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COMMUTATORS INVOLVING MATRIX FUNCTIONS*

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Abstract. Some results are obtained for matrix commutators involving matrix exponentials $\left(\left[e^A, B\right], \left[e^A, e^B\right]\right)$ and their norms.

Key words. Matrix commutators, Matrix exponential, Norm.

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1. Introduction. The commutator of two matrices A and B is defined as $[A, B] = AB - BA$ and plays an important role in many branches of mathematics, mathematical physics, quantum physics and quantum chemistry. Most of the studies on matrix commutators have been focused on norm inequalities in recent years [1, 2, 3, 4]. Böttcher and Wenzel obtained a nice inequality for $A, B \in M_n(\mathbb{C})$

$$\|[A, B]\|_F \leq \sqrt{2} \|A\|_F \|B\|_F , \tag{1.1}$$

where $\|.\|_F$ stands for the Frobenius norm [1].

Matrix functions are used in many areas of linear algebra and arise in numerous applications in science and engineering. Square root, polynomial, trigonometric functions, exponential and logarithm of matrices are widely used in matrix theory. The matrix exponential is by far most studied matrix function. There are many various ways in the literature to compute it. One of them is as follows:

$$\exp(A) = I + A + \frac{A^2}{2!} + \cdots + \frac{A^s}{s!} + \cdots = \sum_{s=0}^{\infty} \frac{A^s}{s!} \tag{1.2}$$

for $A \in M_n(\mathbb{C})$[7].

Our motivation in the present study is to obtain some results concerning relations between $[f(A), B]$, $[f(A), f(B)]$ and $[A, B]$, where $f(x)$ stands for e^x, and to obtain inequalities for the Frobenius norms of matrix commutators (1.1) such as

$$\|[f(A), B]\|_F \leq c \|A\|_F \|B\|_F \text{ and } \|[f(A), f(B)]\|_F \leq k \|A\|_F \|B\|_F .$$

Let us give some easy properties of matrix commutators and matrix exponential before we present our main results. For $A, B \in M_n(\mathbb{C})$, we have

- $[A + B, C] = [A, C] + [B, C]$
- $[AB, C] = A[B, C] + [A, C]B$

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- $[A, BC] = [A, B]\, C + B\, [A, C]$
- $[A^s, B] = sA^{s-1}\, [A, B]$, where $[A, [A, B]] = 0$

- $e^0 = I$, where 0 denotes zero matrix
- $e^A e^B = e^{A+B}$, if $AB = BA$
- $e^A e^B = e^B e^A$, if $AB = BA$
- $e^{P^{-1}AP} = P^{-1} e^A P$, where P is invertible

2. Main Results. In this section, we will present our results in two subsections.The results concerning the relations between $\left[e^A, B\right]$, $\left[e^A, e^B\right]$ and $[A, B]$ will be presented in Section 2.1 and the results related to norm inequalities will be presented in Section 2.2.

2.1. Properties of Commutators Involving Exponential Matrices. Let us start with an easy-to-prove lemma which we will use in main results.

LEMMA 2.1. *Let A be an arbitrary n-square matrix. Then*

$$\left[e^A, B\right] = \sum_{s=1}^{\infty} \frac{[A^s, B]}{s!}.$$

Proof. We have $\left[e^A, B\right] = e^A B - B e^A$. If we replace e^A by its Taylor expansion (1.2) in the right hand-side of the equation, we get

$$\left[e^A, B\right] = \left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^s}{s!} + \cdots\right) B - B\left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^s}{s!} + \cdots\right),$$

$$\left[e^A, B\right] = \left(B + AB + \frac{A^2}{2!}B + \cdots + \frac{A^s}{s!}B + \cdots\right) - \left(B + BA + B\frac{A^2}{2!} + \cdots + B\frac{A^s}{s!} + \cdots\right),$$

$$\left[e^A, B\right] = AB - BA + \frac{A^2}{2!}B - B\frac{A^2}{2!} + \cdots + \frac{A^s}{s!}B - B\frac{A^s}{s!} + \cdots,$$

$$\left[e^A, B\right] = \sum_{s=1}^{\infty} \frac{[A^s, B]}{s!},$$

and this concludes the proof. ∎

A nice corollary can be obtained for a special case of A.

COROLLARY 2.2. *Let A be an n-square matrix satisfying* $[A, [A, B]] = 0$. *Then*

$$\left[e^A, B\right] = e^A\, [A, B]\,.$$

Proof. In the previous lemma it was shown that $\left[e^A, B\right] = \sum_{s=1}^{\infty} \frac{[A^s, B]}{s!}$. On the other hand, we know that $[A^s, B] = sA^{s-1}\, [A, B]$ when $[A, [A, B]] = 0$. Combining both results, we get

$$\left[e^A, B\right] = \sum_{s=1}^{\infty} \frac{sA^{s-1}\, [A, B]}{s!},$$

$$\left[e^A, B\right] = \sum_{s=1}^{\infty} \frac{A^{s-1}\, [A, B]}{(s-1)!},$$

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$$\left[e^A, B\right] = e^A \left[A, B\right],$$

and this concludes the proof. ∎

Theorem 2.3. *Let A and B be complex n-square matrices. Then*

$$\left[e^A, e^B\right] = \sum_{s,t=1}^{\infty} \frac{\left[A^s, B^t\right]}{s!t!}.$$

Proof. As $\left[e^A, e^B\right] = e^A e^B - e^B e^A$, replacing e^A and e^B by their Taylor expansions (1.2) in the right hand-side of the equation, we get

$$e^A e^B - e^B e^A = \left(I + A + \frac{A^2}{2!} + \cdots\right)\left(I + B + \frac{B^2}{2!} + \cdots\right) - \left(I + B + \frac{B^2}{2!} + \cdots\right)\left(I + A + \frac{A^2}{2!} + \cdots\right)$$

$$e^A e^B - e^B e^A = AB - BA + \frac{A^2B - BA^2}{2!} + \frac{AB^2 - B^2A}{2!} + \frac{A^2B^2 - B^2A^2}{2!2!} + \cdots$$

$$e^A e^B - e^B e^A = [A, B] + \frac{\left[A^2, B\right]}{2!} + \frac{\left[A, B^2\right]}{2!} + \frac{\left[A^2, B^2\right]}{2!2!} + \cdots$$

$$\left[e^A, e^B\right] = \sum_{s,t=1}^{\infty} \frac{\left[A^s, B^t\right]}{s!t!},$$

and this concludes the proof. ∎

We can obtain nice corollaries related to this theorem by using some special A and B.

Corollary 2.4. *Let A and B be n-square matrices such that $A^2 = 0$, $B^2 = 0$. Then*

$$\left[e^A, e^B\right] = [A, B].$$

Proof. This result is an immediate consequence of Theorem 2.3. Because $A^2 = 0$, $A^n = 0$ for $n \geq 2$ for any square matrix, the Taylor expansion of e^A is equal to $I + A$. Thus,

$$\left[e^A, e^B\right] = e^A e^B - e^B e^A$$

$$\left[e^A, e^B\right] = (I + A)(I + B) - (I + B)(I + A)$$

$$\left[e^A, e^B\right] = AB - BA = [A, B].$$

Corollary 2.5. *Let A and B be n-square matrices such that $A^2 = I$, $B^2 = I$. Then*

$$\left[e^A, e^B\right] = \sum_{s=1}^{\infty} \frac{[A, B]}{(2s-1)!(2t-1)!}.$$

Proof. In this case, it is obvious that $A^n = A$, $B^n = B$ for odd n and $A^n = I$, $B^n = I$ for even n because $A^2 = I$ and $B^2 = I$. So, the proof can easily be concluded using the Taylor expansions of e^A and e^B. ∎

Our next two results are related to $\left[e^A, e^B\right]$, where A and B both commute with their commutator. There are some studies related to exponential of matrices which commute with their commutator in the literature. One of them is as follows.

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Lemma 2.6. [6] *If $A, B \in M_n(\mathbb{C})$ commute with their commutator, then*

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Theorem 2.7. *Let A and B be complex n-square matrices which commute with their commutator, then*

$$\left[e^A, e^B\right] = 2e^{A+B}\sinh\left(\frac{[A,B]}{2}\right).$$

Proof. Let A and B commute with their commutator, then

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Since $[A,B]$=$-$ $[B,A]$, $e^B e^A = e^{A+B-\frac{1}{2}[A,B]}$,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

$$e^B e^A = e^{A+B-\frac{1}{2}[A,B]}$$

$$e^A e^B - e^B e^A = e^{A+B+\frac{1}{2}[A,B]} - e^{A+B-\frac{1}{2}[A,B]}.$$

It is clear that $A+B$ commutes with $\pm\frac{1}{2}[A,B]$ because A and B commute with their commutator. So,

$$e^{A+B\pm\frac{1}{2}[A,B]} = e^{A+B}e^{\pm\frac{1}{2}[A,B]}$$

$$\left[e^A, e^B\right] = e^{A+B}e^{\frac{1}{2}[A,B]} - e^{A+B}e^{-\frac{1}{2}[A,B]}$$

$$\left[e^A, e^B\right] = e^{A+B}\left(e^{\frac{1}{2}[A,B]} - e^{-\frac{1}{2}[A,B]}\right).$$

Since $e^{\frac{1}{2}[A,B]} - e^{-\frac{1}{2}[A,B]}$ can be written as $\sinh\left(\frac{[A,B]}{2}\right)$ in view of $\sinh(x) = \frac{e^x - e^{-x}}{2}$,

$$\left[e^A, e^B\right] = 2e^{A+B}\sinh\left(\frac{[A,B]}{2}\right).$$

Theorem 2.8. *Let the complex n-square matrices A, B commute with their commutator, Then*

$$\left[e^A, e^B\right] = \sum_{s,t=1}^{\infty}\sum_{r=0}^{s-1}\frac{A^{s-r-1}B^{t-1}A^r}{s!\,(t-1)!}\,[A,B].$$

Proof. We have

$$\begin{aligned}\left[A^s, B^t\right] &= A^{s-1}\left[A, B^t\right] + \left[A^{s-1}, B^t\right]A \\ &= A^{s-1}\left[A, B^t\right] + \left(A^{s-2}\left[A, B^t\right] + \left[A^{s-2}, B^t\right]A\right)A \\ &= A^{s-1}\left[A, B^t\right]A^0 + A^{s-2}\left[A, B^t\right]A + \left[A^{s-2}, B^t\right]A^2.\end{aligned}$$

If we continue reducing the power of A in the commutator, we get

$$\left[A^s, B^t\right] = \sum_{s,t=1}^{\infty}\sum_{r=0}^{s-1}A^{s-r-1}\left[A, B^t\right]A^r.$$

On the other hand, we know that $\left[A, B^t\right] = t\,[A,B]\,B^{t-1}$ when $[B,[A,B]] = 0$. So,

$$\left[A^s, B^t\right] = \sum_{s,t=1}^{\infty}\sum_{r=0}^{s-1}tA^{s-r-1}B^{t-1}A^r\,[A,B],$$

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because of A and B commute with their commutator. Applying this equality to $\left[e^A, e^B\right]$, we get

$$\left[e^A, e^B\right] = \sum_{s,t=1}^{\infty} \frac{\left[A^s, B^t\right]}{s!t!},$$

$$\left[e^A, e^B\right] = \sum_{s,t=1}^{\infty} \sum_{r=0}^{s-1} \frac{A^{s-r-1} B^{t-1} A^r}{s!\,(t-1)!} \left[A, B\right].$$

This conclues the proof. ∎

2.2. Norm Inequalities. In this section we will present our results related to the norm inequalities of $A, B \in M_2(\mathbb{C})$ such as $\|[f(A), B]\|_F \leq c \|A\|_F \|B\|_F$ and $\|[f(A), f(B)]\|_F \leq k \|A\|_F \|B\|_F$ in the cases where their eigenvalues are real and complex numbers respectively.

LEMMA 2.9. *Putzer's Spectral Formula for 2×2 matrices.*

- *If the 2×2 matrix A has real distinct eigenvalues λ_1 and λ_2, then*

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(A - \lambda_1 I\right).$$

- *If the 2×2 matrix A has one double real eigenvalue λ_1, then*

$$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t} \left(A - \lambda_1 I\right).$$

- *If the 2×2 matrix A has complex eigenvalues $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$, then*

$$e^{At} = e^{at} \cos(bt) I + \frac{e^{at} \sin(bt)}{b} \left(A - aI\right).$$

THEOREM 2.10. *Let the 2×2 matrix A have real distinct eigenvalues λ_1 and λ_2. Then the following norm inequality holds, where $c = \sqrt{2} \left| \left(\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} \right) \right|$.*

$$\left\| \left[e^A, B\right] \right\|_F \leq c \|A\|_F \|B\|_F.$$

Proof. Let the 2×2 matrix A have real distinct eigenvalues. Then

$$\left[e^{At}, B\right] = \left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(A - \lambda_1 I\right) \right) B - B \left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(A - \lambda_1 I\right) \right)$$

$$\left[e^{At}, B\right] = e^{\lambda_1 t} B + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(A - \lambda_1 I\right) B - \left(e^{\lambda_1 t} B + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} B \left(A - \lambda_1 I\right) \right)$$

$$\left[e^{At}, B\right] = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(\left(A - \lambda_1 I\right) B - B \left(A - \lambda_1 I\right) \right)$$

$$\left[e^{At}, B\right] = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left[A, B\right]$$

$$\left\| \left[e^{At}, B\right] \right\|_F = \left| \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right| \left\| [A, B] \right\|_F.$$

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If we combine this with (1.1), we get

$$\left\| \left[e^{At}, B\right] \right\|_F \leq \underbrace{\sqrt{2}\left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right|}_{c} \|A\|_F \|B\|_F .$$

The proof can be concluded by setting $t = 1$.

$$\left\| \left[e^{A}, B\right] \right\|_F \leq \underbrace{\sqrt{2}\left|\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2}\right|}_{c} \|A\|_F \|B\|_F . \qquad \square$$

THEOREM 2.11. *Let the 2×2 matrix A have a double real eigenvalue λ. Then the following norm inequality holds, where $c = \sqrt{2}e^{\lambda}$.*

$$\left\| \left[e^{A}, B\right] \right\|_F \leq c \|A\|_F \|B\|_F .$$

Proof. We have

$$\left[e^{At}, B\right] = \left(e^{\lambda t} I + t e^{\lambda t} (A - \lambda I)\right) B - B \left(e^{\lambda t} I + t e^{\lambda t} (A - \lambda I)\right)$$

$$\left[e^{At}, B\right] = \left(e^{\lambda t} B + t e^{\lambda t} (A - \lambda I) B\right) - \left(e^{\lambda_1 t} B + t e^{\lambda t} (A - \lambda I)\right)$$

$$\left[e^{At}, B\right] = t e^{\lambda t} \left((A - \lambda I) B - B (A - \lambda I)\right)$$

$$\left[e^{At}, B\right] = t e^{\lambda t} [A, B]$$

$$\left\| \left[e^{At}, B\right] \right\|_F = e^{\lambda t} |t| \, \|[A, B]\|_F .$$

If we combine this with (1.1), we get

$$\left\| \left[e^{At}, B\right] \right\|_F \leq \underbrace{\sqrt{2} e^{\lambda t} |t|}_{c} \|A\|_F \|B\|_F .$$

The proof can be concluded by setting $t = 1$.

$$\left\| \left[e^{A}, B\right] \right\|_F \leq \underbrace{\sqrt{2} e^{\lambda}}_{c} \|A\|_F \|B\|_F . \qquad \square$$

THEOREM 2.12. *Let $A, B \in M_2(\mathbb{C})$ and A have complex eigenvalues $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$. Then the following norm inequality holds, where $c = \sqrt{2}e^{a}\left|\frac{\sin(b)}{b}\right|$*

$$\left\| \left[e^{A}, B\right] \right\|_F \leq c \|A\|_F \|B\|_F .$$

Proof. Let the 2×2 matrix A have complex eigenvalues $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$. Then

$$\left[e^{At}, B\right] = \left(e^{at}\cos(bt) I + \frac{e^{at}\sin(bt)}{b}(A - aI)\right) B - B\left(e^{at}\cos(bt) I + \frac{e^{at}\sin(bt)}{b}(A - aI)\right),$$

$$\left[e^{At}, B\right] = e^{at}\cos(bt) B + \frac{e^{at}\sin(bt)}{b}(A - aI) B - B e^{at}\cos(bt) - \frac{e^{at}\sin(bt)}{b} B (A - aI),$$

$$\left[e^{At}, B\right] = \frac{e^{at}\sin(bt)}{b}\left[(A - aI), B\right] = \frac{e^{at}\sin(bt)}{b}[A, B],$$

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$$\left\| \left[e^{At}, B \right] \right\|_F = e^{at} \left| \frac{\sin(bt)}{b} \right| \left\| [A,B] \right\|_F .$$

If we combine this with (1.1), we get

$$\left\| \left[e^{At}, B \right] \right\|_F \leq \underbrace{\sqrt{2} e^{at} \left| \frac{\sin(bt)}{b} \right|}_{c} \left\| A \right\|_F \left\| B \right\|_F .$$

The proof can be concluded by setting $t = 1$.

$$\left\| \left[e^{A}, B \right] \right\|_F \leq \underbrace{\sqrt{2} e^{a} \left| \frac{\sin(b)}{b} \right|}_{c} \left\| A \right\|_F \left\| B \right\|_F . \qquad \square$$

Theorem 2.13. *Let the 2×2 matrices A and B have real distinct eigenvalues. Then the following norm inequality holds*

$$\left\| \left[e^{A}, e^{B} \right] \right\|_F \leq k \left\| A \right\|_F \left\| B \right\|_F ,$$

where $k = \sqrt{2} \left| \left(\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} \right) \left(\frac{e^{\mu_1} - e^{\mu_2}}{\mu_1 - \mu_2} \right) \right|$, *$\lambda$'s and μ's are eigenvalues of A and B respectively.*

Proof. We have

$$e^{At} e^{Bs} = \left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \left(A - \lambda_1 I \right) \right) \left(e^{\mu_1 s} I + \frac{e^{\mu_1 s} - e^{\mu_2 s}}{\mu_1 - \mu_2} \left(B - \mu_1 I \right) \right).$$

Let $\frac{e^{\lambda_1 t} - e^{\lambda_2 s}}{\lambda_1 - \lambda_2} = x$ and $\frac{e^{\mu_1 s} - e^{\mu_2 s}}{\mu_1 - \mu_2} = y$, where x and y are real numbers. Then we get

$$e^{At} e^{Bs} = e^{\lambda_1 t + \mu_1 s} + e^{\lambda_1 t} y \left(B - \mu_1 I \right) + e^{\mu_1 s} x \left(A - \lambda_1 I \right) + xy \left(A - \lambda_1 I \right) \left(B - \mu_1 I \right),$$

$$e^{Bs} e^{At} = e^{\lambda_1 t + \mu_1 s} + e^{\mu_1 s} x \left(A - \lambda_1 I \right) + e^{\lambda_1 t} y \left(B - \mu_1 I \right) + xy \left(B - \mu_1 I \right) \left(A - \lambda_1 I \right),$$

$$\left[e^{At}, e^{Bs} \right] = xy \left(\left(A - \lambda_1 I \right) \left(B - \mu_1 I \right) - \left(B - \mu_1 I \right) \left(A - \lambda_1 I \right) \right),$$

$$\left[e^{At}, e^{Bs} \right] = xy \left[\left(A - \lambda_1 I \right), \left(B - \mu_1 I \right) \right] = xy \left[A, B \right],$$

$$\left\| \left[e^{At}, e^{Bs} \right] \right\|_F = \left| xy \right| \left\| [A,B] \right\|_F .$$

If we combine this with (1.1), we get

$$\left\| \left[e^{At}, e^{Bs} \right] \right\|_F \leq \sqrt{2} \left| xy \right| \left\| A \right\|_F \left\| B \right\|_F ,$$

$$\left\| \left[e^{At}, e^{Bs} \right] \right\|_F \leq \underbrace{\sqrt{2} \left| \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \left(\frac{e^{\mu_1 s} - e^{\mu_2 s}}{\mu_1 - \mu_2} \right) \right|}_{k} \left\| A \right\|_F \left\| B \right\|_F .$$

The proof can be concluded by setting $s, t = 1$.

$$\left\| \left[e^{A}, e^{B} \right] \right\|_F \leq \underbrace{\sqrt{2} \left| \left(\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} \right) \left(\frac{e^{\mu_1} - e^{\mu_2}}{\mu_1 - \mu_2} \right) \right|}_{k} \left\| A \right\|_F \left\| B \right\|_F . \qquad \square$$

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THEOREM 2.14. *Let the 2×2 matrices A and B have real double eigenvalues, λ and μ, respectively. Then the following norm inequality holds*

$$\left\| \left[e^{A}, e^{B}\right] \right\|_{F} \leq k \left\| A \right\|_{F} \left\| B \right\|_{F},$$

where $k = \sqrt{2}e^{\lambda+\mu}$.

Proof. According to the hypothesis we have

$$e^{At}e^{Bs} = \left(e^{\lambda t}I + te^{\lambda t}\left(A - \lambda I\right)\right)\left(e^{\mu s}I + se^{\mu s}\left(B - \mu I\right)\right),$$

$$e^{At}e^{Bs} = e^{\lambda t+\mu s}I + se^{\lambda t+\mu s}\left(B - \mu I\right) + te^{\lambda t+\mu s}\left(A - \lambda I\right) + ste^{\lambda t+\mu s}\left(A - \lambda I\right)\left(B - \mu I\right),$$

$$e^{Bs}e^{At} = e^{\mu s+\lambda t}I + te^{\mu s+\lambda t}\left(A - \lambda I\right) + se^{\mu s+\lambda t+}\left(B - \mu I\right) + ste^{\mu s+\lambda t}\left(B - \mu I\right)\left(A - \lambda I\right),$$

$$\left[e^{At}, e^{Bs}\right] = e^{At}e^{Bs} - e^{Bs}e^{At} = ste^{\lambda t+\mu s}\left(\left(A - \lambda I\right)\left(B - \mu I\right) - \left(B - \mu I\right)\left(A - \lambda I\right)\right),$$

$$\left[e^{At}, e^{Bs}\right] = ste^{\lambda t+\mu s}\left[\left(A - \lambda I\right), \left(B - \mu I\right)\right],$$

$$\left[e^{At}, e^{Bs}\right] = ste^{\lambda t+\mu s}\left[A, B\right],$$

$$\left\| \left[e^{At}, e^{Bs}\right] \right\|_{F} = \left|st\right| e^{\lambda t+\mu s} \left\| \left[A, B\right] \right\|_{F}.$$

If we combine this with (1.1) we get,

$$\left\| \left[e^{At}, e^{Bs}\right] \right\|_{F} \leq \underbrace{\sqrt{2}e^{\lambda t+\mu s}\left|st\right|}_{k} \left\| A \right\|_{F} \left\| B \right\|_{F}.$$

The proof follows by setting $s, t = 1$.

$$\left\| \left[e^{A}, e^{B}\right] \right\|_{F} \leq \underbrace{\sqrt{2}e^{\lambda+\mu}}_{k} \left\| A \right\|_{F} \left\| B \right\|_{F}.$$

□

THEOREM 2.15. *Let the 2×2 matrices A and B have complex eigenvalues $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$ and $\mu_1 = \overline{\mu_2} = c + id$, $d > 0$, respectively. Then, the following norm inequality holds*

$$\left\| \left[e^{A}, e^{B}\right] \right\|_{F} \leq k| \left\| A \right\|_{F} \left\| B \right\|_{F},$$

where $k = \sqrt{2}e^{a+c}\left|\frac{\sin(b)\sin(d)}{bd}\right|$.

Proof. Expressing e^{At} and e^{Bs} using Putzer's Spectral Formula for 2×2 matrices with complex eigenvalues and then computing $e^{At}e^{Bs} - e^{Bs}e^{At}$ as in the previous theorems, the result follows. □

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