphi_1(\omega_2) I & & 0 \\ & \ddots & \\ 0 & & \varphi_k(\omega_2) I \end{bmatrix},$$

from which it follows that

$$\Phi(\omega_1) = \begin{bmatrix} \varphi_1(\omega_1) I & & 0 \\ & \ddots & \\ 0 & & \varphi_k(\omega_1) I \end{bmatrix},$$

where $\varphi_i : \mathcal{S}^1 \to \mathcal{S}^1$ are multiplicative continuous maps, since $\Phi(\omega_2)^{s'} = \Phi(\omega_1)$. Hence $\varphi_i(\omega) = \omega^{\alpha_i}$ for some $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$. □

Write any $z \in \mathbb{C} \setminus \{0\}$ as $z = a\omega$, where $a > 0$ and $\omega \in \mathcal{S}^1$.

THEOREM 3.11. *Let $\Phi : \mathbb{C} \to \mathcal{M}_n(\mathbb{C})$ be a continuous JTP homomorphism with $\Phi(0) = 0$ and $\Phi(1) = I$. Then there exists an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that*

$$\Phi(a\omega) = S \begin{bmatrix} \omega^{\alpha_1} \Phi_1(a) & & 0 \\ & \ddots & \\ 0 & & \omega^{\alpha_k} \Phi_k(a) \end{bmatrix} S^{-1},$$

where $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$ and $\Phi_i : \mathbb{R}^+ \to \mathcal{M}_{n_i}(\mathbb{C})$ are JTP homomorphisms such that $\Phi_i(1) = I$.

Proof. We use the previous lemma. Assume $S = I$. Order $\alpha_1, \dots, \alpha_k$ so that $\alpha_1, \dots, \alpha_j$ are odd and $\alpha_{j+1}, \dots, \alpha_k$ are even. Then

$$\Phi(-1) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

Hence

$$\Phi(a) = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \Phi((-1)a(-1)) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} B_1 & -B_2 \\ -B_3 & B_4 \end{bmatrix}.$$

Then $B_2, B_3 = 0$.

We may therefore assume that either all $\alpha_1, \dots, \alpha_k$ are odd or all are even. Suppose that all are odd. Then we can write

$$\Phi(\omega) = \omega \begin{bmatrix} \omega^{\alpha_1 - 1} I & & 0 \\ & \ddots & \\ 0 & & \omega^{\alpha_k - 1} I \end{bmatrix},$$

where $\alpha_1 - 1, \dots, \alpha_k - 1$ are even.

So, without the loss of generality we may assume that $\alpha_1, \dots, \alpha_k$ are even. Write $\alpha_i = 2^{s_i} t_i$, where t_i is odd. Suppose all $\alpha_1, \dots, \alpha_k$ are of the form $\alpha_i = 2^s t_i$ with t_i odd. Then

$$\Phi(\omega) = \omega^{2^s} \begin{bmatrix} \omega^{\alpha_1 - 2^s} I & & 0 \\ & \ddots & \\ 0 & & \omega^{\alpha_k - 2^s} I \end{bmatrix},$$

where $\alpha_i - 2^s$ are all even of the form $2^{s+1} m_i$. If all m_i are odd or all are even, repeat the procedure until some have distinct parities.

Denote $s = \min\{s_1, \dots, s_k\}$. Reorder $\alpha_1, \dots, \alpha_k$ so that $s_1 = \ldots = s_j = s$ and $s_{j+1}, \dots, s_k > s$. Take ω the 2^{s+1}-th root of unity. Then $\omega^{\alpha_1} = \cdots = \omega^{\alpha_j} = -1$ and $\omega^{\alpha_{j+1}} = \cdots = \omega^{\alpha_k} = 1$.

Write

$$\Phi(\omega) = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \qquad \text{and} \qquad \Phi(a) = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then

$$\begin{aligned} \Phi(\omega^2 a) &= \Phi(\omega a \omega) = \Phi(\sqrt{a}\omega^2\sqrt{a}) = \Phi(\sqrt{a})\Phi(\omega^2)\Phi(\sqrt{a}) = \Phi(a) \\ &= \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

from which it follows that $B_2, B_3 = 0$. Thus $\Phi(a)$ is block diagonal. To finish the proof, proceed inductively. ∎

Corollary 3.12. *Let $\Phi : \mathbb{C} \to \mathcal{T}_n(\mathbb{C})$ be a continuous JTP homomorphism with $\Phi(0) = 0$ and $\Phi(1) = I$. Then there exists an invertible $S \in \mathcal{M}_n(\mathbb{C})$ such that*

$$\Phi(z) = S \begin{bmatrix} \Phi_1(z) & & 0 \\ & \ddots & \\ 0 & & \Phi_k(z) \end{bmatrix} S^{-1},$$

where

$$\Phi_i(z) = \begin{bmatrix} \varphi_i(z) & & \bigstar \\ & \ddots & \\ 0 & & \varphi_i(z) \end{bmatrix}$$

with $\varphi_i : \mathbb{C} \to \mathbb{C}$ multiplicative maps.

Proof. Combine the use of Theorem 3.11 and Proposition 3.6. ∎

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