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ON THE INVERSE OF A CLASS OF WEIGHTED GRAPHS*

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Abstract. In this article, only connected bipartite graphs G with a unique perfect matching $\mathcal{M}$ are considered. Let G_{w} denote the weighted graph obtained from G by giving weights to its edges using the positive weight function $\mathrm{w} : E(G) \to (0, \infty)$ such that $\mathrm{w}(e) = 1$ for each $e \in \mathcal{M}$. An unweighted graph G may be viewed as a weighted graph with the weight function $\mathrm{w} \equiv \mathbf{1}$ (all ones vector). A weighted graph G_{w} is nonsingular if its adjacency matrix $A(G_{\mathrm{w}})$ is nonsingular. The *inverse* of a nonsingular weighted graph G_{w} is the unique weighted graph whose adjacency matrix is similar to the inverse of the adjacency matrix $A(G_{\mathrm{w}})$ via a diagonal matrix whose diagonal entries are either 1 or -1. In [S.K. Panda and S. Pati. On some graphs which possess inverses. *Linear and Multilinear Algebra*, 64:1445–1459, 2016.], the authors characterized a class of bipartite graphs G with a unique perfect matching such that G is invertible. That class is denoted by $\mathcal{H}_{nmc}$. It is natural to ask whether G_{w} is invertible for each invertible graph $G \in \mathcal{H}_{nmc}$ and for each weight function $\mathrm{w} \not\equiv \mathbf{1}$. In this article, first an example is given to show that there is an invertible graph $G \in \mathcal{H}_{nmc}$ and a weight function $\mathrm{w} \not\equiv \mathbf{1}$ such that G_{w} is not invertible. Then the weight functions w for each graph $G \in \mathcal{H}_{nmc}$ such that G_{w} is invertible, are characterized.

Key words. Adjacency matrix, Inverse graph, Weighted graph, Unique perfect matching.

AMS subject classifications. 05C50, 15A09.

1. Introduction.

Let G be a simple, undirected graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $[i, j]$ to denote an edge between the vertices i and j. By G_{w} we denote the weighted graph obtained from G by assigning weights to its edges using the weight function $\mathrm{w} : E(G) \to (0, \infty)$. The unweighted graph G may be viewed as a weighted graph, where each edge has weight 1. Let G_{w} be a weighted graph on vertices $1, \ldots, n$. The adjacency matrix $A(G_{\mathrm{w}})$ of G_{w} is the square symmetric matrix of size n whose (i, j)th entry a_{ij} is given by

$$a_{ij} = \begin{cases} \mathrm{w}([i,j]), & \text{if } [i,j] \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

A *perfect matching* in a graph G is a spanning forest whose components are paths on two vertices. Note that G can have more than one perfect matching. If G has a unique perfect matching, then we denote it by $\mathcal{M}$. Furthermore, when v is a vertex, we shall always use v' to denote the matching mate for v, that is, v' is the vertex for which the edge $[v, v'] \in \mathcal{M}$. Let G be a graph with a unique perfect matching $\mathcal{M}$. An edge $e \in \mathcal{M}$ is called a *matching edge*, while an edge (of G) $e \notin \mathcal{M}$ is called a *nonmatching* edge. Let $\mathcal{H}$ be the class of connected bipartite graphs G with a unique perfect matching $\mathcal{M}$. A weighted graph G_{w} is *nonsingular* if $A(G_{\mathrm{w}})$ is nonsingular. A weighted bipartite graph with a unique perfect matching is nonsingular.

The inverse of a graph was first introduced by Godsil [3]. The weighted version was supplied in [6].

Definition 1.1. [6] Let G_{w} be a nonsingular weighted graph. Suppose that there is a signature matrix S (a diagonal matrix with diagonal entries 1 or -1) such that $SA(G_{\mathrm{w}})^{-1}S$ is nonnegative. Consider the weighted graph H such that $A(H) = SA(G_{\mathrm{w}})^{-1}S$. Then H is called the inverse graph of G_{w}, and it is denoted by G_{w}^{+}.

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Let $\mathcal{H}_g = \{G \in \mathcal{H} \mid G/\mathcal{M} \text{ is bipartite}\}$, where $G/\mathcal{M}$ is the graph obtained from G by contracting each matching edge to a vertex. In [3], Godsil showed that if $G \in \mathcal{H}_g$, then G^+ exists. He posed the problem of characterizing the graphs in $\mathcal{H}$ which possess inverses. In [1], Akbari and Kirkland characterized the unicyclic graphs $G \in \mathcal{H}$ which possess inverses. In [8], Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses, utilizing constructions derived from the graph itself. In [5], Panda and Pati characterized a class of bipartite graphs G with a unique perfect matching such that G is invertible. This class contains the class $\mathcal{H}_g$ and the unicyclic graphs. In [6], Panda and Pati extended the notion of an inverse graph to positively weighted graphs. They showed that for each $G \in \mathcal{H}_g$, the inverse graph G^+_{w} exists for each weight function w such that $\mathrm{w}(e) = 1$ for each $e \in \mathcal{M}$.

Graphs G and H are isomorphic ($G \cong H$) if one can be obtained by relabeling the vertices of the other. An invertible graph G is said to be a *self-inverse* graph if G is isomorphic to its inverse graph. Characterizing self-inverse graphs in $\mathcal{H}$ is also a challenging problem. This question, for the class $\mathcal{H}_g$ was asked by Godsil in 1985 and has already been answered by Simion and Cao in [7]. In [8], Tifenbach and Kirkland supplied necessary and sufficient conditions for a unicyclic graph $G \in \mathcal{H}$ to be self-inverse. In [9], Tifenbach supplied a necessary and sufficient condition for a graph $G \in \mathcal{H}$ to satisfy $G \cong G^+$ via a particular isomorphism. In [6], the authors have proved many different characteristics of the inverse graphs of the graphs $G \in \mathcal{H}_g$.

To proceed further we need the following known definitions.

Definition 1.2. [5] Consider a graph G with a unique perfect matching $\mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *alternating path* if the edges on P are alternately matching and nonmatching edges, that is, for each i, if $[u_i, u_{i+1}]$ is a matching (resp., nonmatching) edge and $[u_{i+1}, u_{i+2}] \in E(G)$, then $[u_{i+1}, u_{i+2}]$ is a nonmatching (resp., matching) edge. Let $P = [u_1, u_2, \ldots, u_{2k}]$ be an alternating path. We say P is an *mm-alternating path* (matching-matching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \in \mathcal{M}$. We say P is an *nn-alternating path* (nonmatching-nonmatching-alternating path) if $[u_1, u_2], [u_{2k-1}, u_{2k}] \notin \mathcal{M}$.

Example 1.3. Consider the graph G shown in Figure 1. The graph G has a unique perfect matching $\mathcal{M} = \{[1, 1'], [2, 2'], [3, 3'], [4, 4']\}$. The alternating paths $[1, 1', 2, 2', 3, 3']$, $[2', 4, 4', 5, 5', 3]$, $[1', 2, 2', 3, 3']$ and $[1, 1', 2, 2', 3]$ in G are examples of mm-alternating path, nn-alternating path, nm-alternating path and mn-alternating path, respectively.

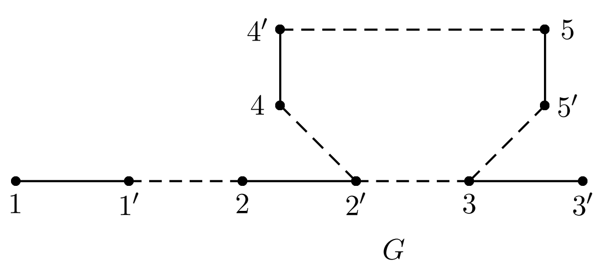

Figure 1. *Here, the solid edges are the matching edges.*

Definition 1.4. [5] Let G be a connected graph with a unique perfect matching $\mathcal{M}$ and $[u, v] \notin \mathcal{M}$. An *extension* at $[u, v]$ is an nn-alternating u-v-path other than $[u, v]$. An extension at $[u, v]$ is called *even type* (resp., *odd type*) if the number of nonmatching edges on that extension is even (resp., odd). For example, in the graph G shown in Figure 1, the path $[2', 4, 4', 5, 5', 3]$ is an extension at $[2', 3]$.

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Definition 1.5. [5] The nonmatching edge $[u, v]$ is said to be an *odd type* edge, if either there are no extensions at $[u, v]$ or each extension at $[u, v]$ is odd type. An odd type nonmatching edge $[u, v]$ is said to be *simple odd type* if there is no extension at $[u, v]$. We say $[u, v]$ is an *even type* edge, if each extension at $[u, v]$ is even type. We say $[u, v]$ is *mixed type*, if it has an even type extension and an odd type extension. Let $\mathcal{E}$ be the set of all even type edges of G.

Definition 1.6. [5] By $\mathcal{H}_{nmc}$ we denote the class of graphs G in $\mathcal{H}$ such that G has no mixed type edges and G satisfies the condition

C: The extensions at two distinct even type edges never have an odd type edge in common.

Here 'nmc' is an abbreviation of 'no mixed type edges and a condition'. Thus,

$$\mathcal{H}_{nmc} = \{G \in \mathcal{H} \mid G \text{ has no mixed type edges and } G \text{ satisfies condition C}\}.$$

Definition 1.7. [5] Let $G \in \mathcal{H}_{nmc}$ and $\mathcal{E}$ be the set of all even type edges. Then by $(G-\mathcal{E})/\mathcal{M}$ denote the graph obtained by deleting all the even type edges and then contracting each matching edge to a single vertex.

Theorem 1.8. [5] *Let $G \in \mathcal{H}_{nmc}$. Then the inverse G^+ exists if and only if $(G-\mathcal{E})/\mathcal{M}$ is bipartite.*

Definition 1.9. [6] Let $G \in \mathcal{H}$. We shall consider weight functions w such that $\mathrm{w}(e) = 1$ for each matching edge e. Let $\mathcal{W}_G$ be the class of such weight functions on G.

The following result can be found in [6].

Theorem 1.10. *Let $G \in \mathcal{H}_g$ and $\mathrm{w} \in \mathcal{W}_G$. Then the inverse G^+_{w} exists.*

Having considered Theorems 1.8 and 1.10, it is natural to ask the following questions.

a) *Does G^+_{w} exist for each invertible graph $G \in \mathcal{H}_{nmc}$ and for each $\mathrm{w}(\not\equiv \mathbb{1})$ in $\mathcal{W}_G$* (see, Example 2.3)?
b) *If the answer of question a) is negative, then characterize all the weight functions* w *for each graph $G \in \mathcal{H}_{nmc}$ such that G^+_{w} exists* (see, Theorem 2.11).

We supply answers to both these questions in Section 2. Finally, we show that if $G \in \mathcal{H}_{nmc}$ and G^+_{w} exists for some weight function $\mathrm{w} \in \mathcal{W}_G$, then $(G-\mathcal{E})/\mathcal{M}$ is bipartite. That is, there is no weight function $\mathrm{w} \in \mathcal{W}_G$ such that G^+_{w} exists for some $G \in \{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M} \text{ is nonbipartite}\}$.

2. Inverses of weighted graphs.

To state the next result, we need the following definition.

Definition 2.1. Let G be a graph. Assume that P is a path in G. We use $\mathrm{w}(P)$ to mean the weight of P, which is the product of the weights of the edges on P. That is $\mathrm{w}(P) = \prod_{e \in E(P)} \mathrm{w}(e)$.

The following is essentially contained in [2, Theorem 1] and [4, Lemma 2.1]. We note that the mm-alternating paths have been termed as alternating paths in [2, 4].

Lemma 2.2. *Consider G_{w}, where $G \in \mathcal{H}$ and $\mathrm{w} \in \mathcal{W}_G$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P(i,j) \in \mathcal{P}(i,j)} (-1)^{(\|P(i,j)\|-1)/2} \mathrm{w}(P),$$

where $\mathcal{P}(i,j)$ is the set of mm-alternating i-j-paths in G_{w} and $\|P(i,j)\|$ is the number of edges in the i-j-path $P(i,j)$. Then $B = A(G_{\mathrm{w}})^{-1}$.

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The following example tells us that there is an invertible graph G in $\mathcal{H}_{nmc}$ such that $G^+_{\rm w}$ does not exist for some ${\rm w} \in \mathcal{W}_G$.

EXAMPLE 2.3. Consider the graph G shown in Figure 2. Notice that $G \in \mathcal{H}_{nmc}$. By Theorem 1.8, G^+ exists. We consider the weight function ${\rm w} : E(G) \to (0, \infty)$ such that ${\rm w}(e) = 1$ for each edge in $G - [1', 3]$ and ${\rm w}([1', 3]) = 2$. Suppose that $G^+_{\rm w}$ exists. Then there is a signature matrix S such that $SA(G_{\rm w})^{-1}S \geq 0$. We use s_i and $A(G_{\rm w})^{-1}_{i,j}$ to denote the ith diagonal entry of S and ijth entry of the matrix $A(G_{\rm w})^{-1}$, respectively. Notice that $A(G_{\rm w})^{-1}_{1,2'} = -1 = A(G_{\rm w})^{-1}_{1,2'} = A(G_{\rm w})^{-1}_{1,3'}$ and $A(G_{\rm w})^{-1}_{2,2'} = 1$. Then we have $s_1 s'_2 = s_2 s'_3 = s_1 s'_3 = -1$ and $s_2 s'_2 = 1$. Therefore, $s_1^2 s_2^2 s_{2'}^2 s_{3'}^2 = -1$ which is not possible. Hence, the answer to question a) is negative.

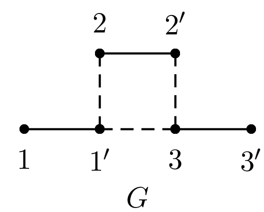

FIGURE 2. *Here the solid edges are the matching edges.*

Next we give an unexpected combinatorial answer to the following question. *Let* $G \in \{G \in \mathcal{H}_{nmc} \mid (G - \mathcal{E})/\mathcal{M}$ *is bipartite*$\}$. *What are those weight functions* ${\rm w} \in \mathcal{W}_G$ *for which* $G^+_{\rm w}$ *exists?* In order to answer this we need the following definition and results from the literature.

DEFINITION 2.4. Let $G \in \mathcal{H}$ and suppose that u and v are two distinct vertices in G. Following [5], we call an mm-alternating u-v-path a *minimal path*, if this path does not contain any even type extensions (of any nonmatching edge in G).

To proceed further we need the following three known results.

LEMMA 2.5. [5] *Let* $G \in \mathcal{H}_{nmc}$. *Let* $P(i, j)$ *be an mm-alternating* i-j-*path. Then there exists a unique minimal* i-j-*path* $P_m(i, j)$ *and a set* F *of even type edges on* $P_m(i, j)$ *such that* $P(i, j)$ *is created from* $P_m(i, j)$ *by replacing each edge* $f \in F$ *with an even type extension* Q_f *at* f.

LEMMA 2.6. [5] *Let* $G \in \mathcal{H}_{nmc}$ *with* $(G - \mathcal{E})/\mathcal{M}$ *is bipartite. Then* G *does not contain a cycle which has an odd number of odd type edges. In particular, if one path from* u *to* v *contains an odd (resp., even) number odd type edges, then each path from* u *to* v *must contain an odd (resp., even) number odd type edges.*

LEMMA 2.7. [5] *Let* $G \in \mathcal{H}$ *and* $P(i, j)$ *be an mm-alternating* i-j-*path. Let* $[u, v]$ *be a nonmatching edge on* $P(i, j)$ *and* $Q(u, v)$ *be an extension at* $[u, v]$. *Then* $Q(u, v)$ *contains no vertex of* $P(i, j)$ *other than* u *and* v. *That is,* $V(P(i, j)) \cap V(Q(u, v)) = \{u, v\}$.

We need another definition which is given below.

DEFINITION 2.8. Let $G \in \mathcal{H}$, ${\rm w} \in \mathcal{W}_G$ and e be an even type edge in G. We define $W(e) = \sum\limits_{Q(e)} {\rm w}(Q(e))$, where the sum is taken over all extensions at e. That is, $W(e)$ is the sum of the weights of all extensions at e.

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The following result supplies a necessary condition on weight functions w such that G^+_{w} exists for each $G \in \{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M} \text{ is bipartite}\}$.

Theorem 2.9. *Let $G \in \mathcal{H}_{nmc}$ for which $(G-\mathcal{E})/\mathcal{M}$ is bipartite and let $\mathrm{w} \in \mathcal{W}_G$. Assume that G^+_{w} exists. Then $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$*

Proof. As $G \in \mathcal{H}_{nmc}$ and $(G-\mathcal{E})/\mathcal{M}$ is bipartite, by Theorem 1.8, G^+ exists. Now suppose that $\mathrm{w} \in \mathcal{W}_G$ is a weight function such that G^+_{w} exists. We shall show that $\mathrm{w}(e) \leq W(e)$ holds for each even type edge e in G. Proceeding by the way of contradiction, let if possible, $\mathrm{w}(e) > W(e)$ hold for some even type edge $e = [u, v]$. Let $Q(u, v) = [u, u_1, u'_1, u_2, u'_2, \ldots, u_{2k-1}, u'_{2k-1}, v]$ be a maximum length even type extension at $[u, v]$. Let $[x, y]$ be a nonmatching edge on $Q(u, v)$. The edge $[x, y]$ is odd type.

Claim. The edge $[x, y]$ is simple odd type.

Proof of the Claim. Suppose that the edge $[x, y]$ is not simple odd type. Then there is an extension $Q(x, y)$ at $[x, y]$, and by Lemma 2.7, x and y are the only common points on the paths $Q(x, y)$ and $[u', Q(u, v), v']$. In that case, by replacing $[x, y]$ with $Q(x, y)$ in $Q(u, v)$, we get a larger length even type extension at $[u, v]$, which is a contradiction. So the claim is justified.

Thus, each nonmatching edge on $Q(u, v)$ is simple odd type. Consider $B = A(G_{\mathrm{w}})^{-1}$. By using Lemma 2.2, we see that

i) $b_{u_i,u'_i} = 1$ for all $i = 1, \ldots, 2k-1$;
ii) $-\mathrm{w}([u', u'_1]) = b_{u',u'_1}$, $b_{v',u_{2k-1}} = -\mathrm{w}([v', u'_{2k-1}])$ and $b_{u_i,u'_{i+1}} = -\mathrm{w}([u_i, u'_{i+1}])$ for all $i = 1, \ldots, 2k-2$, as each nonmatching edge on $Q(u, v)$ is simple odd type; and
iii) $b_{u',v'} = -\mathrm{w}(e) + W(e) < 0$.

Since G^+_{w} exists, there is a signature matrix S such that $SA(G_{\mathrm{w}})^{-1}S \geq 0$. Then we have

i) $s_{u_i}s_{u'_i} = 1$ for all $i = 1, \ldots, 2k-1$;
ii) $-1 = s_{u'}s_{u'_1} = s_{v'}s_{u_{2k-1}} = s_{u_i}s_{u'_{i+1}}$ for all $i = 1, \ldots, 2k-2$; and
iii) $s_{u'}s_{v'} = -1$.

Therefore, $s^2_{u'}s^2_{v'}s^2_{u_1} \cdots s^2_{u_{2k-1}}s^2_{u'_1} \cdots s^2_{u'_{2k-1}} = -1$ which is not possible. This is a contradiction to our hypothesis that G^+_{w} exists. Hence, $\mathrm{w}(e) \leq W(e)$ for all $e \in \mathcal{E}$. ∎

The following result tells us that the above necessary condition on weight functions w is also sufficient for the existence of G^+_{w} for a graph $G \in \{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M} \text{ is bipartite}\}$.

Theorem 2.10. *Let $G \in \mathcal{H}_{nmc}$ for which $(G-\mathcal{E})/\mathcal{M}$ is bipartite and $\mathrm{w} \in \mathcal{W}_G$. Assume that $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$. Then G^+_{w} exists.*

Proof. Suppose that $G \in \mathcal{H}_{nmc}$ with $(G-\mathcal{E})/\mathcal{M}$ bipartite. Take a weight function $\mathrm{w} \in \mathcal{W}_G$ such that $\mathrm{w}(e) \leq W(e)$ holds for each $e \in \mathcal{E}$. We shall show that G^+_{w} exists. Let S be the signature matrix defined by $s_1 = 1$ and $s_i = (-1)^z$, where z is the number of odd type edges on a i-1-path. This matrix is well defined, in view of Lemma 2.6. Suppose that $SA(G_{\mathrm{w}})^{-1}S \not\geq 0$. That is, there exist i and j such that $s_iA(G_{\mathrm{w}})^{-1}_{i,j}s_j < 0$. We have two possibilities.

Case I. The entry $A(G_{\mathrm{w}})^{-1}_{i,j} < 0$. Then $s_i = s_j$. By Lemma 2.6, the parity of the number of odd type edges on any path from 1 to i is the same with that of any path from 1 to j. It follows that any path from i to j must contain an even number of odd type edges.

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Let $P_m^1(i,j), P_m^2(i,j), \ldots, P_m^t(i,j)$ be the minimal paths from i to j. Let $\mathcal{P}^r(i,j)$ be the set of all mm-alternating i-j-paths which are created from $P_m^r(i,j)$, for $r = 1, \ldots, t$. Using Lemma 2.5, we have $|\mathcal{P}(i,j)| = \sum_{r=1}^t |\mathcal{P}^r(i,j)|$. Using Lemma 2.2, we have

$$A(G_{\mathrm{w}})^{-1}_{i,j} = \sum_{r=1}^{t} \sum_{P(i,j)\in\mathcal{P}^r(i,j)} \left[(-1)^{\frac{\|P(i,j)\|-1}{2}} \mathrm{w}(P(i,j))\right], \tag{2.1}$$

where $\sum_{P(i,j)\in\mathcal{P}^r(i,j)} \left[(-1)^{\frac{\|P(i,j)\|-1}{2}} \mathrm{w}(P(i,j))\right]$ is the contribution to $A(G_{\mathrm{w}})^{-1}_{i,j}$ coming from the rth minimal path $P_m^r(i,j)$.

Assume first that $P_m^r(i,j)$ contains an odd number of nonmatching edges. As any i-j-path contains an even number of odd type edges, we must have an odd number of even type edges on $P_m^r(i,j)$. Let $e_1, e_2, \ldots, e_k$ be the even type edges on the rth minimal path $P_m^r(i,j)$, where k is odd. Let $m_l \geq 1$ be the number of extensions (these are even type) at the edge e_l, for $l = 1, \ldots, k$. Suppose that we choose the even type edges $e_{i_1}, \ldots, e_{i_p}$ from $e_1, e_2, \ldots, e_k$ and create an mm-alternating i-j-path by using one extension for each of the chosen even type edges. Then we can create $m_{i_1} \cdots m_{i_p}$ many such mm-alternating i-j-paths and each such path has an odd (resp., even) number of nonmatching edges if p is even (resp., odd). Thus, the contribution of the mm-alternating paths that are created from $P_m^r(i,j)$ by choosing p many edges out of $e_1, e_2, \ldots, e_k$, to $A(G_{\mathrm{w}})^{-1}_{i,j}$ is

$$(-1)^{p+1}\mathrm{w}(P_m^r(i,j)) \sum_{\{e_{i_1},\ldots,e_{i_p}\}\subseteq\{e_1,e_2,\ldots,e_k\}} \frac{W(e_{i_1})W(e_{i_2})\cdots W(e_{i_p})}{\mathrm{w}(e_{i_1})\mathrm{w}(e_{i_2})\cdots \mathrm{w}(e_{i_p})}.$$

Hence, the total contribution of $\mathcal{P}^r(i,j)$, the set of mm-alternating i-j-paths that are created from $P_m^r(i,j)$, to $A(G_{\mathrm{w}})^{-1}_{i,j}$ is

$$\begin{aligned}
&\sum_{p=0}^{k}(-1)^{p+1}\mathrm{w}(P_m^r(i,j)) \sum_{\{e_{i_1},\ldots,e_{i_p}\}\subseteq\{e_1,e_2,\ldots,e_k\}} \frac{W(e_{i_1})W(e_{i_2})\cdots W(e_{i_p})}{\mathrm{w}(e_{i_1})\mathrm{w}(e_{i_2})\cdots \mathrm{w}(e_{i_p})}\\
&= -\mathrm{w}(P_m^r(i,j)) \sum_{p=0}^{k} \sum_{\substack{T\subseteq\{e_1,e_2,\ldots,e_k\}\\ |T|=p}} \prod_{e_i\in T} \frac{-W(e_i)}{\mathrm{w}(e_i)}\\
&= -\mathrm{w}(P_m^r(i,j)) \prod_{i=1}^{k}\left[1-\frac{W(e_i)}{\mathrm{w}(e_i)}\right] \geq 0,
\end{aligned}$$

as k is odd. Similarly, if $P_m^r(i,j)$ contains an even number of nonmatching edges, then also the contribution $P_m^r(i,j)$, to $A(G_{\mathrm{w}})^{-1}_{i,j}$ is nonnegative. Hence, $A(G_{\mathrm{w}})^{-1}_{i,j} \geq 0$, by (2.1). This contradicts the hypothesis that $A(G_{\mathrm{w}})^{-1}_{i,j} < 0$.

Case II. The entry $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$. Carrying the arguments in a way similar to CASE I, we get a contradiction to our hypothesis that $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$.

Hence, we conclude that $SA(G_{\mathrm{w}})^{-1}S \geq 0$. That is, G_{w}^+ exists. □

The main theorem of this section is the following.

THEOREM 2.11. *Let $G \in \mathcal{H}_{nmc}$ for which $(G-\mathcal{E})/\mathcal{M}$ is bipartite and $\mathrm{w} \in \mathcal{W}_G$. Then G_{w}^+ exists if and only if $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$, where $\mathcal{E}$ is the set of all even type edges.*

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Remark 2.12. It is clear that Theorems 1.8 and 1.10 are particular cases of Theorem 2.11.

Instead of looking at $\{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M} \text{ is bipartite}\}$, let us look at the larger class $\mathcal{H}_{nmc}$ itself. Suppose that for $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$, the inverse G^+_{w} exists. What can be said about such a graph G? The following result says that in that case the graph G must belong to $\{G \in \mathcal{H}_{nmc} \mid (G-\mathcal{E})/\mathcal{M} \text{ is bipartite}\}$. In other words, these are the only graphs in $\mathcal{H}_{nmc}$ which have inverses for some weight functions.

Proposition 2.13. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$. If G^+_{w} exists, then $(G-\mathcal{E})/\mathcal{M}$ is bipartite.*

Proof. Let $G \in \mathcal{H}_{nmc}$, $\mathrm{w} \in \mathcal{W}_G$ for which G^+_{w} exists. Let S be the signature matrix such that $SA(G_{\mathrm{w}})^{-1}S \geq 0$. As $G \in \mathcal{H}_{nmc}$, deleting the even type edges, we see that $(G-\mathcal{E})$ has no even type edges. Then by using Lemma 2.2, we have

i) $A(G_{\mathrm{w}})^{-1}_{u',v'} < 0$ for any nonmatching edge $[u,v] \in (G-\mathcal{E})$ and
ii) $A(G_{\mathrm{w}})^{-1}_{x,x'} = 1$ for any matching edge $[x,x'] \in (G-\mathcal{E})$.

Let $[u,v] \in (G-\mathcal{E})$ be a nonmatching edge. So $A(G_{\mathrm{w}})^{-1}_{u',v'} < 0$. Since $s_{u'}A(G_{\mathrm{w}})^{-1}_{u',v'}s_{v'} \geq 0$, we have that $s_{u'}s_{v'} = -1$. Let $[x,x']$ be a matching edge in $(G-\mathcal{E})$. By similar arguments, we have $s_x s_{x'} = 1$. Taking $X = \{u \in (G-\mathcal{E})/\mathcal{M} \mid s_u > 0\}$ and $Y = \{u \in (G-\mathcal{E})/\mathcal{M} \mid s_u < 0\}$, we get a bipartition. ☐

We summarize our observation of this article by the following result which also addresses question b).

Theorem 2.14. *Let $G \in \mathcal{H}_{nmc}$ and $\mathrm{w} \in \mathcal{W}_G$.*

i) If $(G-\mathcal{E})/\mathcal{M}$ is bipartite, then G^+_{w} exists if and only if $\mathrm{w}(e) \leq W(e)$ for each $e \in \mathcal{E}$.
ii) If $(G-\mathcal{E})/\mathcal{M}$ is not bipartite, then G^+_{w} does not exist.

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