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APPLICATION OF JORDAN ALGEBRA FOR TESTING HYPOTHESES ABOUT STRUCTURE OF MEAN VECTOR IN MODEL WITH BLOCK COMPOUND SYMMETRIC COVARIANCE STRUCTURE*

ROMAN ZMYŚLONY†, IVAN ŽEŽULA‡, AND ARKADIUSZ KOZIOŁ†

Abstract. In this article authors derive test for structure of mean vector in model with block compound symmetric covariance structure for two-level multivariate observations. One possible structure is so called structured mean vector when its components remain constant over sites or over time points, so that mean vector is of the form $\mathbf{1}_u \otimes \boldsymbol{\mu}$ with $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)' \in \mathbb{R}^m$. This hypothesis is tested against alternative of unstructured mean vector, which can change over sites or over time points.

Key words. Best unbiased estimator, testing structured mean vector, blocked compound symmetric covariance structure, doubly multivariate data, coordinate free approach, unstructured mean vector.

AMS subject classifications. 62J10, 62F03 62F05, 62H15.

1. Introduction.

This article deals with testing the hypothesis of so called structured mean vector based on the best unbiased estimator (BUE) for covariance parameters and mean vector ([13], [7] and [19]). Arnold considered some testing problems in multivariate data with block compound symmetry (BCS) covariance structure. He proposed using some orthogonal transformation for data to solve the problem of testing the hypothesis, which led to test based on statistics distributed as a product of independent beta-variates. Another contribution was made also by Arnold, who proposed a general method of testing of certain class of models applicable also to BCS structure, see [2]. Overview of all previous results was given by Szatrowski in [18]. Problem of testing hypotheses about mean vector in model with BCS covariance structure was considered among others by Szatrowski [17] and Roy [11]. Some testing problems in models with special block structures were considered by Fleiss [4] and Arnold [1]. Fleiss derived likelihood ratio test (LRT) for testing the hypothesis about structured mean vector. In this paper we deal with testing the above mentioned hypothesis using Jordan Algebra properties and we construct test based on best quadratic unbiased estimators (BQUE). Changing linear function of mean vector in null hypothesis into equivalent quadratic function of mean parameters, we show that both hypotheses are equivalent. Applying idea of positive and negative part of quadratic estimators, given by [10], after an orthogonal transformation we get the test statistic which has F distribution under the null hypothesis.

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†Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Szafrana 4a, 65-516 Zielona Góra, Poland (r.zmyslony@wmie.uz.zgora.pl, a.koziol@wmie.uz.zgora.pl).

‡Institute of Mathematics, Faculty of Science, P. J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia (ivan.zezula@upjs.sk).

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2. Doubly exchangeable covariance structure. The $(mu \times mu)-$dimensional BCS covariance structure is defined as

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_0 & \boldsymbol{\Gamma}_1 & \dots & \boldsymbol{\Gamma}_1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Gamma}_1 & \boldsymbol{\Gamma}_1 & \dots & \boldsymbol{\Gamma}_0 \end{bmatrix}.$$

It means that formula for $\boldsymbol{\Gamma}$ can be represented as

$$\boldsymbol{\Gamma} = \boldsymbol{I}_u \otimes (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) + \boldsymbol{J}_u \otimes \boldsymbol{\Gamma}_1, \tag{2.1}$$

where $\boldsymbol{I}_u$ is the $u \times u$ identity matrix, $\mathbf{1}_u$ is a $u \times 1$ vector of ones, $\boldsymbol{J}_u = \mathbf{1}_u \mathbf{1}_u'$ and $\otimes$ represents the Kronecker product. The above BCS structure $\boldsymbol{\Gamma}$ can equivalently be written as follows

$$\boldsymbol{\Gamma} = \boldsymbol{I}_u \otimes \boldsymbol{\Gamma}_0 + (\boldsymbol{J}_u - \boldsymbol{I}_u) \otimes \boldsymbol{\Gamma}_1, \tag{2.2}$$

or written as a sum of two strong orthogonal matrices (i.e. the product of such matrices is equal to the matrix $\mathbf{0}$)

$$\boldsymbol{\Gamma} = \left(\boldsymbol{I}_u - \frac{1}{u} \boldsymbol{J}_u \right) \otimes (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) + \frac{1}{u} \boldsymbol{J}_u \otimes (\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1). \tag{2.3}$$

Note that the previous representation of $\boldsymbol{\Gamma}$ (using rank argument and strong orthogonality) implies the following Proposition:

Proposition 2.1. *The matrix $\boldsymbol{\Gamma}$ is positive definite if and only if $\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1$ are positive definite.*

For another proof of above fact see Lemma 2.1 in [12].

3. Model with unstructured mean vector. The BCS model can be written in the following way:

$$\underset{num \times 1}{\boldsymbol{y}} = \mathrm{vec}(\underset{um \times n}{\boldsymbol{Y}}) = \mathrm{vec}\,[\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_n] \sim N\big((\mathbf{1}_n \otimes \boldsymbol{I}_{um})\boldsymbol{\mu}, \boldsymbol{I}_n \otimes \boldsymbol{\Gamma}_{um}\big). \tag{3.4}$$

It means that matrix $\boldsymbol{Y}$ contains n independent normally distributed random column vectors which are identically distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Gamma}$.

We want to make use of coordinate-free approach. That is why we use operator $\odot$ instead of Kronecker product ($\otimes$). It can be defined in the following way:

Definition 3.1. *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices with such dimensions that multiplication $\boldsymbol{ACB}$ is possible. Then:*

$$(\boldsymbol{A} \odot \boldsymbol{B})\boldsymbol{C} = \boldsymbol{ACB}.$$

$\boldsymbol{A} \odot \boldsymbol{B}$ is well-defined operator. If we consider space of all $u \times m$ matrices with inner product $(\boldsymbol{U}, \boldsymbol{V}) = \mathrm{tr}(\boldsymbol{UV}')$ then covariance operator $\boldsymbol{A} \odot \boldsymbol{B}$ is self adjoint linear operator which transform $u \times m$ matrices into matrices with the same size such that $Var((\boldsymbol{C}, \boldsymbol{Y}), (\boldsymbol{D}, \boldsymbol{Y})) = (\boldsymbol{C}, \boldsymbol{ADB})\ \forall \boldsymbol{C}, \boldsymbol{D}$ (see [3] and [8]).

In the next part of the paper we deal with the following operator vec^{-1}. For completeness we remind definition of operator vec.

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Definition 3.2. *Let $\boldsymbol{X}$ be a matrix of size $k \times l$. The vectorization of $\boldsymbol{X}$, denoted by* $\operatorname{vec}(\boldsymbol{X})$, *is transformation which converts $\boldsymbol{X}$ into a $kl \times 1$ column vector by stacking the column vectors of $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l]$ on top of one another*

$$\operatorname{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_l \end{bmatrix}.$$

Definition 3.3. *Let $\boldsymbol{x}$ be a column vector of size $m \times 1$, where $m = k \cdot l$ and $k, l \in \mathbb{N}$. For $\boldsymbol{x}$ the inverse transformation of the vectorization operator, denoted by $\operatorname{vec}_k^{-1}(\boldsymbol{x})$, is transformation which converts column vector $\boldsymbol{x}$ into a matrix of dimensions $k \times l$*

$$\operatorname{vec}_k^{-1}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_{k+1} & \ldots & \boldsymbol{x}_{k(l-1)+1} \\ \boldsymbol{x}_2 & \boldsymbol{x}_{k+2} & \ldots & \boldsymbol{x}_{k(l-1)+2} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{x}_k & \boldsymbol{x}_{2k} & \ldots & \boldsymbol{x}_{kl} \end{bmatrix}.$$

Remark 3.4. *Let $\boldsymbol{Y}$ be a random matrix of size $k \times l$. Operator $\odot$ has a following properties:*

- $(\boldsymbol{A} \otimes \boldsymbol{B}) \operatorname{vec}(\boldsymbol{Y}) = \operatorname{vec}\big((\boldsymbol{B}' \odot \boldsymbol{A})\boldsymbol{Y}\big)$;
- $\operatorname{vec}_k^{-1}((\boldsymbol{A} \otimes \boldsymbol{B}) \operatorname{vec}(\boldsymbol{Y})) = (\boldsymbol{B}' \odot \boldsymbol{A})\boldsymbol{Y}$;
- $(\boldsymbol{A} \odot \boldsymbol{B})(\boldsymbol{C} \odot \boldsymbol{D}) = \boldsymbol{AC} \odot \boldsymbol{DB}$.

Remark 3.5. *From the first property in Remark 3.4 it follows that for model with unstructured mean vector:*

$$E(\boldsymbol{y}) = (\boldsymbol{1}_n \otimes \boldsymbol{I}_{um})\boldsymbol{\mu} \Rightarrow E(\boldsymbol{Y}) = (\boldsymbol{I}_{um} \odot \boldsymbol{1}_n')\boldsymbol{\mu} = \boldsymbol{I}\boldsymbol{\mu}\boldsymbol{1}'.$$
$$Var(\boldsymbol{Y}) = \boldsymbol{\Gamma}_{um} \odot \boldsymbol{I}_n.$$

Now we rewrite model's structure using this operator:

$$\underset{um \times n}{\boldsymbol{Y}} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n] \sim N\big((\boldsymbol{I}_{um} \odot \boldsymbol{1}_n')\boldsymbol{\mu}, \boldsymbol{\Gamma}_{um} \odot \boldsymbol{I}_n\big).$$

Let us consider transformation $\boldsymbol{I}_{um} \odot \boldsymbol{Q}_n$ on $\boldsymbol{Y}_{um \times n}$ where $\boldsymbol{Q}$ is an orthogonal matrix, i.e. $\boldsymbol{QQ'} = \boldsymbol{Q'Q} = \boldsymbol{I}$.

Proposition 3.6. *If $Var(\boldsymbol{Y}) = \boldsymbol{\Sigma} \odot \boldsymbol{I}$ with any covariance matrix $\boldsymbol{\Sigma}$ then this covariance matrix is invariant with respect to transformation $\boldsymbol{I} \odot \boldsymbol{Q}$.*

Proof. Let $\boldsymbol{U} = (\boldsymbol{I} \odot \boldsymbol{Q})\boldsymbol{Y}$. Then

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{U}} &= Var((\boldsymbol{I} \odot \boldsymbol{Q})\boldsymbol{Y}) = (\boldsymbol{I} \odot \boldsymbol{Q})Var(\boldsymbol{Y})(\boldsymbol{I} \odot \boldsymbol{Q})' \\ &= (\boldsymbol{I} \odot \boldsymbol{Q})(\boldsymbol{\Sigma} \odot \boldsymbol{I})(\boldsymbol{I} \odot \boldsymbol{Q}') = (\boldsymbol{I} \odot \boldsymbol{Q})(\boldsymbol{\Sigma} \odot \boldsymbol{Q}') \\ &= \boldsymbol{\Sigma} \odot \boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{\Sigma} \odot \boldsymbol{I}. \qquad \square \end{aligned}$$

Proposition 3.7. *Let $\boldsymbol{\vartheta}_{\boldsymbol{\Sigma}_{\boldsymbol{Y}}}$ be the space generated by covariance matrices $\boldsymbol{\Sigma}$ and let $\boldsymbol{P}_{E(\boldsymbol{Y})}$ denote orthogonal projector on the subspace of mean matrix of a random matrix $\boldsymbol{Y}$. Moreover let $\boldsymbol{U} = \boldsymbol{Q}(\boldsymbol{Y})$, where $\boldsymbol{Q}$ is an arbitrary orthogonal operator. Then we have*

$$\text{If } \boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \boldsymbol{\Sigma}_{\boldsymbol{Y}}\boldsymbol{P}_{E(\boldsymbol{Y})} \text{ then } \boldsymbol{P}_{E(\boldsymbol{U})}\boldsymbol{\Sigma}_{\boldsymbol{U}} = \boldsymbol{\Sigma}_{\boldsymbol{U}}\boldsymbol{P}_{E(\boldsymbol{U})}. \tag{3.5}$$

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(3.6) *If $\boldsymbol{\vartheta}_{\boldsymbol{\Sigma_Y}}$ is a quadratic subspace then $\boldsymbol{\vartheta}_{\boldsymbol{\Sigma_U}}$ is also a quadratic subspace.*

Proof. It is easy to prove that the orthogonal projector on the subspace of mean random matrix $\boldsymbol{U}$ is $\boldsymbol{P}_{E(\boldsymbol{U})} = \boldsymbol{Q}\boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{Q}'$. Moreover, one can show that $\boldsymbol{\Sigma_U} = \boldsymbol{Q}\boldsymbol{\Sigma_Y}\boldsymbol{Q}'$. Then it holds $\boldsymbol{P}_{E(\boldsymbol{U})}\boldsymbol{\Sigma_U} = \boldsymbol{Q}\boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{\Sigma_Y}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{\Sigma_Y}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{\Sigma_Y}\boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{\Sigma_Y}\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{P}_{E(\boldsymbol{Y})}\boldsymbol{Q}' = \boldsymbol{\Sigma_U}\boldsymbol{P}_{(\boldsymbol{U})}$ which implies (3.5). To prove (3.6) using $\boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{I}$ note that $(\boldsymbol{\Sigma_U})^2 = \boldsymbol{Q}\left(\boldsymbol{\Sigma_Y}\right)^2\boldsymbol{Q}'$. Since $\boldsymbol{\vartheta}_{\boldsymbol{\Sigma_Y}}$ is a quadratic subspace then $\boldsymbol{\vartheta}_{\boldsymbol{\Sigma_U}}$ is also a quadratic subspace. ∎

For the special case of $\boldsymbol{Q} = \boldsymbol{Q}_1 \odot \boldsymbol{Q}_2$ we get the following:

Lemma 3.8. *Since the space $\boldsymbol{\vartheta}_{Var(\boldsymbol{Y})}$ generated by covariance matrices $\boldsymbol{\Gamma} \odot \boldsymbol{I}$ is a quadratic subspace and orthogonal projector $\boldsymbol{P}_{E(\boldsymbol{Y})} = \boldsymbol{I}_{um} \odot \frac{1}{n}\boldsymbol{J}_n$ commutes with covariance matrices, we have*

$$\boldsymbol{P}_{E(\boldsymbol{U})}Var(\boldsymbol{U}) = Var(\boldsymbol{U})\boldsymbol{P}_{E(\boldsymbol{U})} \text{ and } \boldsymbol{\vartheta}_{Var(\boldsymbol{U})} \text{ is also a quadratic subspace.}$$

Remark 3.9. *The proof that for the model (3.4) $\boldsymbol{\vartheta}_{Var(\boldsymbol{Y})}$ is a quadratic subspace and assumption that commutativity of $\boldsymbol{P}_{E(\boldsymbol{Y})}$ holds see [13].*

Using Proposition 3.6, we can easily prove the following:

Lemma 3.10. *Let $\boldsymbol{U} = (\boldsymbol{I}_{um} \odot \boldsymbol{Q}_2)\boldsymbol{Y}$, where $\boldsymbol{Q}_2 = \left[\frac{1}{\sqrt{n}}\mathbf{1}_n \vdots \boldsymbol{K}_{\mathbf{1}_n}\right]$ is Helmert matrix, so that $\boldsymbol{K}'_{\mathbf{1}_n}\boldsymbol{K}_{\mathbf{1}_n} = \boldsymbol{I}_{n-1}$ and $\boldsymbol{K}'_{\mathbf{1}_n}\mathbf{1}_n = \mathbf{0}$. Then $\boldsymbol{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$ has independent column vectors, where*

$$\boldsymbol{u}_1 \sim N(\sqrt{n}\boldsymbol{\mu}, \boldsymbol{\Gamma}) \text{ and } \boldsymbol{u}_i \sim N(\mathbf{0}, \boldsymbol{\Gamma}) \text{ for } i = 2, \ldots n.$$

Proof. Since transformation $\boldsymbol{I}_{um} \odot \boldsymbol{Q}_2$ is linear, the matrix $\boldsymbol{U}$ is normally distributed with independent column vectors. In view of Preposition 3.6 the covariance structure is unchanged. It is clear from structure of $\boldsymbol{Q}_2$ that for $i = 2, \ldots, n$, $E(\boldsymbol{u}_i) = \sum_{j=1}^n \boldsymbol{k}_{ji-1}\boldsymbol{\mu} = \boldsymbol{\mu}\sum_{j=1}^n \boldsymbol{k}_{ji-1} = 0$, where $\boldsymbol{k}_{ji}$ is ji-th element of $\boldsymbol{K}_{\mathbf{1}_n}$. ∎

For convenience we will use operator vec^{-1} for vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ given in previous lemma. For each $\boldsymbol{u}_i$ with dimension $um \times 1$ we define matrix $\boldsymbol{U}_i$ of size $m \times u$ dividing vector $\boldsymbol{u}_i$ using vec_m^{-1} for column vector of dim $m \times 1$ i.e.

$$\boldsymbol{U}_i = \left[\boldsymbol{u}_1^{(i)}, \ldots, \boldsymbol{u}_u^{(i)}\right]$$

with distribution

$$\boldsymbol{U}_1 \sim N\left(\sqrt{n}\left[\boldsymbol{\mu}_1^{(1)}, \ldots, \boldsymbol{\mu}_u^{(1)}\right], (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) \odot (\boldsymbol{I}_u - \frac{1}{u}\boldsymbol{J}_u) + (\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1) \odot \frac{1}{u}\boldsymbol{J}_u\right),$$

$$\boldsymbol{U}_i \sim N\left(\mathbf{0}_{m\times u}, (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) \odot (\boldsymbol{I}_u - \frac{1}{u}\boldsymbol{J}_u) + (\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1) \odot \frac{1}{u}\boldsymbol{J}_u\right) \text{ for } i = 2, \ldots, n.$$

Now we use the same orthogonal mapping for each matrix $\boldsymbol{U}_i$ which according to the Proposition 3.7 saves the property of quadratic subspace generated by covariance structure. Let $\boldsymbol{W}_i = (\boldsymbol{I} \odot \boldsymbol{Q}_1)\boldsymbol{U}_i$, where $\boldsymbol{Q}_1 = \left[\frac{1}{\sqrt{u}}\mathbf{1}_u \vdots \boldsymbol{K}_{\mathbf{1}_u}\right]$. Each matrix $\boldsymbol{W}_i$ can be expressed as

$$\boldsymbol{W}_i = \left[\boldsymbol{w}_1^{(i)}, \ldots, \boldsymbol{w}_u^{(i)}\right],$$

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where $\boldsymbol{w}_j^{(i)}$ is $m \times 1$ vector. On can easily prove that

$$Var(\boldsymbol{W}_i) = (\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1) \odot \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \boldsymbol{I}_{u-1} \end{bmatrix} + (\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1) \odot \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \mathbf{0}_{u-1} \end{bmatrix}$$

so that we have the following:

COROLLARY 3.11. *Vectors $\boldsymbol{w}_j^{(i)}$ are independent and*

$$\boldsymbol{w}_1^{(1)} \sim N\left(\sqrt{nu}\sum_{j=1}^{u}\boldsymbol{\mu}_j, \boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1\right), \tag{3.7}$$

$$\boldsymbol{w}_1^{(i)} \sim N\left(\mathbf{0}, \boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1\right) \text{ for } i = 2, \ldots, n, \tag{3.8}$$

$$\boldsymbol{w}_j^{(1)} \sim N\left(\sqrt{nu}\sum_{l=1}^{u}\boldsymbol{k}_{l,j-1}\boldsymbol{\mu}_l, \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1\right) \text{ for } j = 2, \ldots, u, \tag{3.9}$$

where $\boldsymbol{k}_{lj}$ is lj-th element of $\boldsymbol{K}_{\mathbf{1}_u}$.

$$\boldsymbol{w}_j^{(i)} \sim N\left(\mathbf{0}, \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1\right) \text{ for } i = 2, \ldots, n, \ j = 2, \ldots, u. \tag{3.10}$$

REMARK 3.12. *According to full characterization of Jordan Algebra, note that covariance structure is isomorphic to Cartesian product of Jordan Algebra of $n(u-1)$ and n full $m \times m$ symmetric matrices $\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_0 + (u-1)\boldsymbol{\Gamma}_1$, respectively, see [6].*

Now we formulate null hypothesis for structure of mean

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \ldots = \boldsymbol{\mu}_u,$$

This hypothesis can be written equivalently as

$$H_0 : \boldsymbol{\mu}_2^{(c)} = \boldsymbol{\mu}_3^{(c)} = \ldots = \boldsymbol{\mu}_u^{(c)} = 0,$$

where $\boldsymbol{\mu}_j^{(c)} = \sqrt{nu}\sum_{l=1}^{u}\boldsymbol{k}_{l,j-1}\boldsymbol{\mu}_l$.
Following idea of [10] this hypothesis is equivalent

$$H_0 : \sum_{j=2}^{u}\boldsymbol{\mu}_j^{(c)}\boldsymbol{\mu}_j^{(c)'} = 0.$$

One can prove that quadratic estimator of $\sum_{j=2}^{u}\boldsymbol{\mu}_j^{(c)}\boldsymbol{\mu}_j^{(c)'}$ is a function of complete sufficient statistics (see [13]) and has the following form:

$$\widehat{\sum_{j=2}^{u}\boldsymbol{\mu}_j^{(c)}\boldsymbol{\mu}_j^{(c)'}} = \sum_{j=2}^{u}\widehat{\boldsymbol{\mu}}_j^{(c)}\widehat{\boldsymbol{\mu}}_j^{(c)'} - (u-1)\widehat{\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}}_1. \tag{3.11}$$

Note that

$$\sum_{j=2}^{u}\widehat{\boldsymbol{\mu}}_j^{(c)}\widehat{\boldsymbol{\mu}}_j^{(c)'} \stackrel{\text{df}}{=} (u-1)\widehat{\boldsymbol{\Delta}}_2 \tag{3.12}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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is positive part and

$$(u-1)\widehat{\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1} = \frac{u-1}{(n-1)(u-1)} \sum_{i=2}^{n} \sum_{j=2}^{u} w_j^{(i)} w_j^{(i)\prime} \stackrel{\text{df}}{=} (u-1)\widehat{\boldsymbol{\Delta}}_1 \tag{3.13}$$

is negative part of estimator in (3.11). Moreover, under null hypothesis positive part has Wishart distribution and negative part multiplied by $(n-1)$ is Wishart distributed with the same covariance matrix $\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1$.

Now we prove the following:

Lemma 3.13. *If $\boldsymbol{W}_1 \sim \mathcal{W}_m(\boldsymbol{\Sigma}, n_1)$ and $\boldsymbol{W}_2 \sim \mathcal{W}_m(\boldsymbol{\Sigma}, n_2)$ are independent, then for every fixed vector $\boldsymbol{x} \neq 0 \in \mathbb{R}^m$:*

$$T = \frac{n_2 \boldsymbol{x}'\boldsymbol{W}_1\boldsymbol{x}}{n_1 \boldsymbol{x}'\boldsymbol{W}_2\boldsymbol{x}} \sim F_{n_1,n_2}.$$

Proof. According to Theorem 3.4.2 in [9], if $\boldsymbol{W} \sim \mathcal{W}_m(\boldsymbol{\Sigma}, n)$ then for every $\boldsymbol{x} \neq \mathbf{0} \in \mathbb{R}^m$:

$$\frac{\boldsymbol{x}'\boldsymbol{W}\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}} \sim \chi^2_n.$$

Now if we calculate ratio of $\frac{\boldsymbol{x}'\boldsymbol{W}_1\boldsymbol{x}}{n_1}$ and $\frac{\boldsymbol{x}'\boldsymbol{W}_2\boldsymbol{x}}{n_2}$ we get:

$$\frac{\frac{\boldsymbol{x}'\boldsymbol{W}_1\boldsymbol{x}}{n_1}}{\frac{\boldsymbol{x}'\boldsymbol{W}_2\boldsymbol{x}}{n_2}} = \frac{\frac{\boldsymbol{x}'\boldsymbol{W}_1\boldsymbol{x}}{n_1\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}{\frac{\boldsymbol{x}'\boldsymbol{W}_2\boldsymbol{x}}{n_2\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \sim \frac{\frac{\chi^2_{n_1}}{n_1}}{\frac{\chi^2_{n_2}}{n_2}} \sim F_{n_1,n_2}. \qquad \square$$

Using Lemma 3.13 we get the following result:

Theorem 3.14. *Under null hypothesis test statistic*

$$T = \frac{\boldsymbol{x}' \sum_{j=2}^{u} \widehat{\boldsymbol{\mu}}_j^{(c)} \widehat{\boldsymbol{\mu}}_j^{(c)\prime} \boldsymbol{x}}{(u-1)\boldsymbol{x}'\widehat{\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1}\boldsymbol{x}} = \frac{\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_2\boldsymbol{x}}{\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_1\boldsymbol{x}} \tag{3.14}$$

has F distribution with $(u-1)$ and $(n-1)(u-1)$ degrees of freedom for any fixed $\boldsymbol{x}$.

From the above theorem we have the following

Corollary 3.15. *Since under alternative hypothesis expectation of $\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_2\boldsymbol{x}$ is bigger than expectation of $\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_1\boldsymbol{x}$, the null hypothesis is rejected if*

$$T > F_{\alpha,u-1,(n-1)(u-1)}.$$

4. Alternative tests.

4.1. Roy's test. One can ask what is the optimal choice of $\boldsymbol{x}$ in the previous test statistic. Since the distribution of

$$T = \frac{\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_2\boldsymbol{x}}{\boldsymbol{x}'\widehat{\boldsymbol{\Delta}}_1\boldsymbol{x}}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 33, pp. 41-52, August 2018.

is the same for any $\boldsymbol{x}$, we can look for higher values of T in order to get higher power of the test. Let us denote $\boldsymbol{y} = \widehat{\boldsymbol{\Delta}}_1^{1/2}\boldsymbol{x}$. This is a regular transformation, since we assume $\boldsymbol{\Delta}_1 > \mathbf{0}$. If the number of degrees of freedom is greater than the dimensionality, i.e. $(n-1)(u-1) > m$, then also $\widehat{\boldsymbol{\Delta}}_1 > \mathbf{0}$ with probability 1. That is why

$$T_m \overset{\text{df}}{=} \max_{\boldsymbol{x}} T = \max_{\boldsymbol{y}} \frac{\boldsymbol{y}'\widehat{\boldsymbol{\Delta}}_1^{-1/2}\widehat{\boldsymbol{\Delta}}_2\widehat{\boldsymbol{\Delta}}_1^{-1/2}\boldsymbol{y}}{\boldsymbol{y}'\boldsymbol{y}} = \lambda_{max}\left(\widehat{\boldsymbol{\Delta}}_1^{-1/2}\widehat{\boldsymbol{\Delta}}_2\widehat{\boldsymbol{\Delta}}_1^{-1/2}\right) = \lambda_{max}\left(\widehat{\boldsymbol{\Delta}}_2\widehat{\boldsymbol{\Delta}}_1^{-1}\right).$$

We know that under null hypothesis

$$(n-1)(u-1)\widehat{\boldsymbol{\Delta}}_1 \sim \mathcal{W}_m\left((n-1)(u-1), \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1\right),$$
$$(u-1)\widehat{\boldsymbol{\Delta}}_2 \sim \mathcal{W}_m\left(u-1, \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1\right),$$

where $\widehat{\boldsymbol{\Delta}}_1$ and $\widehat{\boldsymbol{\Delta}}_2$ are independent.

Using the Definition 3.7.2 and Equation 3.7.12 of [9], we can tell that the distribution of

$$R = \frac{\frac{1}{(n-1)}T_m}{1 + \frac{1}{(n-1)}T_m}$$

is Roy's largest root distribution with parameters m, $(n-1)(u-1)$, and $u-1$ if $n-1 > m$. Thus, the hypothesis can also be tested using critical values of Roy's distribution.

However, one has to bear in mind that the maximizing vector $\boldsymbol{x}$ is the eigenvector $\boldsymbol{u}_1$ corresponding to the largest eigenvalue, which is no more fixed but depends on the data. As a consequence, Roy's test does not necessarily have higher power than the F-test. Practical experience e.g. in MANOVA designs show that Roy's test performs better than other ones only when the largest eigenvalue is substantially greater than the remaining ones.

4.2. Likelihood ratio test. There is one more test we have to compare our test with - the likelihood ratio test. LRT is preferred by many statisticians for its optimal asymptotic properties. However, when the sample size is not high, properties of LRT can be far from the optimal ones. So that we again need some practical computational comparison.

LRT for this situation was developed by Fleiss in [4]. The test statistic is of the form

$$L = \frac{\left|\widehat{\boldsymbol{\Delta}}_1\right|}{\left|\widehat{\boldsymbol{\Delta}}_1 + \frac{1}{n}\widehat{\boldsymbol{\Delta}}_2\right|},$$

where $\frac{1}{n}\widehat{\boldsymbol{\Delta}}_2 = \frac{1}{n(u-1)}\sum_{j=2}^{u}\widehat{\boldsymbol{\mu}}_j^{(c)}\widehat{\boldsymbol{\mu}}_j^{(c)'} = \frac{1}{u-1}\overline{X}\left(I - \frac{1}{u}J_u\right)\overline{X}'$, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $X_i = \text{vec}^{-1}\,\boldsymbol{y}_i$. This statistic has under H_0 Wilks lambda distribution with parameters m, $u-1$, and $(n-1)(u-1)$ if $n-1 > m$ (compare with Definition 3.7.1 in [9]). We obtain critical values for both tests by 1 000 000 simulations using Monte Carlo method.

5. Simulation study. In our test statistic we take vector $\boldsymbol{x} = \mathbf{1}_m$, so we consider sum of elements of positive and negative part of estimator $\widehat{\sum_{j=2}^{u}\boldsymbol{\mu}_j^{(c)}\boldsymbol{\mu}_j^{(c)'}}$. Using argument of minimal sufficiency we need only to generate independently $\boldsymbol{w}_2^{(1)}, \ldots, \boldsymbol{w}_u^{(1)}$ according $N(\mathbf{0}_m, \boldsymbol{I}_m)$ and random matrix with Wishart distribution

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 33, pp. 41-52, August 2018.

$\mathcal{W}_m((n-1)(u-1), \boldsymbol{I}_m)$ because the test statistic is under null hypothesis independent of the choice of covariance structure. In each step of simulation we add randomly chosen vectors $\boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_u$ to the vectors $\boldsymbol{w}_j^{(1)}$ for $j = 2, \ldots, u$ multiplied by fixed λ to obtain power function of the test. Here λ is between 0 and some suitable value Λ, which is chosen empirically (using small number of simulation) such that power is close to 1. Naturally, for $\lambda = 0$ we have null hypothesis. When λ increases then power should also increase. We have compared powers of all three tests as a function of λ.

All three tests are functions of complete sufficient statistics (see [14], [15], [16], [20]). Because there is no uniformly most powerful test it is natural that those tests are not comparable with respect to their powers, i.e. any of them can be the most powerful in a specific case. However, we can conclude that in case $u = 2$ Roy's test is equivalent to LRT because maximum eigenvalue is only one that is greater than zero. Our simulation study has confirmed this assertion. For real data example (mineral contents in bones) taken from [5] on page 43 we calculated p-values for all three tests. For F test p-value is equal to 0.0363 and for LRT and Roy's test equals 0.1725, so that we make different conclusions on standard 5% level of significance. Figure 1 shows comparison of tests powers in the case when estimated parameters are the true ones. Figures 2 and 3 display superiority of F test over the other ones when all elements of the contrast vector are all positive or all negative. Figures also differ in sample size. Figure 4 illustrates the situation when components of contrast vector have different signs. In this case F test has the smallest power. The figures referenced above are included after the bibliography.

6. Conclusion. In paper we present F test which is a new alternative for testing the hypothesis of structured mean vector under BCS covariance structure. We compare it with other known tests of the hypothesis which can be found in the literature, and are used in practice. Simulations show that any of the three existing tests can have the largest power in a specific case.

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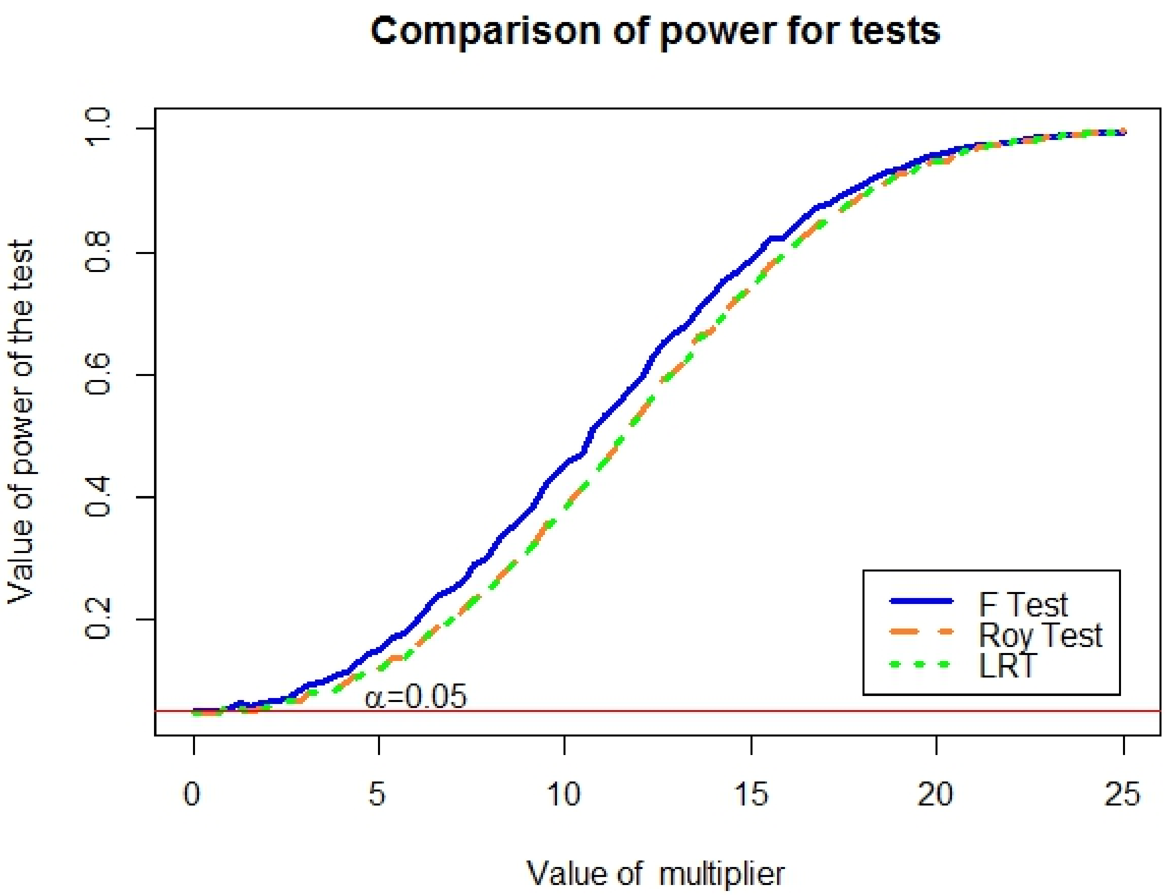

Figure 1. $n = 25, u = 2, m = 3$

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Volume 33, pp. 41-52, August 2018.

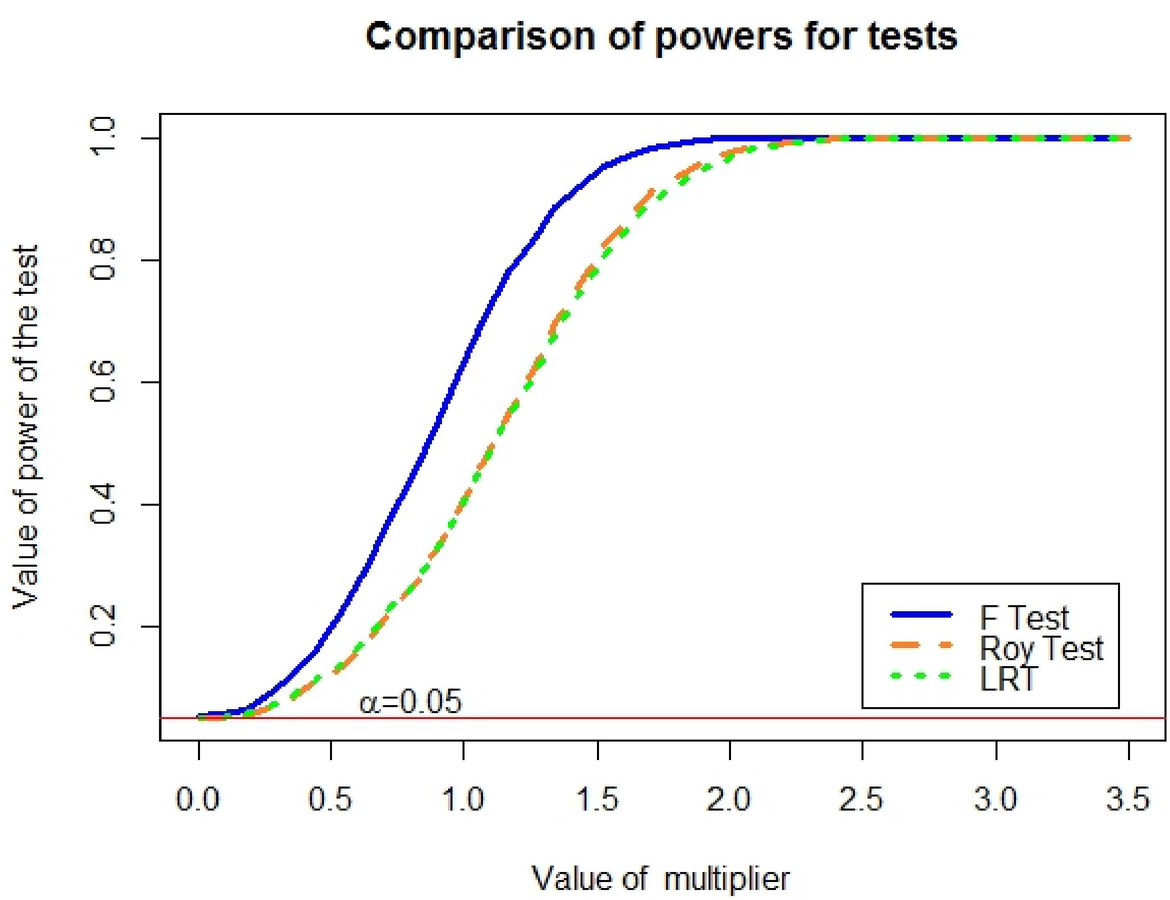

FIGURE 2. $n = 10, u = 3, m = 3$

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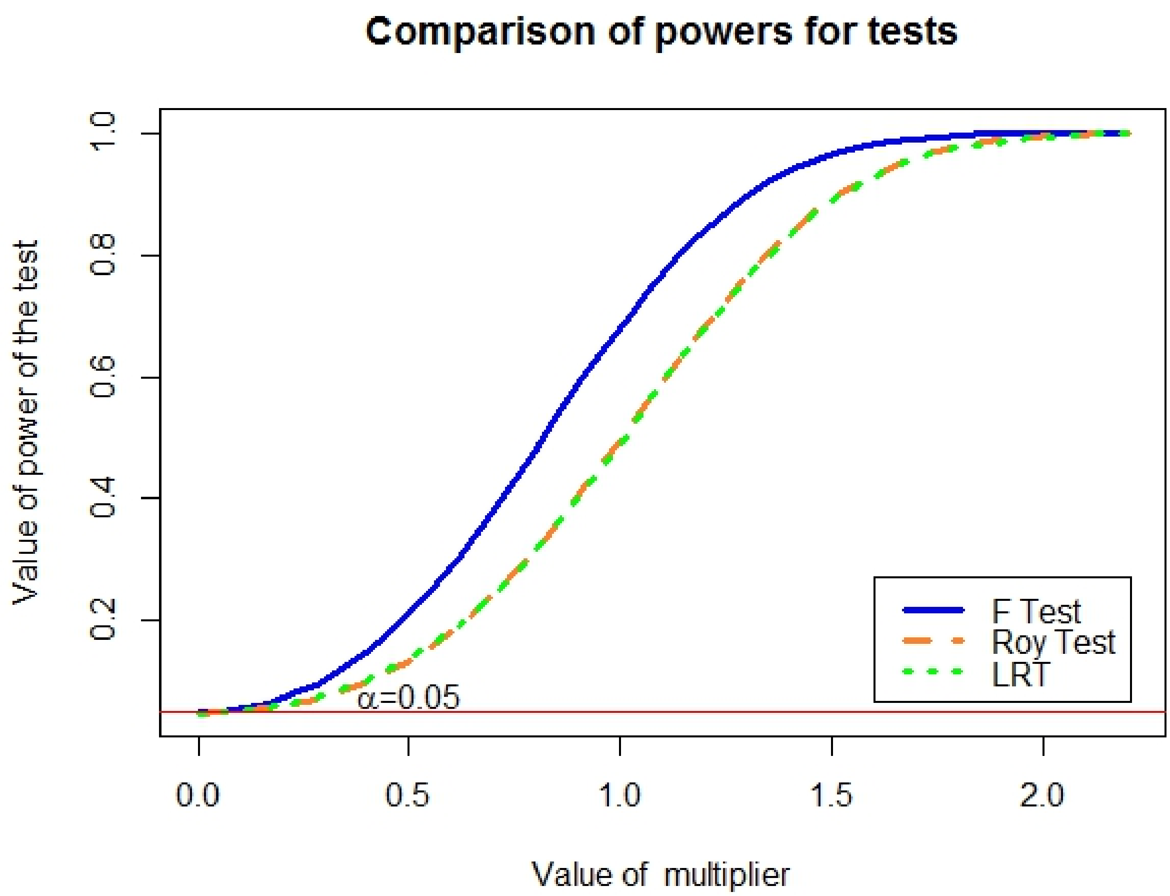

FIGURE 3. $n = 25, u = 3, m = 3$

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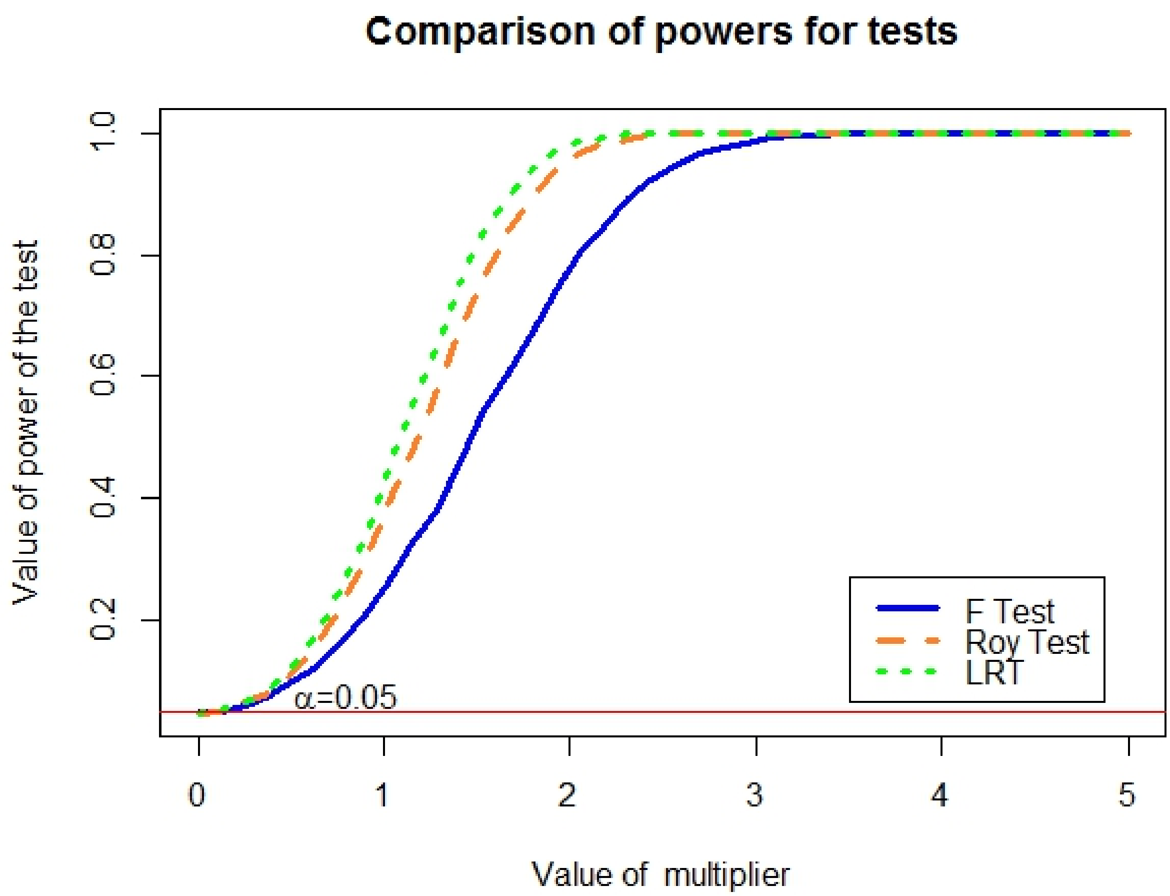

FIGURE 4. $n = 25, u = 3, m = 3$