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# THE PROPERTIES OF PARTIAL TRACE AND BLOCK TRACE OPERATORS OF PARTITIONED MATRICES*

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**Abstract.** The aim of this paper is to give the properties of two linear operators defined on non-square partitioned matrix: the partial trace operator and the block trace operator. The conditions for symmetry, nonnegativity, and positive-definiteness are given, as well as the relations between partial trace and block trace operators with standard trace, vectorizing and the Kronecker product operators.

Both partial trace as well as block trace operators can be widely used in statistics, for example in the estimation of unknown parameters under the multi-level multivariate models or in the theory of experiments for the determination of an optimal designs under the linear models.

**Key words.** Partial trace operator, Block trace operator, Block matrix.

**AMS subject classifications.** 15A15, 47B99.

**1. Introduction.** Throughout the paper we use "vec" operator which stacks the columns of the matrix one below another. The $i$-th column of an identity matrix $\mathbf{I}_u$ is denoted by $\mathbf{e}_{i,u}$, moreover recall that $\mathbf{I}_u = \sum_{i=1}^{u} \mathbf{e}_{i,u}\mathbf{e}'_{i,u}$ and $\operatorname{vec}\mathbf{I}_u = \sum_{i=1}^{u} \mathbf{e}_{i,u} \otimes \mathbf{e}_{i,u}$.

In this paper we consider two linear operators from the $mp \times np$ dimensional space $\mathcal{V}$ of matrices, into the $m \times n$ space $\mathcal{U}$ of matrices, defined on specific partitions of $\mathbf{A} = (\mathbf{A}_{ij}) \in \mathcal{V}$, where $\mathbf{A}_{ij}$ are either $p \times p$ or $m \times n$ blocks of $\mathbf{A}$.

Definition 1.1. *For an arbitrary matrix $\mathbf{A} = (\mathbf{A}_{ij}) : mp \times np$ the $m \times n$ **partial trace** operator, denoted by $\operatorname{PTr}_p \mathbf{A}$, and the $m \times n$ **block trace** operator, denoted by $\operatorname{BTr}_{m,n} \mathbf{A}$, can be defined in the following equivalent forms:*

**(i)** *the matrix of the traces of $p \times p$ blocks of $\mathbf{A}$, that is*

$$\operatorname{PTr}_p \mathbf{A} = (\operatorname{tr} \mathbf{A}_{ij}), \quad i = 1, \ldots, m, \; j = 1, \ldots, n,$$

*and, respectively, the sum of all diagonal $m \times n$ blocks of $\mathbf{A}$, that is*

$$\operatorname{BTr}_{m,n} \mathbf{A} = \sum_{i=1}^{p} \mathbf{A}_{ii};$$

**(ii)** $\operatorname{PTr}_p \mathbf{A} = \sum_{i=1}^{p} (\mathbf{I}_m \otimes \mathbf{e}'_{i,p})\mathbf{A}(\mathbf{I}_n \otimes \mathbf{e}_{i,p})$,
$\operatorname{BTr}_{m,n} \mathbf{A} = \sum_{i=1}^{p} (\mathbf{e}'_{i,p} \otimes \mathbf{I}_m)\mathbf{A}(\mathbf{e}_{i,p} \otimes \mathbf{I}_n)$;

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**(iii)** $\mathrm{PTr}_p\,\mathbf{A} = (\mathbf{I}_m \otimes \mathrm{vec}'\,\mathbf{I}_p)(\mathbf{A} \otimes \mathbf{I}_p)(\mathbf{I}_n \otimes \mathrm{vec}\,\mathbf{I}_p)$,
$\mathrm{BTr}_{m,n}\,\mathbf{A} = (\mathrm{vec}'\,\mathbf{I}_p \otimes \mathbf{I}_m)(\mathbf{I}_p \otimes \mathbf{A})(\mathrm{vec}\,\mathbf{I}_p \otimes \mathbf{I}_n)$.

Since both of the operators correspond to two different partitions of the $mp \times np$ matrix $\mathbf{A}$, the subscripts used in the notation indicate the size of blocks of $\mathbf{A}$ due to the respective partition. For convenience, if $n = m$ we denote $\mathrm{BTr}_{m,n}\,\mathbf{A}$ as $\mathrm{BTr}_m\,\mathbf{A}$.

The partial trace and block trace operators of square matrices have been studied in the physics and mathematics before, though not necessarily under these names and using different notations. De Pillis [3] studied $mn \times mn$ block matrices with blocks of order $n$. He showed that replacing every block of positive semi-definite matrix by its trace preserves positive semi-definiteness. Zhang [30] considered square block matrices with blocks replaced by their functions. Among others he proved the same result as de Pillis [3], denoting the block matrix with blocks replaced by their traces by $T$. Note, that the operation considered by de Pillis [3] as well as Zhang [30] correspond to the partial trace operator defined in Definition 1.1 (i) for particular case $m = n$, and the property of preserving positive semi-definiteness corresponds to the property given in Lemma 2.4 (iii) of this paper.

For an arbitrary Kronecker product of two Hilbert spaces $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ of dimensions $m$ and $n$, respectively, Bhatia [1] and Petz [21] defined two different matrix operations as linear maps from the space of linear operators on the Kronecker product $\boldsymbol{\mathcal{H}}_1 \otimes \boldsymbol{\mathcal{H}}_2$ onto the space of linear operators on $\boldsymbol{\mathcal{H}}_1$ and on $\boldsymbol{\mathcal{H}}_2$. Both of these linear maps were called partial trace operators, and are widely used in physical sciences. Bhatia [1] denoted partial trace operators by $\mathbf{A}_1 = \mathrm{tr}_{\mathcal{H}_1}\,\mathbf{A}$ and by $\mathbf{A}_2 = \mathrm{tr}_{\mathcal{H}_2}\,\mathbf{A}$, whilst Petz [21] used the notation $\mathrm{Tr}_2$ and $\mathrm{Tr}_1$, respectively. Both of these authors have shown several inequalities used in quantum mechanics.
Note that the first and third definition of $\mathbf{A}_1$ of Bhatia [1, p. 126, 127] correspond respectively to the partial trace operator $\mathrm{PTr}_p\,\mathbf{A}$ given in Definition 1.1 (ii) and (i) for $m = n$. Moreover, the definitions of $\mathbf{A}_2$ of Bhatia [1, p. 126, 127] correspond to the partial trace operator $\mathrm{PTr}_m\,\mathbf{A}$ defined as in Definition 1.1 (ii) and (i) but with different partition of $\mathbf{A}$ (the blocks of $\mathbf{A}$ are $m \times m$ dimensional here).
Petz [21] defined respective partial traces for a Kronecker product using the properties of standard trace operator. The definition of $\mathrm{Tr}_2$ corresponds to the property of partial trace operator given in Lemma 2.13, whilst the definition of $\mathrm{Tr}_1$ corresponds to the property of block trace operator given in Lemma 2.13, with $m = n$ in both cases. It means that for $\mathbf{A}$ being a Kronecker product of two square matrices, $\mathbf{H}_1 : m \times m$ and $\mathbf{H}_2 : p \times p$, we get $\mathrm{BTr}_p(\mathbf{H}_1 \otimes \mathbf{H}_2) = \mathrm{tr}(\mathbf{H}_1)\mathbf{H}_2$, thus, $\mathrm{Tr}_1$ of Petz [21] can be also expressed in terms of block trace operator given in Definition 1.1 with different partition, i.e., $\mathrm{BTr}_p\,\mathbf{A}$.

Recently, some inequalities related to partial trace operators defined by Bhatia [1] or Petz [21] were studied by Choi [2]. He considered partitioned matrices not necessarily having Kronecker product structure and denoted the operators respectively by $\mathrm{tr}_2\mathbf{A}$ and $\mathrm{tr}_1\mathbf{A}$.

Vitória [29] and Martins et al. [20] considered the block trace operator $tr_b(A_b)$ of square block matrix $A_b$ in context of differential equations. Recently, Jackson et al. [11] and Jackson et al. [12] used the block trace operator $\mathrm{btr}(\mathbf{A})$ of a square block matrix $\mathbf{A}$ in meta-analysis. Considering some statistical properties of physical model Nehorai and Paldi [19] defined the block trace operator $\mathrm{btr}[\mathbf{A}]$ of a square matrix $\mathbf{A}$. This definition corresponds in fact to the definition of partial trace given in Definition 1.1 (i).

In statistical research Filipiak and Markiewicz [7, 8] denoted the partial trace operator on an $mp \times mp$ matrix $\mathbf{A}$ by $\mathrm{Tr}\mathbf{A}$ and $\mathrm{Tr}_p\mathbf{A}$ and used it to show optimality of some experimental designs. Moreover, Filipiak and Klein [5] proved some properties of partial trace operator to express the maximum likelihood estimators

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of unknown parameters under the generalized growth curve model. Roy et al. [23] used the block trace operator, denoted by blktr$\mathbf{A}$, for solving the problem of testing the mean vector for doubly multivariate observations. The same notation for block trace was used also by Roy et al. [24] for maximum likelihood estimation problem for doubly exchangeable covariance structure.

In some statistical applications it is comfortable to use the partial trace of non-square matrices (see e.g. Section 3.1), which can be defined for every block matrix with square blocks. Since there is very strong relation between considered operators, and since also non-square matrices can be summed up, we have decided to define the block trace operator (as a generalization of the standard trace operator applied on the blocks of a matrix) also for non-square matrices. Therefore in this paper we extend the operators presented in the literature into the case of arbitrary matrix (not necessarily square).

The aim of this paper is to unify the notation used for partial trace and block trace operators and to show the relation between them as well as some of their properties. We characterize some sequential properties of these two operators as well as some preserved (and not preserved) properties, such as symmetry, nonnegativity, positive semi-definiteness, M-matrix property, singularity, and commutativity. We also study the relations between considered operators and "vec" and Kronecker product operators, which are very useful especially in statistics. Finally, we show the conditions for which the partial trace and block trace operator of a product of matrices can be presented as the product of matrices and we present some properties in relation to Bhatia's [1] definition of partial trace operator.

Both operators can be widely used for multi-level multivariate models in statistics, for example in the estimation of unknown parameters. They allow an elegant presentation of a complex formulas for estimators as well as an easier and faster computation of estimates. Possible applications of these operators in statistics are presented in Section 3.

## 2. Properties of partial trace and block trace operators.

Let $\mathbf{K}_{m,n}$ be a commutation matrix defined as a matrix which transforms $\operatorname{vec}\mathbf{X}$ into $\operatorname{vec}(\mathbf{X}')$ for any matrix $\mathbf{X} : m \times n$ (for the details see e.g. Magnus and Neudecker [18] or Ghazal and Neudecker [9]). The following lemma shows that the commutation matrix provides a unique relation between partial trace and block trace operators.

LEMMA 2.1. *For any matrix $\mathbf{A} : mp \times np$ the following relations between partial trace and block trace operators hold:*

$$\mathrm{PTr}_p\,\mathbf{A} = \mathrm{BTr}_{m,n}(\mathbf{K}_{p,m}\mathbf{A}\mathbf{K}_{n,p}) \quad \textit{and} \quad \mathrm{BTr}_{m,n}\,\mathbf{A} = \mathrm{PTr}_p(\mathbf{K}_{m,p}\mathbf{A}\mathbf{K}_{p,n}).$$

*Proof.* We show only the first equality, as the second one can be obtained by analogy. Using Definition 1.1 (ii) and Magnus and Neudecker [18, formula (24)] we obtain

$$\begin{aligned}\mathrm{PTr}_p\,\mathbf{A} &= \textstyle\sum_{i=1}^p(\mathbf{I}_m\otimes\mathbf{e}'_{i,p})\mathbf{A}(\mathbf{I}_n\otimes\mathbf{e}_{i,p})\\ &= \textstyle\sum_{i=1}^p(\mathbf{e}'_{i,p}\otimes\mathbf{I}_m)\mathbf{K}_{p,m}\mathbf{A}\mathbf{K}_{n,p}(\mathbf{e}_{i,p}\otimes\mathbf{I}_n)\\ &= \mathrm{BTr}_{m,n}(\mathbf{K}_{p,m}\mathbf{A}\mathbf{K}_{n,p}).\end{aligned}$$ ∎

In the following we study the properties of partial trace and block trace operators, such as non-negativity (non-negativity of the entries), positive semi-definiteness, transpose, symmetry, or connections between the "vec" operator and Kronecker product. However, we start with the following sequential properties of operators.

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LEMMA 2.2. *For an arbitrary matrix* $\mathbf{B}: nps \times mps$ *the following relations hold:*

$$\begin{aligned}
\mathrm{PTr}_s(\mathrm{PTr}_p\,\mathbf{B}) &= \mathrm{PTr}_{ps}\,\mathbf{B},\\
\mathrm{BTr}_{m,n}(\mathrm{BTr}_{mp,np}\,\mathbf{B}) &= \mathrm{BTr}_{m,n}\,\mathbf{B},\\
\mathrm{PTr}_p(\mathrm{BTr}_{mp,np}\,\mathbf{B}) &= \mathrm{PTr}_{ps}\left[\mathbf{K}_{mp,s}\mathbf{B}\mathbf{K}_{s,np}\right]\\
&= \mathrm{PTr}_{ps}\left[(\mathbf{K}_{m,s}\otimes\mathbf{I}_p)\mathbf{B}(\mathbf{K}_{s,n}\otimes\mathbf{I}_p)\right]\\
&= \mathrm{BTr}_{m,n}\left[(\mathbf{I}_s\otimes\mathbf{K}_{p,m})\mathbf{B}(\mathbf{I}_s\otimes\mathbf{K}_{n,p})\right],\\
\mathrm{BTr}_{m,n}(\mathrm{PTr}_s\,\mathbf{B}) &= \mathrm{BTr}_{m,n}\left[\mathbf{K}_{s,mp}\mathbf{B}\mathbf{K}_{np,s}\right]\\
&= \mathrm{BTr}_{m,n}\left[(\mathbf{I}_p\otimes\mathbf{K}_{s,m})\mathbf{B}(\mathbf{I}_p\otimes\mathbf{K}_{n,s})\right]\\
&= \mathrm{PTr}_{ps}\left[(\mathbf{K}_{m,p}\otimes\mathbf{I}_s)\mathbf{B}(\mathbf{K}_{p,n}\otimes\mathbf{I}_s)\right].
\end{aligned}$$

*Proof.* First two relations follow directly from Definition 1.1 (ii). We show only the first part of the third equality, as the remaining ones can be obtained by analogy. Using Definition 1.1 (ii) and Magnus and Neudecker [18, formula (24)] we obtain

$$\begin{aligned}
\mathrm{PTr}_p(\mathrm{BTr}_{mp,np}\,\mathbf{B}) &= \sum_{j=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}'_{j,p})\cdot\sum_{i=1}^{s}(\mathbf{e}'_{i,s}\otimes\mathbf{I}_{mp})\mathbf{B}(\mathbf{e}_{i,s}\otimes\mathbf{I}_{np})\cdot(\mathbf{I}_n\otimes\mathbf{e}_{j,p})\\
&= \sum_{j=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}'_{j,p})\cdot\sum_{i=1}^{s}(\mathbf{I}_{mp}\otimes\mathbf{e}'_{i,s})\mathbf{K}_{mp,s}\mathbf{B}\mathbf{K}_{s,np}(\mathbf{I}_{np}\otimes\mathbf{e}_{i,s})\cdot(\mathbf{I}_n\otimes\mathbf{e}_{j,p})\\
&= \sum_{j=1}^{p}\sum_{i=1}^{s}(\mathbf{I}_m\otimes\mathbf{e}'_{j,p}\otimes\mathbf{e}'_{i,s})\mathbf{K}_{mp,s}\mathbf{B}\mathbf{K}_{s,np}(\mathbf{I}_n\otimes\mathbf{e}_{j,p}\otimes\mathbf{e}_{i,s})\\
&= \sum_{i=1}^{ps}(\mathbf{I}_m\otimes\mathbf{e}'_{i,ps})\mathbf{K}_{mp,s}\mathbf{B}\mathbf{K}_{s,np}(\mathbf{I}_n\otimes\mathbf{e}_{i,ps})\\
&= \mathrm{PTr}_{ps}\left(\mathbf{K}_{mp,s}\mathbf{B}\mathbf{K}_{s,np}\right).
\end{aligned}$$

∎

Next lemma follows directly from Definition 1.1.

LEMMA 2.3. *For an arbitrary matrix* $\mathbf{A}: mp \times np$ *the following relations hold:*

$$\mathrm{PTr}'_p\,\mathbf{A} = \mathrm{PTr}_p(\mathbf{A}') \qquad \text{and} \qquad \mathrm{BTr}'_{m,n}\,\mathbf{A} = \mathrm{BTr}_{n,m}(\mathbf{A}').$$

The following lemma shows that for the square matrix $\mathbf{A}$ of order $mp$ the partial trace and block trace operators preserve symmetry, nonnegativity, positive semi-definiteness and M-matrix property of symmetric matrices.

LEMMA 2.4. *Let* $\mathbf{A}$ *be an arbitrary* $mp \times mp$ *matrix. Then*

**(i)** *if* $\mathbf{A}$ *is symmetric, then the partial trace and block trace operators are symmetric;*
**(ii)** *if* $\mathbf{A}$ *is nonnegative (nonpositive), then the partial trace and block trace operators are nonnegative (nonpositive);*
**(iii)** *if* $\mathbf{A}$ *is positive semi-definite, then the partial trace and block trace operators are positive semi-definite;*
**(iv)** *if* $\mathbf{A}$ *is a symmetric M-matrix, then the partial trace and block trace operators are symmetric M-matrices.*

*Proof.* The first two properties are trivial, for the third one it is enough to represent the positive semi-definite matrix $\mathbf{A} = \mathbf{B}\mathbf{B}'$, where $\mathbf{B}$ is an $mp \times t$ matrix of full column rank, and substitute it into Definition 1.1 (ii): both operators must be positive semi-definite as the products of a matrix and its transpose. Note, that the proof of positive semi-definiteness of the partial trace operator can also be found in de Pillis [3]

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or Zhang [30]. For the fourth property first recall that one of the conditions for a Z-matrix being an M-matrix is the positivity of all real eigenvalues, which for symmetric matrix means its positive definiteness. Trivially, partial trace and block trace operator preserves Z-matrix property and since they preserves also positive definiteness, the resulting partial trace and block trace matrix is M-matrix. ◻

It is worth to note that if M-matrix is not symmetric then partial and block trace operators do not preserve M-matrix property. As a counterexample one can see that for

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & -1.5 & -0.5 \\ 0 & 1 & 0 & -0.5 \\ -0.5 & 0 & 1 & 0 \\ 0 & -1.5 & 0 & 1 \end{pmatrix}$$

we have $\mathrm{PTr}_2\, \mathbf{M}$ is singular. Moreover, this example also shows that partial trace operator does not preserve non-singularity. Using Lemma 2.1 similar counterexample for block trace operator can be given. From the opposite way singularity is not preserved neither. As an example it is enough to take a singular block diagonal matrix $\mathbf{A}$ with blocks $\mathbf{I}_p$ and $\mathbf{J}_p$, where $\mathbf{J}_p$ is a matrix of ones, for which both $\mathrm{PTr}_p\, \mathbf{A}$ and $\mathrm{BTr}_2\, \mathbf{A}$ are not singular.

We now focus on the relations between the partial trace and block trace operators for a square matrix $\mathbf{A}$ of order $mp$ and a standard trace operator. The following properties are trivial and follows directly from Definition 1.1 (i).

Lemma 2.5. *For any matrix $\mathbf{A} : mp \times mp$ the following relation holds:*

**(i)** $\mathrm{PTr}_{mp}\, \mathbf{A} = \mathrm{BTr}_1\, \mathbf{A} = \mathrm{tr}\, \mathbf{A}$;
**(ii)** $\mathrm{tr}(\mathrm{PTr}_p\, \mathbf{A}) = \mathrm{tr}(\mathrm{PTr}_m\, \mathbf{A}) = \mathrm{tr}(\mathrm{BTr}_p\, \mathbf{A}) = \mathrm{tr}(\mathrm{BTr}_m\, \mathbf{A}) = \mathrm{tr}\, \mathbf{A}$.

It is well-known that $\mathrm{tr}(\mathbf{XY}) = \mathrm{tr}(\mathbf{YX})$ for arbitrary matrices of proper sizes. We study the assumptions under which such a relation can hold also for partial trace and block trace operators. The following corollary follows directly from Lemmas 2.3 and 2.4.

Corollary 2.6. *For arbitrary symmetric matrices $\mathbf{A}, \mathbf{B} : mp \times mp$ the following relations hold:*

**(i)** $\mathrm{PTr}_p(\mathbf{AB}) = \mathrm{PTr}'_p(\mathbf{BA})$,
$\mathrm{BTr}_m(\mathbf{AB}) = \mathrm{BTr}'_m(\mathbf{BA})$;
**(ii)** $\mathrm{PTr}_p(\mathbf{AB}) = \mathrm{PTr}'_p(\mathbf{AB}) \quad \Leftrightarrow \quad \mathrm{PTr}_p(\mathbf{AB}) = \mathrm{PTr}_p(\mathbf{BA})$,
$\mathrm{BTr}_m(\mathbf{AB}) = \mathrm{BTr}'_m(\mathbf{AB}) \quad \Leftrightarrow \quad \mathrm{BTr}_m(\mathbf{AB}) = \mathrm{BTr}_m(\mathbf{BA})$;
**(iii)** $\mathbf{AB} = \mathbf{BA} \quad \Rightarrow \quad \mathrm{PTr}_p(\mathbf{AB}) = \mathrm{PTr}_p(\mathbf{BA})$,
$\mathbf{AB} = \mathbf{BA} \quad \Rightarrow \quad \mathrm{BTr}_m(\mathbf{AB}) = \mathrm{BTr}_m(\mathbf{BA})$.

We now prove the following property of commutativity of two matrices under the partial trace and block trace operators. It shows that if one of the matrices has particular Kronecker product structure, then the matrices under the partial trace and block trace operators commute.

Lemma 2.7. *For arbitrary matrices $\mathbf{A} : mp \times np$ and $\mathbf{C} : n \times m$ the following relations hold:*

$$\mathrm{PTr}_m[\mathbf{A}(\mathbf{I}_p \otimes \mathbf{C})] = \mathrm{PTr}_n[(\mathbf{I}_p \otimes \mathbf{C})\mathbf{A}],$$
$$\mathrm{BTr}_p[\mathbf{A}(\mathbf{C} \otimes \mathbf{I}_p)] = \mathrm{BTr}_p[(\mathbf{C} \otimes \mathbf{I}_p)\mathbf{A}].$$

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*Proof.* Let $\mathbf{A}_{ij}$, $i,j=1,\dots,p$, be the $m\times n$ blocks of $\mathbf{A}$. Then

$$\mathrm{PTr}_m[\mathbf{A}(\mathbf{I}_p\otimes\mathbf{C})]=(\mathrm{tr}(\mathbf{A}_{ij}\mathbf{C}))_{1\le i,j\le p}=(\mathrm{tr}(\mathbf{C}\mathbf{A}_{ij}))_{1\le i,j\le p}=\mathrm{PTr}_m[(\mathbf{I}_p\otimes\mathbf{C})\mathbf{A}].$$

Let now $\mathbf{A}_{ij}$ and $c_{ij}$ be respectively the $(i,j)$th block of $\mathbf{A}$ and the $(i,j)$th entry of $\mathbf{C}$, $i=1,\dots,m$, $j=1,\dots,n$. Then

$$\mathrm{BTr}_p[\mathbf{A}(\mathbf{C}\otimes\mathbf{I}_p)]=\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{c}_{ij}\mathbf{A}_{ji}=\mathrm{BTr}_p[(\mathbf{C}\otimes\mathbf{I}_p)\mathbf{A}].$$

∎

Next properties present the connections between the partial trace and block trace operators with the "vec" operator (Lemma 2.8), "vec" and Kronecker product operators (Lemma 2.9), and Kronecker product operator (Lemma 2.11). All of them can be useful especially in matrix differentiation.

Lemma 2.8. *For arbitrary matrices $\mathbf{B}:\ p\times m$ and $\mathbf{C}:\ p\times n$ the following relations hold:*

$$\begin{aligned}\mathrm{PTr}_p\left(\mathrm{vec}\,\mathbf{B}\,\mathrm{vec}'\,\mathbf{C}\right)&=\mathbf{B}'\mathbf{C},\\ \mathrm{BTr}_{m,n}\left(\mathrm{vec}\,\mathbf{B}'\,\mathrm{vec}'\,\mathbf{C}'\right)&=\mathbf{B}'\mathbf{C}.\end{aligned}$$

*Proof.* We show only the first equality, as the second one can be proved by analogy. From Definition 1.1 (ii) we obtain

$$\begin{aligned}\mathrm{PTr}_p[\mathrm{vec}\,\mathbf{B}\,\mathrm{vec}'\,\mathbf{C}]&=\sum_{i=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}_{i,p}')\,\mathrm{vec}\,\mathbf{B}\,\mathrm{vec}'\,\mathbf{C}(\mathbf{I}_n\otimes\mathbf{e}_{i,p})\\ &=\sum_{i=1}^{p}\mathrm{vec}(\mathbf{e}_{i,p}'\mathbf{B})\,\mathrm{vec}'(\mathbf{e}_{i,p}'\mathbf{C})\\ &=\sum_{i=1}^{p}\mathbf{B}'\mathbf{e}_{i,p}\mathbf{e}_{i,p}'\mathbf{C}=\mathbf{B}'\mathbf{C}.\end{aligned}$$

∎

Lemma 2.9. *For an arbitrary matrix $\mathbf{A}:\ mp\times np$ the following relations hold:*

$$\begin{aligned}\mathrm{vec}(\mathrm{PTr}_p\,\mathbf{A})&=(\mathbf{I}_{mn}\otimes\mathrm{vec}'\,\mathbf{I}_p)(\mathbf{I}_n\otimes\mathbf{K}_{m,p}\otimes\mathbf{I}_p)\,\mathrm{vec}\,\mathbf{A},\\ \mathrm{vec}(\mathrm{BTr}_{m,n}\,\mathbf{A})&=(\mathrm{vec}'\,\mathbf{I}_p\otimes\mathbf{I}_{mn})(\mathbf{I}_p\otimes\mathbf{K}_{p,n}\otimes\mathbf{I}_m)\,\mathrm{vec}\,\mathbf{A}.\end{aligned}$$

*Proof.* We show only the first equality, as the second one can be obtained by analogy. From Definition 1.1 (ii) and using Magnus and Neudecker [18, Lemma 4]: $\mathbf{e}_{i,p}'\otimes\mathbf{I}_m=(\mathbf{I}_m\otimes\mathbf{e}_{i,p}')\mathbf{K}_{m,p}$, we get

$$\begin{aligned}\mathrm{vec}(\mathrm{PTr}_p\,\mathbf{A})&=\mathrm{vec}\left[\sum_{i=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}_{i,p}')\mathbf{A}(\mathbf{I}_n\otimes\mathbf{e}_{i,p})\right]\\ &=\sum_{i=1}^{p}(\mathbf{I}_n\otimes\mathbf{e}_{i,p}'\otimes\mathbf{I}_m\otimes\mathbf{e}_{i,p}')\,\mathrm{vec}\,\mathbf{A}\\ &=\sum_{i=1}^{p}\left[\mathbf{I}_n\otimes(\mathbf{I}_m\otimes\mathbf{e}_{i,p}')\mathbf{K}_{m,p}\otimes\mathbf{e}_{i,p}'\right]\mathrm{vec}\,\mathbf{A}\\ &=(\mathbf{I}_n\otimes\mathbf{I}_m\otimes\mathrm{vec}'\,\mathbf{I}_p)(\mathbf{I}_n\otimes\mathbf{K}_{m,p}\otimes\mathbf{I}_p)\,\mathrm{vec}\,\mathbf{A}.\end{aligned}$$

∎

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From the above lemma next corollary follows.

Corollary 2.10. *For arbitrary matrices* $\mathbf{A} : mp \times np$ *and* $\mathbf{B}, \mathbf{C} : p \times p$*, the following relations hold:*

$$\begin{aligned}
\operatorname{vec}\left\{\mathrm{PTr}_p\left[(\mathbf{I}_m \otimes \mathbf{B})\mathbf{A}(\mathbf{I}_n \otimes \mathbf{C}')\right]\right\} &= [\mathbf{I}_{mn} \otimes \operatorname{vec}'(\mathbf{B}'\mathbf{C})](\mathbf{I}_n \otimes \mathbf{K}_{m,p} \otimes \mathbf{I}_p)\operatorname{vec}\mathbf{A},\\
\operatorname{vec}\left\{\mathrm{BTr}_{m,n}\left[(\mathbf{B} \otimes \mathbf{I}_m)\mathbf{A}(\mathbf{C}' \otimes \mathbf{I}_n)\right]\right\} &= [\operatorname{vec}'(\mathbf{B}'\mathbf{C}) \otimes \mathbf{I}_{mn}](\mathbf{I}_p \otimes \mathbf{K}_{p,n} \otimes \mathbf{I}_m)\operatorname{vec}\mathbf{A}.
\end{aligned}$$

Note that Roy et al. [24, Lemma A.1 and Corollary A.2] gave two properties of block trace operator, which can be proven using the above corollary.

Lemma 2.11. *For arbitrary matrices* $\mathbf{A} : mp \times np$, $\mathbf{B} : s \times m$*, and* $\mathbf{C} : n \times t$ *the following relations hold:*

$$\begin{aligned}
\mathrm{PTr}_p[(\mathbf{B} \otimes \mathbf{I}_p)\mathbf{A}(\mathbf{C} \otimes \mathbf{I}_p)] &= \mathbf{B} \cdot \mathrm{PTr}_p(\mathbf{A}) \cdot \mathbf{C},\\
\mathrm{BTr}_{s,t}[(\mathbf{I}_p \otimes \mathbf{B})\mathbf{A}(\mathbf{I}_p \otimes \mathbf{C})] &= \mathbf{B} \cdot \mathrm{BTr}_{m,n}(\mathbf{A}) \cdot \mathbf{C}.
\end{aligned}$$

*Proof.* We show only the first equality, as the second one can be obtained by analogy. From Definition 1.1 (ii) we obtain

$$\begin{aligned}
\mathrm{PTr}_p[(\mathbf{B} \otimes \mathbf{I}_p)\mathbf{A}(\mathbf{C} \otimes \mathbf{I}_p)] &= \sum_{i=1}^{p}(\mathbf{I}_s \otimes \mathbf{e}'_{i,p})(\mathbf{B} \otimes \mathbf{I}_p)\mathbf{A}(\mathbf{C} \otimes \mathbf{I}_p)(\mathbf{I}_t \otimes \mathbf{e}_{i,p})\\
&= \sum_{i=1}^{p}\mathbf{B}(\mathbf{I}_m \otimes \mathbf{e}'_{i,p})\mathbf{A}(\mathbf{I}_n \otimes \mathbf{e}_{i,p})\mathbf{C}\\
&= \mathbf{B} \cdot \mathrm{PTr}_p(\mathbf{A}) \cdot \mathbf{C}.
\end{aligned}$$

□

The following corollary is a direct consequence of the previous Lemma and Lemma 2.5. It is worth to note that this property for partial trace operator and square matrices $\mathbf{A}$ and $\mathbf{C}$ was shown in Bhatia [1].

Corollary 2.12. *For arbitrary matrices* $\mathbf{A} : mp \times np$ *and* $\mathbf{C} : n \times m$ *the following relations hold:*

$$\begin{aligned}
\operatorname{tr}[\mathbf{A}(\mathbf{C} \otimes \mathbf{I}_p)] &= \operatorname{tr}[\mathrm{PTr}_p\,\mathbf{A} \cdot \mathbf{C}],\\
\operatorname{tr}[\mathbf{A}(\mathbf{I}_p \otimes \mathbf{C})] &= \operatorname{tr}[\mathrm{BTr}_{m,n}\,\mathbf{A} \cdot \mathbf{C}].
\end{aligned}$$

Bhatia [1] originally defined the partial trace and block trace operators for the Kronecker product of two matrices. Thus, in Lemmas 2.13 and Lemma 2.14 we give the properties of operators in mentioned case and their extensions, respectively. Observe that the properties given in Lemma 2.13 are related to the definitions of partial traces given in Petz [21].

Lemma 2.13. *For arbitrary matrices* $\mathbf{B} : m \times n$ *and* $\mathbf{C} : q \times q$ *the following relations hold:*

$$\begin{aligned}
\mathrm{PTr}_q(\mathbf{B} \otimes \mathbf{C}) &= \operatorname{tr}\mathbf{C} \cdot \mathbf{B},\\
\mathrm{BTr}_{m,n}(\mathbf{C} \otimes \mathbf{B}) &= \operatorname{tr}\mathbf{C} \cdot \mathbf{B}.
\end{aligned}$$

*Proof.* Follows directly from Definition 1.1 (ii). □

Lemma 2.14. *For arbitrary matrices* $\mathbf{A} : mp \times np$, $\mathbf{C} : q \times q$*, and* $\mathbf{D} : s \times t$*, the following relations hold:*

**(i)** $\mathrm{PTr}_{pq}(\mathbf{A} \otimes \mathbf{C}) = \operatorname{tr}\mathbf{C} \cdot \mathrm{PTr}_p\,\mathbf{A}$,
$\mathrm{BTr}_{m,n}(\mathbf{C} \otimes \mathbf{A}) = \operatorname{tr}\mathbf{C} \cdot \mathrm{BTr}_{m,n}\,\mathbf{A}$;

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**(ii)** $\mathrm{PTr}_p(\mathbf{D}\otimes\mathbf{A}) = \mathbf{D}\otimes\mathrm{PTr}_p\,\mathbf{A}$,
$\mathrm{BTr}_{ms,nt}(\mathbf{A}\otimes\mathbf{D}) = \mathrm{BTr}_{m,n}\,\mathbf{A}\otimes\mathbf{D}$.

*Proof.* We prove only the relations for partial trace operator, as the respective properties for block trace operator are analogous. From Definition 1.1 (ii) we get

$$\begin{aligned}
\mathrm{PTr}_{pq}(\mathbf{A}\otimes\mathbf{C}) &= \sum_{i=1}^{pq}(\mathbf{I}_m\otimes\mathbf{e}'_{i,pq})(\mathbf{A}\otimes\mathbf{C})(\mathbf{I}_n\otimes\mathbf{e}_{i,pq})\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}(\mathbf{I}_m\otimes\mathbf{e}'_{i,p}\otimes\mathbf{e}'_{j,q})(\mathbf{A}\otimes\mathbf{C})(\mathbf{I}_n\otimes\mathbf{e}_{i,p}\otimes\mathbf{e}_{j,q})\\
&= \sum_{i=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}'_{i,p})\mathbf{A}(\mathbf{I}_n\otimes\mathbf{e}_{i,p})\otimes\sum_{j=1}^{q}\mathbf{e}'_{j,q}\mathbf{C}\mathbf{e}_{j,q}\\
&= \mathrm{tr}\,\mathbf{C}\cdot\mathrm{PTr}_p\,\mathbf{A},
\end{aligned}$$

$$\begin{aligned}
\mathrm{PTr}_p(\mathbf{D}\otimes\mathbf{A}) &= \sum_{i=1}^{p}(\mathbf{I}_{sm}\otimes\mathbf{e}'_{i,p})(\mathbf{D}\otimes\mathbf{A})(\mathbf{I}_{tn}\otimes\mathbf{e}_{i,p})\\
&= \mathbf{D}\otimes\sum_{i=1}^{p}(\mathbf{I}_m\otimes\mathbf{e}'_{i,p})\mathbf{A}(\mathbf{I}_n\otimes\mathbf{e}_{i,p})\\
&= \mathbf{D}\otimes\mathrm{PTr}_p\,\mathbf{A}.
\end{aligned}$$

∎

Next lemma shows the relation between block and partial trace operators for Kronecker product of two square matrices.

LEMMA 2.15. *For arbitrary square matrices* $\mathbf{B} : m\times m$ *and* $\mathbf{C} : q\times q$

$$\mathrm{PTr}_q(\mathbf{B}\otimes\mathbf{C}) = \mathrm{BTr}_m(\mathbf{C}\otimes\mathbf{B}).$$

*Proof.* Follows directly from Lemma 2.1 and the properties of the commutation matrix. ∎

## 3. Applications in statistics.

**3.1. Optimality of designs.** A basic problem in the theory of experimental designs is to characterize optimal designs. If the response to a treatment in an experiment is affected by the other treatments, then the universal optimality of designs under an interference model is usually studied; see e.g. Druilhet [4], Kunert and Martin [16], Filipiak and Markiewicz [7, 8] and many others. One of the condition of universal optimality in Kiefer's sense [13] is expressed as the maximal trace of the information matrix over all possible designs. Since in the interference model the information matrix is the Schur complement of a non-negative definite partitioned matrix, the condition for universal optimality can be expressed using partial trace operators.

Filipiak and Markiewicz [7, 8] used the partial trace operator of a Schur complement of a partitioned matrix to prove universal optimality of some circular designs under the mixed interference models. The same property was used also by Kunert and Martin [16] to prove optimality of some non-circular designs under the interference model, however, the partial trace operator was not mentioned there. We now prove this property in a general case.

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For any partitioned matrix $\mathbf{A} = (\mathbf{A}_{ij})_{i,j=1,2}$, where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square blocks not necessarily of the same dimension, the Schur complement $\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$ of $\mathbf{A}_{22}$ in $\mathbf{A}$ will be denoted as $[\mathbf{A}/\mathbf{A}_{22}]$.

THEOREM 3.1. *For any non-negative definite $mp \times mp$ matrix $\mathbf{A} = (\mathbf{A}_{ij})_{1\le i,j\le p}$ let us denote the right $m(p-1)\times m(p-1)$ lower corner block as $\mathbf{B}$. Then*

$$\operatorname{tr}[\mathbf{A}/\mathbf{B}] \le [\operatorname{PTr}_m \mathbf{A}/\operatorname{PTr}_m \mathbf{B}].$$

*Proof.* Let $\boldsymbol{\mathcal{H}}$ be a set of all $m\times m$ orthogonal matrices. For any symmetric matrix $\mathbf{C}$ the balancing operator, or center matrix, under the compact group $\boldsymbol{\mathcal{H}}$ is defined to be

$$\overline{\mathbf{C}} = \int_{\boldsymbol{\mathcal{H}}} \mathbf{H}\mathbf{C}\mathbf{H}'\mathrm{d}\mathbf{H},$$

where the integral is taken with respect to the unique invariant probability measure on the group $\boldsymbol{\mathcal{H}}$ (cf. Pukelsheim [22]). The $\boldsymbol{\mathcal{H}}$-center matrix $\overline{\mathbf{C}}$ is $\boldsymbol{\mathcal{H}}$-invariant, i.e., $\overline{\mathbf{C}} = \mathbf{H}\overline{\mathbf{C}}\mathbf{H}'$ for all $\mathbf{H}\in\boldsymbol{\mathcal{H}}$. Since the set of all matrices that are invariant under the orthogonal transformation is the set $\{\alpha\mathbf{I}_m : \alpha\in\mathbb{R}\}$ and obviously $\operatorname{tr}\overline{\mathbf{C}} = \operatorname{tr}\mathbf{C}$, we have $\overline{\mathbf{C}} = m^{-1}\operatorname{tr}\mathbf{C}\cdot\mathbf{I}_m$.

Let us denote the center matrix with respect to the group $\mathbf{I}_p\otimes\boldsymbol{\mathcal{H}}$ by $\overline{\overline{\mathbf{A}}}$. Obviously, $\overline{\mathbf{A}}_{ij}$ are the center matrices with respect to the group $\boldsymbol{\mathcal{H}}$ and hence $\overline{\overline{\mathbf{A}}} = m^{-1}\operatorname{PTr}_m\mathbf{A}\otimes\mathbf{I}_m$. Moreover, from the concavity of the Schur complement (cf. Li and Mathias [17]) we have

$$\overline{[\mathbf{A}/\mathbf{B}]} \le_L [\overline{\overline{\mathbf{A}}}/\overline{\overline{\mathbf{B}}}],$$

where $\le_L$ means the Löwner ordering. Thus

$$\begin{aligned}\operatorname{tr}[\mathbf{A}/\mathbf{B}] &= \operatorname{tr}\overline{[\mathbf{A}/\mathbf{B}]} \le \operatorname{tr}[\overline{\overline{\mathbf{A}}}/\overline{\overline{\mathbf{B}}}] = \operatorname{tr}[m^{-1}\operatorname{PTr}_m\mathbf{A}\otimes\mathbf{I}_m/m^{-1}\operatorname{PTr}_m\mathbf{B}\otimes\mathbf{I}_m]\\ &= m^{-1}\operatorname{tr}\left([\operatorname{PTr}_m\mathbf{A}/\operatorname{PTr}_m\mathbf{B}]\otimes\mathbf{I}_m\right) = [\operatorname{PTr}_m\mathbf{A}/\operatorname{PTr}_m\mathbf{B}].\end{aligned}$$ ∎

**3.2. Multi-level multivariate models.** Filipiak et al. [6] considered the problem of estimation and hypothesis testing under the doubly-multivariate model with specified, separable, covariance structure. Let $\mathbf{Y}_i$ for $i=1,\dots,n$ be independent and identically matrix normally distributed $(q\times p)$-dimensional observation matrices. It means that $\mathbf{Y}_i\sim\mathcal{N}_{q,p}(\mathbf{M},\boldsymbol{\Sigma},\boldsymbol{\Psi})$, where $\mathbf{M}: q\times p$ is any real matrix and $\boldsymbol{\Psi}: p\times p$ and $\boldsymbol{\Sigma}: q\times q$ are symmetric, positive-definite matrices (cf. e.g. Kollo and von Rosen [15]). Since usually matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$ are unknown, the researcher is interested in their estimation. Filipiak et al. [6] have shown that the maximum likelihood estimators of unknown covariance matrices can be obtained as a solution of the following system of matrix equations:

$$\left\{\begin{array}{lcl} nq\ \operatorname{vec}\boldsymbol{\Psi} &=& (\mathbf{I}_{p^2}\otimes\operatorname{vec}'\boldsymbol{\Sigma}^{-1})(\mathbf{I}_p\otimes\mathbf{K}_{p,q}\otimes\mathbf{I}_q)\operatorname{vec}\mathbf{S}\\ np\ \operatorname{vec}\boldsymbol{\Sigma} &=& (\operatorname{vec}'\boldsymbol{\Psi}^{-1}\otimes\mathbf{I}_{q^2})(\mathbf{I}_p\otimes\mathbf{K}_{p,q}\otimes\mathbf{I}_q)\operatorname{vec}\mathbf{S}\end{array}\right. \tag{3.1}$$

with $\mathbf{S} = \mathbf{Y}(\mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\mathbf{Y}'$, where $\mathbf{Y} = (\operatorname{vec}\mathbf{Y}_1, \operatorname{vec}\mathbf{Y}_2,\dots,\operatorname{vec}\mathbf{Y}_n)$ and $\mathbf{1}_n$ is an $n$-dimensional vector of ones. Note, that (3.1) follows from differentiation of likelihood function with respect to $\boldsymbol{\Psi}$ and $\boldsymbol{\Sigma}$.

Observe, that using Corollary 2.10, the above system can be presented as

$$\left\{\begin{array}{lcl} nq\ \boldsymbol{\Psi} &=& \operatorname{PTr}_q\left[(\mathbf{I}_p\otimes\boldsymbol{\Sigma}^{-1})\mathbf{S}\right]\\ np\ \boldsymbol{\Sigma} &=& \operatorname{BTr}_q\left[(\boldsymbol{\Psi}^{-1}\otimes\mathbf{I}_q)\mathbf{S}\right].\end{array}\right. \tag{3.2}$$

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Both systems of matrix equations can be solved only numerically. Moreover, the number of operations in (3.2) is much smaller than in (3.1), which makes computing the estimates much faster, especially when $q$ is big. Finally, representation (3.2) is much more elegant and easier to understand comparing to (3.1).

Similar representation can be used for example in the situation of $\mathbf{\Psi}$ structured as autoregression of order one or compound symmetry; cf. Filipiak et al. [6].

If for a doubly multivariate data $\mathbf{Y}_i \sim \mathcal{N}_{pq}(\boldsymbol{\mu}, \mathbf{\Sigma})$ we assume the covariance structure known as a block compound symmetry or block exchangeable structure, the block trace operator can be used in order to express the maximum likelihood estimators of unknown covariance parameters. This structure can be written in the form $\mathbf{\Sigma} = \mathbf{P}_p \otimes \mathbf{\Delta}_1 + \mathbf{Q}_p \otimes \mathbf{\Delta}_2$, where $\mathbf{\Delta}_i : q \times q$ are unknown and $\mathbf{P}_p = p^{-1}\mathbf{1}_p\mathbf{1}_p'$ and $\mathbf{Q}_p = \mathbf{I}_p - \mathbf{P}_p$. Then maximum likelihood estimators of $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ can be expressed as

$$\begin{aligned} \widehat{\mathbf{\Delta}}_1 &= \tfrac{1}{n} \operatorname{BTr}_q \left[(\mathbf{P}_p \otimes \mathbf{I}_q)\mathbf{S}\right], \\ \widehat{\mathbf{\Delta}}_2 &= \tfrac{1}{n(p-1)} \operatorname{BTr}_q \left[(\mathbf{Q}_p \otimes \mathbf{I}_q)\mathbf{S}\right], \end{aligned}$$

with $\mathbf{S} = \mathbf{Y}(\mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\mathbf{Y}'$, where $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n)$; cf. Roy et al. [24].

Let us consider now the multi-level multivariate model, in which $\boldsymbol{\mathcal{Y}}$ is a tensor of order 3, distributed as $\mathcal{N}_{n,p,q}(\llbracket \boldsymbol{\mathcal{X}}; \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket, \mathbf{I}_n, \mathbf{\Psi}, \mathbf{\Sigma})$, where $\llbracket \boldsymbol{\mathcal{X}}; \mathbf{A}, \mathbf{B}, \mathbf{C} \rrbracket$ denotes the Tucker operator, which represents the three-mode multiplication of an unknown tensor $\boldsymbol{\mathcal{X}}$ on its three 'sides' or modes by the known matrices of proper size: $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, respectively; cf. Filipiak and Klein [5], Kolda [14], Savas and Lim [25] or Singull et al. [26].

In order to determine the maximum likelihood estimators of $\mathbf{\Psi}$ and $\mathbf{\Sigma}$, Filipiak and Klein [5, Lemma 3] presented Definition 1.1 (iii) of partial trace operator and proved Lemma 2.9 and a special case of Lemma 2.11. The form of the estimators are given respectively in Theorem 4 and Theorem 6 of Filipiak and Klein [5]. As it can be observed, the estimators presented in Theorem 6 can be obtained only numerically, as the estimators are not given in explicit form. A problem arises here in relation to the convergence of the numerical algorithm to a unique solution giving the maximum likelihood estimators. Considering the algorithm only from algebraic point of view it can be seen from partial trace representation (9) of Filipiak and Klein [5, p. 81], that it gives a unique solution only under special conditions.

Similar problem of uniqueness of the algorithm for the estimation of unknown covariance matrix appears in Srivastava et al. [27]. However, equation (A.7) of Srivastava et al. [27, p. 158], that is

$$\mathbf{0} = (\operatorname{vec}' \mathbf{I}_n \otimes \mathbf{I}_q)(\mathbf{I}_n \otimes (\mathbf{P}_1 - \mathbf{P}_2))(\operatorname{vec} \mathbf{I}_n \otimes \mathbf{I}_q), \tag{3.3}$$

is not so obvious to be satisfied if and only if $\mathbf{P}_1 = \mathbf{P}_2$. As a consequence, the proof of this fact has some weakness, which could be easy noted if (3.3) would be represented using block trace operator (Definition 1.1 (iii)) as

$$\mathbf{0} = \operatorname{BTr}_q(\mathbf{P}_1 - \mathbf{P}_2).$$

**3.3. Distributional properties.** In this section we extend the result of Glueck and Muller [10] about the distribution of the trace of a Wishart matrix to the distribution of block and partial traces. Let $\mathbf{Y} : p \times n$ be the random matrix, normally distributed, that is $\mathbf{Y} \sim \mathcal{N}_{p,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{I}_n)$, where $\mathbf{\Sigma} : p \times p$ is any symmetric, positive-definite matrix, and $\mathbf{M} : p \times n$ is any real matrix (cf. Kollo and von Rosen [15]). Glueck and Muller [10] showed the following theorem, in which they allowed also a singular covariance matrix. We present this theorem assuming only positive-definiteness of this matrix.

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THEOREM 3.2. *Let* $\mathbf{Y} \sim \mathcal{N}_{p,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{I}_n)$*, where* $\mathbf{\Sigma} : p \times p$ *is a symmetric positive-definite matrix with spectral decomposition* $\mathbf{\Sigma} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}'$*, where* $\mathbf{V}'\mathbf{V} = \mathbf{I}_p$ *and* $\mathbf{\Lambda}$ *is a diagonal matrix with positive eigenvalues* $\lambda_1, \dots, \lambda_p$ *on its diagonal. Let* $\mathbf{Q} : n \times n$ *be any constant idempotent matrix of rank* $\nu \geq p$*. Then*

$$\operatorname{tr}(\mathbf{Y}\mathbf{Q}\mathbf{Y}') \sim \sum_{i=1}^{p} \lambda_i \, \chi^2_\nu(\boldsymbol{\delta}_i),$$

*where* $\chi^2_\nu(\boldsymbol{\delta}_i)$ *denotes the non-central* $\chi^2$ *distribution with* $\nu$ *degrees of freedom and non-centrality parameter* $\boldsymbol{\delta}_i = \mathbf{e}'_{i,p}\mathbf{V}'\mathbf{M}\mathbf{Q}\mathbf{M}'\mathbf{V}\mathbf{e}_{i,p}$*.*

We now extend this result to the distribution of the partial and block trace operators, which can be used in multi-level multivariate models.

THEOREM 3.3. *Let assume that* $\mathbf{Y} \sim \mathcal{N}_{pq,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{I}_n)$*, where* $\mathbf{\Sigma} : pq \times pq$ *is a symmetric positive-definite matrix and there exits an orthogonal matrix* $\mathbf{V} : p \times p$ *and a block-diagonal matrix* $\mathbf{\Lambda}$ *with positive-definite blocks* $\mathbf{\Lambda}_i : q \times q$, $i = 1, \dots, p$*, such that* $\mathbf{\Sigma} = (\mathbf{V} \otimes \mathbf{I}_q)\mathbf{\Lambda}(\mathbf{V}' \otimes \mathbf{I}_q)$*. Let* $\mathbf{Q} : n \times n$ *be any constant symmetric idempotent matrix of rank* $\nu \geq pq$*. Then*

$$\operatorname{BTr}_q(\mathbf{Y}\mathbf{Q}\mathbf{Y}') \sim \sum_{i=1}^{p} W_q(\mathbf{\Lambda}_i, \nu, \mathbf{\Delta}_i),$$

*where* $\mathbf{\Delta}_i = (\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{M}\mathbf{Q}\mathbf{M}'(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)$*.*

*Proof.* From Lemma 2.7 and Definition 1.1 (ii) we have:

$$\begin{aligned}
\operatorname{BTr}_q(\mathbf{Y}\mathbf{Q}\mathbf{Y}') &= \operatorname{BTr}_q((\mathbf{I}_p \otimes \mathbf{I}_q)\mathbf{Y}\mathbf{Q}\mathbf{Y}') \\
&= \operatorname{BTr}_q((\mathbf{V}_p \otimes \mathbf{I}_q)(\mathbf{V}_p \otimes \mathbf{I}_q)'\mathbf{Y}\mathbf{Q}\mathbf{Y}') \\
&= \operatorname{BTr}_q((\mathbf{V} \otimes \mathbf{I}_q)'\mathbf{Y}\mathbf{Q}\mathbf{Y}'(\mathbf{V} \otimes \mathbf{I}_q)) \\
&= \sum_{i=1}^{p} (\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{Y}\mathbf{Q}\mathbf{Y}'(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q).
\end{aligned}$$

Observe now, that $(\mathbf{V} \otimes \mathbf{I}_q)'\mathbf{Y} \sim \mathcal{N}_{pq,n}((\mathbf{V} \otimes \mathbf{I}_q)'\mathbf{M}, \mathbf{\Lambda}, \mathbf{I}_n)$ and $\mathbf{\Lambda}$ is block-diagonal, which implies that for every $j, k = 1, \dots, p$, $j \neq k$, $(\mathbf{V}\mathbf{e}_{j,p} \otimes \mathbf{I}_q)'\mathbf{Y}$ is independent of $(\mathbf{V}\mathbf{e}_{k,p} \otimes \mathbf{I}_q)'\mathbf{Y}$ and $(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{Y} \sim \mathcal{N}_{q,n}((\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{M}, \mathbf{\Lambda}_i, \mathbf{I}_n)$, $i = 1, \dots, p$. Therefore $(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{Y}\mathbf{Q}\mathbf{Y}'(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q) \sim \boldsymbol{\mathcal{W}}_q(\mathbf{\Lambda}_i, \nu, \mathbf{\Delta}_i)$ with $\nu$ being equal to rank $\mathbf{Q}$ and $\mathbf{\Delta}_i$ given in the theorem (cf. Vaish and Chaganty [28]). It completes the proof.□

Similar distributional property for partial trace follows from the previous theorem and Lemma 2.1.

COROLLARY 3.4. *Let assume that* $\mathbf{Y} \sim \mathcal{N}_{pq,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{I}_n)$*, where* $\mathbf{\Sigma} : pq \times pq$ *is a symmetric positive-definite matrix and there exits an orthogonal matrix* $\mathbf{V} : p \times p$ *and a matrix* $\mathbf{\Lambda}$*, whose* $p \times p$ *blocks are diagonal, such that* $\mathbf{\Sigma} = (\mathbf{I}_q \otimes \mathbf{V})\mathbf{\Lambda}(\mathbf{I}_q \otimes \mathbf{V}')$*. Let* $\mathbf{Q} : n \times n$ *be any constant symmetric idempotent matrix of rank* $\nu \geq pq$*. Then*

$$\operatorname{PTr}_p(\mathbf{Y}\mathbf{Q}\mathbf{Y}') \sim \sum_{i=1}^{p} W_q(\mathbf{\Lambda}_i, \nu, \mathbf{\Delta}_i),$$

*where*

$$\begin{aligned}
\mathbf{\Lambda}_i &= (\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{K}_{p,q}\mathbf{\Lambda}\mathbf{K}_{q,p}(\mathbf{e}_{i,p} \otimes \mathbf{I}_q), \\
\mathbf{\Delta}_i &= (\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q)'\mathbf{K}_{p,q}\mathbf{M}\mathbf{Q}\mathbf{M}'\mathbf{K}_{q,p}(\mathbf{V}\mathbf{e}_{i,p} \otimes \mathbf{I}_q).
\end{aligned}$$

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